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Shifted Legendre Tau Method for Solving the Fractional Stochastic Integro-Differential Equations

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Abstract In this paper, the Tau method based on shifted Legendre polynomials is proposed for solving a class of fractional stochastic integro-differential equations. For this purpose, shifted Legendre polynomials and their properties are introduced. By using the operational matrices of integration and stochastic Ito-integration we transform the problem into the corresponding linear system of algebraic equations. Finally the efficiency of the proposed method is confirmed by some examples. The results show that this method is very accurate and efficient.

Keywords Shifted Legendre polynomials · Fractional stochastic integro-differential equation · Ito integral.

Mathematics Subject Classification (2010) 65C30 · 60H20

1 Introduction

A probability space (Ω, F, P) consists of three elements. A sample space Ω, which is the set of all possible outcomes. An event space, which is a set of events F, an event being a set of outcomes in the sample space. A probability

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function P, which assigns each event in the event space a probability, which is a number between 0 and 1 [23].

Stochastic integral equations are very important due to their application for modelling stochastic phenomena in physics, fluid mechanics, biology, chemistry, finance, mechanics, microelectronics, etc. see [9,14,17]. The behavior of dynamical systems in these kind of equations are often dependent on a noise source and a Gaussion white noise, governed by certain probability laws. So, modelling such phenomena often requires the use of different stochastic differential equations, stochastic integro-differential equations.

As deriving an explicit form of the solution for stochastic differential and integral equations is difficult, numerical approximation becomes a practical way to face this difficulty. Several numerical methods such as finite-difference method [13], wavelet Galerkin method [5], Haar wavelets method [16], Laplace transforms method [19], Chebyshev wavelet method [15], orthogonal functions [24,21,3,10] have been used for solving fractional differential, integral equation, stochastic integral equations.

The Tau method that is a way to solve linear and nonlinear functional equations is one of the important types of the spectral method that express the solution of the problem as a linear combination of orthogonal or non-orthogonal basis functions. The main advantage of using orthogonal basis is that the problem under consideration is reduced into solving a system of linear or nonlinear algebraic equation [20]. Recently, different orthogonal basis functions such as block pulse functions, Walsh functions, orthogonal polynomials and wavelets, Fourier series, were utilized to approximate solution of functional equations [22,4,8,11,2].

Shifted Legendre polynomials have been widely applied for solving functional equations [7,25]. In this paper the shifted Legendre polynomials will be used for solving the fractional stochastic integro-differential equation which is given in [1] as follows

$$D^{\alpha}y(t) = f(t) + \lambda_1 \int_0^t k_1(t,\tau)y(\tau)d\tau + \lambda_2 \int_0^t k_2(t,\tau)y(\tau)dB(\tau), \quad 0 \le t \le T, \tag{1}$$

subject to the initial conditions

$$y^{(j)}(0) = d_j, \quad j = 0, 1, \dots, r-1, \quad r-1 < \alpha \le r, \quad r \in \mathbb{N}, \tag{2}$$

where $y^{(j)}(t)$ stands for the j-th order derivative of $y(t)$, $D^{\alpha}(.)$ denotes the Caputo fractional order derivate of order α and $y(t)$, $f(t)$, and $k_i(t,\tau)$, $i = 1, 2$ are the stochastic processes defined on the probability space (Ω, F, P), $y(t)$ is unknown and $\int_0^t k_2(t,\tau)y(\tau)dB(\tau)$ is Ito integral. Here, λ_1 and λ_2 are real constants. A real-valued stochastic process $B(t)$, $t \in [0, T]$ is called Brownian motion, if it satisfies the following properties [6,15]

(i) $B(0) = 0$ (with the probability 1).

(ii) For $0 \le s < t \le T$ the random variable given by increment $B(t) - B(s)$ is normally distributed with mean zero and variance $t - s$; equivalently,

$$B(t) - B(s) \sim \sqrt{t-s}N(0,1),$$

where $N(0,1)$ denotes a normally distributed random variable with zero mean and unit variance.

(iii) For $0 \le s < t \le u < v \le T$ the increments $B(t) - B(s)$ and $B(v) - B(u)$ are independent.

(iv) The function $t \to B(t)$ is continuous function of t.

The paper is organized as follows. In the next section, the shifted Legendre polynomials and their properties are described. Section 3 is devoted to some preliminary definitions of BPFs and fractional calculus. In Sections 4 and 5 after expressing the relation between BPFs and shifted Legendre polynomials, Tau method based on shifted Legendre polynomials and their matrices are proposed for solving fractional stochastic integro-differential equation. Some numerical examples are solved using the method of this article in section 6. Finally, a conclusion is given in section 7.

2 Properties of shifted Legendre polynomials

The classical Legendre polynomials are defined on the interval $[-1, 1]$ and can be determined with the aid of the following recurrence formulae

$$P_0(x) = 1,\ \ P_1(x) = x,$$

$$P_{i+1}(x) = \frac{2i+1}{i+1} x\ P_i(x) - \frac{i}{i+1} P_{i-1}(x), \quad i = 1, 2, \ldots .$$

Assume $x \in [a, b]$ and let $\overline{x} = \frac{2x-a-b}{b-a}$. Then $\{P_i(\overline{x})\}$ are called the shifted Legendre polynomials on $[a, b]$. In this paper, we mainly consider the shifted Legendre polynomials defined on $[0, l]$.

