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The Schur Multiplier of Pairs of Nilpotent Lie Algebras

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Abstract The Schur multiplier of a pair of groups was introduced by Ellis in 1998. In this paper, we study the Schur multiplier of a pair of Lie algebras and give some conditions under which the Schur multiplier of a pair of Lie algebras is trivial. Moreover, we give some conditions under which the higher multiplier of a pair of Lie algebras is not trivial.

Keywords Pair of Lie algebras · Schur multiplier · Nilpotent Lie algebras

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1 Introduction

All Lie algebras are considered over a fixed field Λ and $[,]$ denotes the Lie bracket. Let L be a Lie algebra with a free presentation

$$0 \to R \to F \to L \to 0.$$

The Schur multiplier of L is denoted by $\mathcal{M}(L)$ and defined as

$$\mathcal{M}(L) = \frac{R \cap [F, F]}{[R, F]}.$$

One can easily verify that the Schur multiplier of a Lie algebra L is abelian and is independent of the choice of free presentation (see [11] for more information). The notion of the c-nilpotent multiplier of a Lie algebra was introduced by Salemkar et al. in 2009. Let L be a