For $t \in [0, l]$, let $L_{l,i}(t) = P_i(\frac{2t-l}{l})$, $i = 0, 1, 2, \ldots$. Then the shifted Legendre polynomials $\{L_{l,i}(t)\}$ are defined by

$$\begin{aligned} L_{l,0}(t) &= 1, \\ L_{l,1}(t) &= \frac{2t-l}{l}, \\ L_{l,i+1}(t) &= \frac{(2i+1)(2t-l)}{(i+1)l} L_{l,i}(t) - \frac{i}{i+1} L_{l,i-1}(t), \quad i = 1, 2, \ldots . \end{aligned}$$

If $\psi_{l,m}(t)$ be a vector function of shifted Legendre polynomials on the interval $[0, l]$, as

$$\psi_{l,m}(t) = [L_{l,0}, L_{l,1}, \ldots, L_{l,m}]^T, \tag{3}$$

then the set of $L_{l,i}(t)$ is a complete $L^2(0,l)$-orthogonal system, namely

$$\int_0^l L_{l,i}(t)L_{l,j}(t)dt = \begin{cases} \dfrac{l}{2i+1}, & i=j, \\ 0, & i \neq j. \end{cases}$$

So we define $\Pi_m = \text{span}\{L_{l,0}, L_{l,1}, \dots, L_{l,m}\}$. For any $y(t) \in L^2(0,l)$, we write

$$y(t) \simeq \sum_{j=0}^{\infty} c_j L_{l,j}(t),$$

where the coefficients c_j are given by

$$c_j = \frac{2j+1}{l}\int_0^l y(t)L_{l,j}(t)dt, \quad j = 0,1,2,\dots. \tag{4}$$

In practice, only the first $(m+1)$-terms of shifted Legendre polynomials are considered. Hence, we can write

$$y_m(t) \simeq \sum_{j=0}^{m} c_j L_{l,j}(t) = C^T\psi_{l,m}(t) = C^T V X_t,$$

where $C^T = [c_0, c_1, \dots, c_m]$ and V is a non-singular matrix given by

$$\psi_{l,m}(t) = VX_t,$$

with a standard basic vector, $X_t = \left[1, t, t^2, \dots, t^m\right]^T$, where, $(.)^T$ stands for the transpose.

Similarly a function of two independent variables $k(t,\tau)$ may be expressed in terms of the double shifted Legendre polynomials as

$$k(t,\tau) \simeq \sum_{i=0}^{m}\sum_{j=0}^{m} k_{i,j}L_{l,i}(t)L_{l,j}(\tau) = \psi_{l,m}^T(t)K\psi_{l,m}(\tau), \tag{5}$$

where K is a $(m+1)\times(m+1)$ matrix as

$$K = \begin{bmatrix} k_{00} & k_{01} & \dots & k_{0m} \\ k_{10} & k_{11} & \dots & k_{1m} \\ \vdots & \vdots & \dots & \vdots \\ k_{m0} & k_{m1} & \dots & k_{mm} \end{bmatrix},$$

where

$$k_{i,j} = \left(\frac{2i+1}{l}\right)\left(\frac{2j+1}{l}\right)\int_0^l\int_0^l k(t,\tau)L_{l,i}(t)L_{l,j}(\tau)dtd\tau, \quad i,j = 0,1,\dots,m. \tag{6}$$

Also, $k(t,\tau)$ can be expressed as

$$k(t,\tau) \simeq \psi_{l,m}^T(t)K\psi_{l,m}(\tau) = X_t^T V^T K V X_\tau,$$

where $V = [v_{i,j}]_{i,j=0,1,\dots,m}$ is a non-singular matrix given by $\psi_{l,m}(t) = VX_t$ with a standard basic vector, $X_t = \left[1, t, t^2, \dots, t^m\right]^T$. If we take $\overline{K} = V^T K V$ then we can write $k(t,\tau) \simeq X_t^T \overline{K}\, X_\tau$.

Now, we present the shifted Legendre expansion of a function $y(t)$ with bounded second derivative, converges uniformly to $y(t)$.

Theorem 1 *A continuous function $y(t) \in [0,l]$, with bounded second derivative, say $\left|\frac{d^2y(t)}{dt^2}\right| \le \alpha$, can be expanded as an infinite sum of shifted Legendre polynomials and the series $\sum_{i=0}^{\infty} c_i L_{l,i}(t)$ converges uniformly to the $y(t)$. Furthermore, we have*

$$\int_0^l \Big(y(t) - \sum_{i=0}^{m} c_i L_{l,i}(t)\Big)^2 dt \le \alpha l^2 \sqrt{\frac{3l}{8}} \sqrt{\sum_{i=m+1}^{\infty} \frac{1}{(2i-3)^4}}$$

Proof From (4), it follows that

$$c_i = \left(\frac{2i+1}{l}\right) \int_0^l y(t) L_{l,i}(t)dt, \quad i = 0, 1, \dots, m.$$

By partial integration and using following equation

$$L'_{l,i+1}(t) - L'_{l,i-1}(t) = \frac{2}{l}(2i+1)L_{l,i}(t),$$

we have

$$\begin{aligned}
c_i &= \frac{2i+1}{l} \times \frac{l}{2(2i+1)} \int_0^l y(t)\Big(L'_{l,i+1}(t) - L'_{l,i-1}(t)\Big)dt \\
&= \frac{1}{2}\Big(y(t)\Big(L_{l,i+1}(t) - L_{l,i-1}(t)\Big)\Big|_0^l - \int_0^l \Big(L_{l,i+1}(t) - L_{l,i-1}(t)\Big)\frac{dy}{dt}dt \\
&= -\frac{1}{2}\int_0^l \frac{l}{2(2i+3)}\Big(L'_{l,i+2}(t) - L'_{l,i}(t)\Big)\frac{dy}{dt}dt \\
&+ \frac{l}{2}\int_0^l \frac{l}{2(2i-1)}\Big(L'_{l,i}(t) - L'_{l,i-2}(t)\Big)\frac{dy}{dt}dt \\
&= \frac{l}{4}\int_0^l \frac{d^2y(t)}{dt^2}\Big(\frac{L_{l,i+2}(t) - L_{l,i}(t)}{2i+3}\Big)dt \\
&- \frac{l}{4}\int_0^l \frac{d^2y(t)}{dt^2}\Big(\frac{L_{l,i}(t) - L_{l,i-1}(t)}{2i-1}\Big)dt.
\end{aligned}$$

Now, let $Q_{l,i}(t) = (2i-1)L_{l,i+2}(t) - 2(2i+1)L_{l,i}(t) + (2i+3)L_{l,i-2}(t)$ then we have

$$c_i = \frac{l}{4(2i+3)(2i-1)} \int_0^l \frac{d^2y(t)}{dt^2} Q_{l,i}(t)dt,$$

thus

$$\begin{aligned}
|c_i| &\le \frac{l}{4(2i+3)(2i-1)} \int_0^l \Big|\frac{\partial^2 y(t)}{\partial t^2}\Big| |Q_{l,i}(t)| dt \\
&\le \frac{l\alpha}{4(2i+3)(2i-1)} \int_0^l |Q_{l,i}(t)| dt.
\end{aligned}$$

Also we have

$$\begin{aligned}
\Big(\int_0^l |Q_i(t)| dt\Big)^2 &= \Big(\int_0^l |(2i-1)L_{l,i+2}(t) - 2(2i+1)L_{l,i}(t) + (2i+3)L_{l,i-2}(t)| dt\Big)^2 \\
&\le \Big(\int_0^l (1)^2 dt\Big)\Big(\int_0^l (2i-1)^2 L^2_{l,i+2}(t) + (4i+2)^2 L^2_{l,i}(t) + (2i+3)^2 L^2_{l,i-2}(t)\Big) dt \\
&\le l\Big(\frac{(2i-1)^2 l}{2i+5} + \frac{(4i+2)^2 l}{2i+1} + \frac{(2i+3)^2 l}{2i-3}\Big) \\
&\le \frac{6l^2(2i+3)^2}{2i-3}.
\end{aligned}$$

Then we get

$$\int_0^l |Q_i(t)| dt \le \frac{\sqrt{6}\, l(2i+3)}{\sqrt{2i-3}}.$$

Thus we obtain

$$|c_i| \le \frac{l\alpha}{4(2i+3)(2i-1)} \times \frac{\sqrt{6}\, l(2i+3)}{\sqrt{2i-3}} = \frac{l^2\alpha\sqrt{6}}{4\sqrt{(2i-3)^3}}.$$

Consequently, $\sum\limits_{i=0}^{\infty} c_i$ is absolute convergent and thus the expansion of the function converges uniformly. Also, we let

$$\varepsilon_n = \Big(\int_0^l (y(t) - \sum_{i=0}^{m} c_i L_{l,i}(t))^2 dt\Big)^{1/2},$$

where

$$
\begin{aligned}
\varepsilon_n^2 &= \int_0^l \Big(y(t) - \sum_{i=0}^{m} c_i L_{l,i}(t)\Big)^2 dt \\
&= \int_0^l \Big(\sum_{i=0}^{\infty} c_i L_{l,i}(t) - \sum_{i=0}^{m} c_i L_{l,i}(t)\Big)^2 dt \\
&= \int_0^l \Big(\sum_{i=m+1}^{\infty} c_i L_{l,i}(t)\Big)^2 dt \\
&= \int_0^l \sum_{i=m+1}^{\infty} c_i^2 L_{l,i}^2(t) dt \\
&= \sum_{i=m+1}^{\infty} c_i^2 \int_0^l L_{l,i}^2(t) dt \\
&= \sum_{i=m+1}^{\infty} c_i^2 \frac{l}{(2i+1)} \\
&\leq \sum_{i=m+1}^{\infty} \frac{6\alpha^2 l^5}{16(2i-3)^3(2i+1)} \\
&\leq \frac{6\alpha^2 l^5}{16} \sum_{i=m+1}^{\infty} \frac{1}{(2i-3)^4}
\end{aligned}
$$

Then we have

$$
\varepsilon_n \leq \alpha l^2 \sqrt{\frac{3l}{8}} \sqrt{\sum_{i=m+1}^{\infty} \frac{1}{(2i-3)^4}}.
$$

From then on we assume $l = 1$ and let $\psi_{1,m} = \psi_m$, $L_{1,i} = L_i$.

3 Preliminary definitions

3.1 Block Pulse functions

In this paper, it is assumed that $T = 1$. So, the set of BPFs are defined over $[0, 1]$, and $h = \frac{1}{m}$. We define the m-set of BPFs as

$$
\phi_i(t) = \begin{cases} 1, & (i-1)h \leq t < ih, \\ 0, & \text{otherwise.} \end{cases}
$$

The elementary properties of BPFs are as follows

i) Disjointness: The BPFs are disjoint with each other in the interval $[0, 1]$ and

$$\phi_i(t)\phi_j(t) = \delta_{ij}\phi_i(t), \quad i, j = 1, 2, \dots, m.$$

ii) Orthogonality: The set of BPFs defined in the interval $[0, 1]$ are orthogonal with each other, that is

$$\int_0^1 \phi_i(t)\phi_j(t)dt = h\delta_{ij}, \quad i, j = 1, 2, \dots, m.$$

iii) Completeness: If $m \longrightarrow \infty$, then the BPFs set are complete, so an arbitarary real bounded function $f(t)$, which is square integrable in the interval $[0, 1]$, can be expanded into a Block Pulse series as

$$f(t) \simeq \sum_{i=1}^{m} f_i \phi_i(t),$$

where

$$f_i = \frac{1}{h}\int_0^1 \phi_i(t)f(t)dt, \quad i = 1, 2, \dots, m.$$

iv) Vector form: Consider the first m terms of BPFs and write them as

$$\Phi(t) = [\phi_1(t), \phi_2(t), \dots, \phi_m(t)]^T, \quad t \in [0, 1].$$

The above representation and disjointness property follows

$$\Phi(t)\Phi^T(t) = \begin{bmatrix} \phi_1(t) & 0 & \dots & 0 \\ 0 & \phi_2(t) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \phi_m(t) \end{bmatrix},$$

furthermore, we have

$$\Phi^T(t)\Phi(t) = 1.$$

For an m-vector V we have

$$\Phi(t)\Phi^T(t)V = \overline{V}\Phi(t), \tag{7}$$

where $\overline{V}$ is an $m \times m$ matrix, and $\overline{V} = \text{diag}(V)$.
Also, it is easy to show that for an $m \times m$ matrix A

$$\Phi^T(t)A\Phi(t) = \overline{A}^T\Phi(t), \tag{8}$$

where $\overline{A} = \text{diag}(A)$ is a m-vector.

3.2 Fractional calculus

Definition 1 The Riemann-Liouville fractional integral operator J^{α} of order α is given by

$$J^{\alpha}y(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}y(\tau)d\tau, \quad \alpha > 0,$$
$$J^0 y(t) = y(t).$$

Definition 2 The Caputo definition of fractional operator is given by

$$D^{\alpha}y(t) = \begin{cases} \dfrac{d^r y(t)}{dt^r}, & \alpha = r \in \mathbb{N}, \\ \dfrac{1}{\Gamma(r-\alpha)}\int_0^t \dfrac{y^{(r)}(\tau)}{(t-\tau)^{\alpha-r+1}}d\tau, & 0 \le r-1 < \alpha < r. \end{cases}$$

The Caputo fractional derivatives of order α is also defined as

$$D^{\alpha}y(t) = J^{r-\alpha}D^r y(t).$$

The relation between the Caputo operator and the Riemann-Liouville is given by the

$$D^{\alpha}J^{\alpha}y(t) = y(t),$$
$$D^{\alpha}J^{\alpha}y(t) = y(t) - \sum_{k=0}^{r-1} y^{(k)}(0^+)\frac{t^k}{k!}, \quad t > 0.$$

4 Operational matrices of shifted Legendre polynomials

4.1 Relation between the BPFs and shifted Legendre polynomials

In this section we will obtain the relation between the BPFs and shifted Legendre polynomials.

Theorem 2 *Let $\psi_m(t)$ and $\Phi(t)$ be the m-dimensional shifted Legendre polynomials and BPFs vector respectively, the vector $\psi_m(t)$ can be expanded by BPFs vector $\Phi(t)$ as*

$$\psi_m(t) \simeq Q\Phi(t),$$

where Q is an $m \times m$ matrix

$$Q = \Big[\psi_m(\frac{1}{2m}), \psi_m(\frac{3}{2m}), \ldots, \psi_m(\frac{2m-1}{2m})\Big].$$

Proof *Let $L_i(t)$, $i = 1, 2, \ldots, m$ be the i-th element of shifted Legendre polynomial. Expanding $L_i(t)$ into an m-term vector of BPFs, we have*

$$L_i(t) \simeq \sum_{j=1}^{m} Q_{i,j}\phi_j(t) = Q_i^T\Phi(t), \quad i = 1, 2, \ldots, m,$$

where Q_i *is the* i*-th row and* Q_{ij} *is the* (i,j)*-th element of matrix* Q*. By using orthogonality of BPFs we have*

$$Q_{i,j} = \frac{1}{h}\int_0^1 L_i(t)\phi_j(t)dt = \frac{1}{h}\int_{\frac{j-1}{m}}^{\frac{j}{m}} L_i(t)dt.$$

By using mean value theorem for integrals we have

$$Q_{i,j} = \frac{1}{h}\Big(\frac{j}{m} - \frac{j-1}{m}\Big)L_i(\varepsilon_j) = L_i(\varepsilon_j), \quad \varepsilon_j \in \Big(\frac{j}{m}, \frac{j-1}{m}\Big),$$

by taking $\varepsilon_j = \frac{2j-1}{2m}$ *so we have* $Q_{i,j} = L_i(\frac{2j-1}{2m})$.

Lemma 1 *For an* m*-vector* V *we have*

$$\psi_m(t)\psi_m^T(t)V = \overline{V}\psi_m(t),$$

in which $\overline{V}$ *is an* $m \times m$ *matrix as* $\overline{V} = QV_1Q^{-1}$*, where* $V_1 = diag(Q^TV)$*. Moreover, it can be easy to show that for an* $m \times m$ *matrix* A

$$\psi_m^T(t)A\psi_m(t) = \overline{A}^T\psi_m(t),$$

where $\overline{A}^T = UQ^{-1}$ *and* $U = diag(Q^TAQ)$ *is an* m*-vector.*

Proof *The results is the consequence of relations* (7)*,* (8)*, and Theorem* (2)*.*

Lemma 2 *([12]) Let* $\Phi(t)$ *be the BPFs, then integration of this vector can be derived as*

$$\int_0^t \Phi(\tau)d\tau \simeq P\Phi(t),$$

where P *is an* $m \times m$ *matrix given by*

$$P = \frac{h}{2}\begin{bmatrix} 1 & 2 & 2 & \dots & 2 \\ 0 & 1 & 2 & \dots & 2 \\ 0 & 0 & 1 & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 2 \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}.$$

Lemma 3 *([12]) Let* $\Phi(t)$ *be the BPFs, then the Ito integral of this vector can be derived as*

$$\int_0^t \Phi(\tau)dB(\tau) \simeq P_\tau\Phi(t),$$

where P_τ *is an* $m \times m$ *matrix given by*

$$P_\tau = \begin{bmatrix} B(\frac{h}{2}) & B(h) & B(h) & \dots & B(h) \\ 0 & B(\frac{3h}{2}) - B(h) & B(2h) - B(h) & \dots & B(2h) - B(h) \\ 0 & 0 & B(\frac{5h}{2}) - B(2h) & \dots & B(3h) - B(2h) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & B(\frac{(2m-1)h}{2}) - B((m-1)h) \end{bmatrix}.$$

Theorem 3 *Suppose $\psi_m(t)$ be the shifted Legendre vector defined in* (3) *then the integral of this vector can be derived as*

$$\int_0^t \psi_m(\tau)d\tau \simeq (QPQ^{-1})\psi_m(t).$$

Proof From Theorem (2) *and Lemma* (2), *we have*

$$\int_0^t \psi_m(\tau)d\tau \simeq \int_0^t Q\Phi(\tau)d\tau \simeq QP\Phi(t) \simeq (QPQ^{-1})\psi_m(t).$$

Theorem 4 *Suppose $\psi_m(t)$ be the shifted Legendre vector defined in* (3) *then the Ito integral of this vector can be derived as*

$$\int_0^t \psi_m(\tau)dB(\tau) \simeq (QP_\tau Q^{-1})\psi_m(t).$$

Proof Similar to previous Theorem and using Theorem (2) *and Lemma* (3) *it can be easily obtained.*

In this section, we make the operational matrix of fractional integro-differential equation with weakly singular kernel of the shifted Legendre vector.

4.2 Matrix representation of (1)

As a consequence of the previous section, and aid of following Lemma and Theorems we derive formulas for numerical solvability of fractional stochastic integro-differential equations (1) based on shifted Legendre polynomials of the operational Tau method.

Lemma 4 *Let $y_m(t) = C^T V X_t$ be a polynomial where*

$$C^T = [c_0, c_1, \dots, c_m, 0, \dots],$$

and $X_t = [1, t, t^2, \dots]^T$ then we have

$$\frac{d^k}{dt^k} y_m(t) = C^T V \eta^k X_t,$$

$$t^k y_m(t) = C^T V \mu^k X_t,$$

$$k = 0, 1, 2, \dots,$$

where

$$\mu = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots \\ & 0 & 1 & 0 & \\ & & 0 & 1 & \\ & & & & \ddots \end{bmatrix}, \qquad \eta = \begin{bmatrix} 0 & & & \dots \\ 1 & 0 & & \\ 0 & 2 & 0 & \\ 0 & 0 & 3 & \\ & & & \ddots \end{bmatrix}.$$

Proof see [18].

Lemma 5 *If Γ is the Gamma function, then we have*

$$\int_0^t \frac{\tau^m}{(t-\tau)^{\alpha-r+1}}\,\mathrm{d}s = \frac{\Gamma(r-\alpha)\Gamma(m+1)}{\Gamma(m-\alpha+r+1)} t^{m-\alpha+r}, \qquad m = 0, 1, 2, \ldots.$$

Proof With integration by parts and using $\Gamma(\alpha) = (\alpha-1)!$ $\alpha > 0$ it can easily be obtained.

Theorem 5 *Let $\psi_m(t) = VX_t$ be the shifted Legendre vector then*

$$\int_0^t \frac{y^{(r)}(\tau)}{(t-\tau)^{\alpha-r+1}} ds \simeq C^T V \eta^r G\, A\, V\, X_t, \tag{9}$$

where G is a diagonal matrix with elements

$$G_{i,i} = \frac{\Gamma(r-\alpha)\Gamma(i+1)}{\Gamma(i-\alpha+r+1)}, \qquad i = 0, 1, 2, \ldots, m,$$

and

$$A = \Big[B_0, B_1, \ldots, B_m\Big]^T, \qquad B_j = \Big[t_{j,0}, t_{j,1}, \ldots, t_{j,m}\Big],$$

which $t_{j,i}$, $i, j = 0, 1, \ldots, m$ are the coefficients of $L_{l,i}(t)$, $i = 0, 1, \ldots, m$ in expansion of $t^{j-\alpha+r}$.

Proof

$$\begin{aligned}
\int_0^t \frac{y^{(r)}(\tau)}{(t-\tau)^{\alpha-r+1}}\,\mathrm{d}\tau &\simeq \int_0^t \frac{C^T V \eta^r X_\tau}{(t-\tau)^{\alpha-r+1}}\,\mathrm{d}\tau \\
&= C^T V \eta^r \int_0^t \frac{X_\tau}{(t-\tau)^{\alpha-r+1}} \mathrm{d}\tau \\
&= C^T V \eta^r \int_0^t \frac{[1, \tau, \ldots, \tau^m]^T}{(t-\tau)^{\alpha-r+1}}\,\mathrm{d}\tau \\
&= C^T V \eta^r \Bigg[\int_0^t \frac{1}{(t-\tau)^{\alpha-r+1}}\,\mathrm{d}\tau, \int_0^t \frac{\tau}{(t-\tau)^{\alpha-r+1}}\,\mathrm{d}\tau, \\
&\qquad \ldots, \int_0^t \frac{\tau^m}{(t-\tau)^{\alpha-r+1}}\,\mathrm{d}\tau\Bigg]^T,
\end{aligned}$$

by using Lemma (5) we can write

$$\begin{aligned}
\int_0^t \frac{y^{(r)}(\tau)}{(t-\tau)^{\alpha-r+1}}\,\mathrm{d}\tau &\simeq C^T V \eta^r \Bigg[\frac{\Gamma(r-\alpha)\Gamma(1)}{\Gamma(r-\alpha+1)} t^{r-\alpha}, \frac{\Gamma(r-\alpha)\Gamma(2)}{\Gamma(r-\alpha+2)} t^{r-\alpha+1}, \\
&\qquad \ldots, \frac{\Gamma(r-\alpha)\Gamma(m+1)}{\Gamma(m-\alpha+r+1)} t^{m-\alpha+r}\Bigg]^T \\
&= C^T V \eta^r G \Pi,
\end{aligned} \tag{10}$$

where

$$\Pi = \left[t^{r-\alpha}, t^{r-\alpha+1}, \dots, t^{m-\alpha+r}\right]^T.$$

By approximating $t^{j-\alpha+r}$, $j = 0, 1, \dots, m$, we get

$$\begin{aligned} t^{j-\alpha+r} &\simeq \sum_{i=0}^{m} t_{j,i} L_i(t) = B_j \psi_m(t), \\ B_j &= [t_{j,0}, t_{j,1}, \dots, t_{j,m}], \end{aligned}$$

we obtain

$$\Pi = [B_0 V X_t, B_1 V X_t, \dots, B_m V X_t]^T = A\psi_{l,m}(t), \qquad A = [B_0, B_1, \dots, B_m]^T. \tag{11}$$

By substituting (11) into (10) we obtain

$$\int_0^t \frac{y^{(r)}(\tau)}{(t-\tau)^{\alpha-r+1}} ds \simeq C^T V \eta^r G\, A V\, X_t. \tag{12}$$

4.3 Matrix representation for the supplementary conditions

Let $y(t) \simeq \sum_{j=0}^{m} c_j L_{l,j}(t) = C^T V X_t$ on the left hand side of (2), it can be written as

$$\begin{aligned} &y^{(j)}(0) = d_j, && j = 0, 1, \dots, r-1, \\ &C^T V \eta^j X_0 = d_j, && j = 0, 1, \dots, r-1. \end{aligned}$$

Let $H_j = \eta^j X_0$ where $X_0 = [1, 0, 0, \dots, 0]^T$ thus the j-th condition number of (2) is converted to

$$C^T V H_j = d_j, \quad j = 0, 1, \dots, r-1.$$

Now, by setting H as the matrix with columns H_j, $j = 0, 1, \dots, r-1$ and by setting $d = [d_1, d_2, \dots, d_j]$, as the vector that contains right-hand side of supplementary conditions, they take the form

$$C^T V H = d. \tag{13}$$

5 Description of the proposed method

In this section, we apply the operational matrices of integration and stochastic Ito-integration of the shifted Legendre polynomials for solving fractional stochastic integro-differential equation (1) and (2).

For solving the problem by using the stochastic operational matrix of shifted Legendre polynomials, we approximate $y(t)$, $f(t)$, $k_1(t,\tau)$, and $k_2(t,\tau)$ in terms of shifted Legendre vector as follows

$$\begin{aligned} &y(t) \simeq Y^T\psi(t) = \psi^T(t)Y, \\ &f(t) \simeq F\psi(t), \\ &k_1(t,\tau) \simeq \psi^T(t)K_1\psi(\tau), \\ &k_2(t,\tau) \simeq \psi^T(t)K_2\psi(\tau), \end{aligned} \tag{14}$$

where Y and F are shifted Legendre coefficients vector, K_1 and K_2 are shifted Legendre coefficients matrices defined in equation (6).
By using above approximations, Lemma 1 and Theorem 2 we can obtain the first and second term integral in the right hand of the (8) as following

$$\begin{aligned} \int_0^t k_1(t,\tau)y(\tau)d\tau &\simeq \int_0^t \psi^T(t)K_1\psi(\tau)\psi^T(\tau)Y\,d\tau \\ &= \psi^T(t)K_1\int_0^t \psi(\tau)\psi^T(\tau)Y\,d\tau \\ &= \psi^T(t)K_1\int_0^t Q\mathrm{diag}[Q^TY]Q^{-1}\psi(\tau)d\tau \\ &= \psi^T(t)K_1Q\mathrm{diag}[Q^TY]Q^{-1}\int_0^t \psi(\tau)d\tau \\ &= \psi^T(t)K_1Q\mathrm{diag}[Q^TY]Q^{-1}QPQ^{-1}\psi(t) \\ &= \psi^T(t)E_1\psi(t) \\ &= \mathrm{diag}[Q^TE_1Q]Q^{-1}\psi(t) \\ &= B_1\psi(t) \end{aligned} \tag{15}$$

where

$$E_1 = K_1\,Q\,\mathrm{diag}[Q^TY]P\,Q^{-1}, \tag{16}$$

and

$$B_1 = \mathrm{diag}[Q^TE_1Q]\,Q^{-1}. \tag{17}$$

Also we have

$$\begin{aligned}
\int_0^t k_2(t,\tau)y(\tau)dB\tau &\simeq \int_0^t \psi^T(t)K_2\psi(\tau)\psi^T(\tau)YdB\tau \\
&= \psi^T(t)K_2\int_0^t \psi(\tau)\psi^T(\tau)YdB\tau \\
&= \psi^T(t)K_2\int_0^t Q\mathrm{diag}[Q^TY]Q^{-1}\psi(\tau)dB\tau \\
&= \psi^T(t)K_2Q\mathrm{diag}[Q^TY]Q^{-1}\int_0^t \psi(\tau)dB\tau \\
&= \psi^T(t)K_2Q\mathrm{diag}[Q^TY]Q^{-1}QP_\tau Q^{-1}\psi(t) \\
&= \psi^T(t)E_2\psi(t) \\
&= \mathrm{diag}[Q^TE_2Q]Q^{-1}\psi(t) \\
&= B_2\psi(t)
\end{aligned} \tag{18}$$

where

$$E_2 = K_2\,Q\,\mathrm{diag}[Q^TY]P_\tau\,Q^{-1}, \tag{19}$$

and

$$B_2 = \mathrm{diag}[Q^TE_2Q]\,Q^{-1}. \tag{20}$$

Now, using Theorem (5) and equations (14), (15), and (18) and substituting in (8) we obtain

$$\frac{1}{\Gamma(r-\alpha)}Y^TV\eta^rGA\psi(t) \simeq F\psi(t) + \lambda_1B_1\psi(t) + \lambda_2B_2\psi(t),$$

by setting

$$B_3 = \frac{1}{\Gamma(r-\alpha)}Y^TV\eta^rGA,$$

and using the orthogonality of $\{L_i(t)\}_{i=0}^{m-1}$ we have

$$B_3 - \lambda_1B_1 - \lambda_2B_2 \simeq F, \tag{21}$$

this equation is hold for all $t \in [0,1)$, B_1, B_2, and B_3 is linear function of Y, equation (21) is a linear system of equations for unknown vector Y.

Also, from equation (13) and replacing $\simeq$ by $=$, we have following system

$$\begin{cases} B_3 - \lambda_1B_1 - \lambda_2B_2 = F, \\ Y^TVH = d. \end{cases} \tag{22}$$

Now setting

$$\begin{aligned}
\Delta &= B_3 - \lambda_1B_1 - \lambda_2B_2, \\
\overline{H} &= Y^TVH,
\end{aligned}$$

$$G=[\overline{H_1},\overline{H_2},\ldots,\overline{H_r},\Delta_1,\Delta_2,\ldots,\Delta_{m+1-r}],$$

and

$$g=[d_1,d_2,\ldots,d_r,F_0,F_1,\ldots,F_{m-r}],$$

where $\overline{H_i}$ denotes the i-th column of $\overline{H}$, system of (22) can be written as $G=g$, which must be solved for the unknown coefficients $y_0,y_1,\ldots,y_m$.

5.1 Algorithm of shifted Legendre Tau approximation

Step 1. Choose m, form the set of shifted Legendre polynomials $\Big\{L_i(t)\Big\}_{i=0}^{m}$ and let the approximate solution be $y_m(t)\simeq\sum_{i=0}^{m}y_iL_i(t)$.

Step 2. Compute the non singular coefficient matrix V with respect to

$$X_t=\left[1,t,t^2,\ldots,t^m\right]^T,$$

such that $\psi_m(t)=VX_t$.

Step 3. Compute the shifted Legendre vector

$$Q=\Big[\psi_m(\frac{1}{2m}),\psi_m(\frac{3}{2m}),\ldots,\psi_m(\frac{2m-1}{2m})\Big].$$

Step 4. By using orthogonality condition of $\{L_i(t)\}_{i=0}^m$ as

$$f(t)\simeq\sum_{j=0}^{m}f_jL_j(t),$$

where $f_j=\frac{2j+1}{l}\int_0^l f(t)L_j(t)dt$, compute $F=[f_0,f_1,\ldots,f_m]$.

Step 5. Compute the stochastic operational matrices p_τ and p using Lemmas (2) and (3).

Step 6. Compute the matrices G, η, A, E_1, and E_2 from Lemmas (4) and (5), Theorem (5) and equations (16) and (19) then set $B_3=\frac{1}{\Gamma(r-\alpha)}Y^TV\eta^rGA$, $B_1=\operatorname{diag}[Q^TE_1Q]\,Q^{-1}$, and $B_2=\operatorname{diag}[Q^TE_2Q]\,Q^{-1}$.

Step 7. Let $Y^T=[y_0,y_1,\ldots,y_m]$ and obtain the entries of the vector solution Y^T from the $C^TG=g$ where $G=[\overline{H_1},\overline{H_2},\ldots,\overline{H_r},\Delta_1,\Delta_2,\ldots,\Delta_{m+1-r}]$ and $g=[d_1,d_2,\ldots,d_r,F_0,F_1,\ldots,F_{m-r}]$, $\overline{H_i}$ denotes the i-th column of matrix VH and Δ_i denotes the i-th column of matrix $B_3-\lambda_1B_1-\lambda_2B_2$.

6 Numerical results and comparisons

In this section, we present three numerical examples to demonstrate the accuracy of the proposed method. The results show that this method, by selecting a few number of shifted Legendre polynomials is accurate.

First we consider discretized Brownian motion, where $B(t)$ is determined at t distinct values and utilized an spline interpolation to construct $B(t)$. Let $t_i = ih$, $i = 0, 1, 2, \ldots, N$, $h = \frac{T}{N}$ where N denotes the final space level t_N, $N+1$ is the number of nodes and B_i denote $B(t_i)$. Condition (i) in introduction says that $B_0 = 0$ with probability 1, and condition (ii) and (iii) tell us that

$$B_i = B_{i-1} + dB_i, \quad i = 1, 2, \ldots, N,$$

where each dB_i is an independent random variable of the form $\sqrt{h}N(0, 1)$. In order to check the accuracy of the proposed method, the absolute errors and L_2 norm errors between the exact solution $y(t)$ and the approximate solution $y_m(t)$ are given by the following definitions.

Absolute error: $\mid y(t) - y_m(t) \mid$.

L_2 norm error: $\dfrac{1}{N}\left(\sum_{i=0}^{N} \mid y(t_i) - y_m(t_i) \mid^2\right)^{1/2}$.

Example 1 As a first application, we offer the following stochastic fractional integro-differential equation as follows [1]

$$\begin{cases} D^{\alpha} y(t) = f(t) + \int_0^t e^t \tau y(\tau) \mathrm{d}\tau + \lambda_2 \int_0^t e^t \tau y(\tau) \mathrm{d}B\tau, \\ y(0) = 0, \end{cases}$$

the exact solution with $\alpha = 0.75$ and $\lambda_2 = 0$ is $y(t) = t^3$, where $y(t)$ is an unknown stochastic process defined on the probability space (Ω, F, P) and $B(t)$ is a Brownian motion process. In this example we have

$$f(t) = \frac{-t^5 e^t}{5} + \frac{6t^{2.25}}{\Gamma(3.25)}.$$

Table 1 and Table 2 show L_2 errors and absolute errors of the approximation solution for different values of m with $\lambda_2 = 0.0001$ and $N = 10$. From Table 1 we can see clearly that the shifted Legendre Tau (SLT) method can reach a higher degree of accuracy. In [1] the authors obtained the best results in $m = 2^8$ that is 6.3×10^{-5}. The approximation solution and exact solution are shown in Figure 1.

Example 2 Consider the following stochastic fractional integro-differential equation as follows

$$\begin{cases} D^{\alpha} y(t) = f(t) + \int_0^t t \sin \tau y(\tau) \mathrm{d}\tau + \lambda_2 \int_0^t (t + \tau) y(\tau) \mathrm{d}B\tau, \\ y(0) = 0. \end{cases}$$

Table 1 L_2 errors obtained for different values of m.

m	L_2 error
4	3.26×10^{-5}
6	1.08×10^{-5}
8	3.56×10^{-6}
10	1.47×10^{-6}
12	6.03×10^{-7}
14	3.34×10^{-7}

Table 2 Absolute errors obtained for different values of m with $N = 10$ and $\lambda_2 = 0.001$.

t	$m = 8$	$m = 10$	$m = 12$
0.1	5.62×10^{-6}	1.82×10^{-5}	6.31×10^{-6}
0.2	6.63×10^{-6}	3.40×10^{-6}	1.34×10^{-6}
0.3	2.61×10^{-5}	1.32×10^{-6}	8.09×10^{-6}
0.4	8.63×10^{-5}	4.11×10^{-5}	3.20×10^{-5}
0.5	2.17×10^{-4}	1.31×10^{-4}	9.15×10^{-5}
0.6	4.68×10^{-4}	2.90×10^{-4}	2.07×10^{-4}
0.7	9.19×10^{-4}	5.72×10^{-4}	4.08×10^{-4}
0.8	1.68×10^{-3}	1.05×10^{-3}	7.51×10^{-4}
0.9	2.94×10^{-3}	1.83×10^{-3}	1.28×10^{-3}

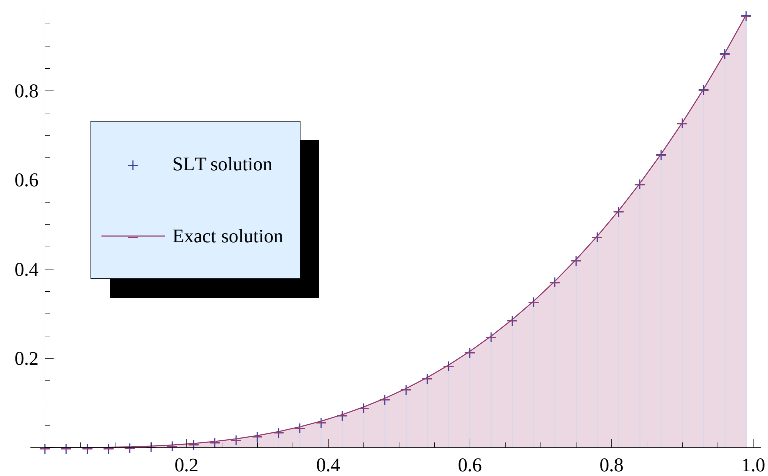

Fig. 1 The absolute error of Example 1 for $m = 14$

In the absence of the noise term ($\lambda_2 = 0$) the exact solution with $\alpha = 0.5$, is $y(t) = t^2 + t$. In this example we have

$$f(t) = \frac{2}{\Gamma(2.5)} t^{1.5} + \frac{1}{\Gamma(1.5)} t^{0.5} + t(2 - 3\cos t - t\sin t + t^2 \cos t) + (\cos t - \sin t)(t^2 + t).$$

We apply shifted Legendre Tau method to solve this problem. The L_2 errors of approximation solution for different values of m with $\lambda_2 = 0.0001$ and $N = 10$

are shown in Table 3. Figure 2 shows the error function of this example with $m = 6$.

Table 3 L_2 errors obtained for different values of m.

m	L_2 error
4	2.02×10^{-5}
6	4.56×10^{-6}
8	1.39×10^{-6}
10	6.83×10^{-7}

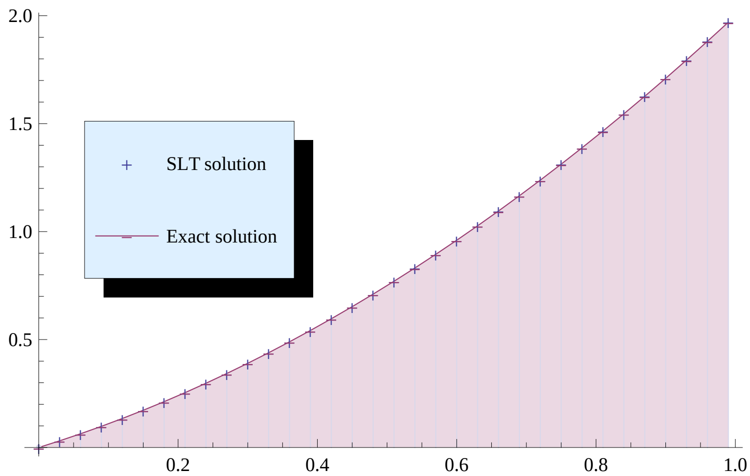

Fig. 2 The graph of the approximate solution and exact solution for $\lambda_2 = 0.0001$ and $m = 10$

Example 3 Consider the following fractional integro-differential equation

$$\begin{cases} D^{\alpha}y(t) = \dfrac{2t^{2-\alpha}}{\Gamma(3-\alpha)} - \dfrac{1}{3}t^3 + \int_0^t y(\tau)\mathrm{d}\tau + \int_0^t (\tau+\tau^2)y(\tau)\mathrm{dB}\tau, \\ y(0) = 0. \end{cases}$$

In the absence of the noise term ($\lambda_2 = 0$) the exact solution, is $u(t) = t^2$. We apply shifted Legendre Tau method to solve this equation and the absolute error are given in Table 4 for different choices of α. The numerical solution have been compared in Figure 3 for $\alpha = 0.25, 0.5$, and 0.95.

Table 4 L_2 errors obtained for different values of α with $m = 10$, $N = 10$, and $\lambda_2 = 0.001$.

t	$\alpha = 0.25$	$\alpha = 0.85$	$\alpha = 0.95$
0.1	1.77×10^{-4}	1.12×10^{-4}	9.30×10^{-6}
0.3	1.21×10^{-3}	2.04×10^{-4}	6.53×10^{-5}
0.5	6.37×10^{-4}	3.30×10^{-4}	2.13×10^{-4}
0.7	1.13×10^{-3}	6.09×10^{-4}	4.17×10^{-4}
0.9	2.86×10^{-3}	1.12×10^{-3}	7.56×10^{-4}

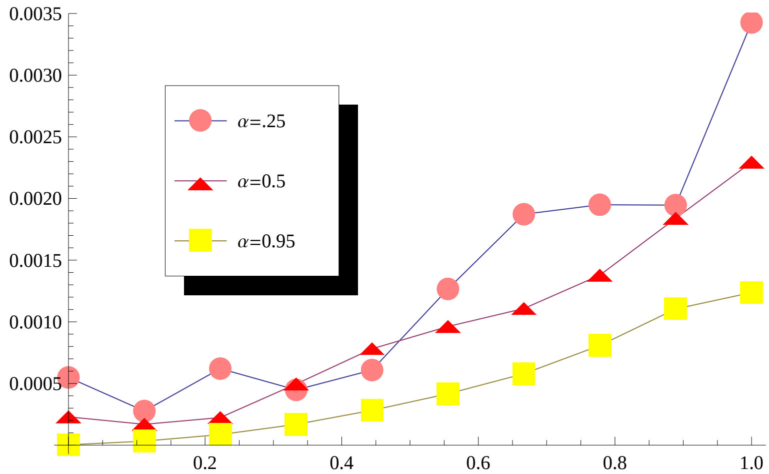

Fig. 3 Comparison of numerical and exact solutions of Example 3 for $m = 8$

7 Conclusion

In this work, a new computational method based on the shifted Legendre polynomials with the Tau method was proposed for solving a class of fractional stochastic integro-differential equation. For this purpose a new stochastic operational matrix for shifted Legendre polynomials is derived. The BPFs and their relations are used to derive this stochastic operational matrix. The most important contribution of our work is that we transform the initial problem into a linear algebraic system equations to obtain the approximation solution. The illustrative examples show the validity of the proposed method. Undoubtedly these examples also exhibit the accuracy of the present method.

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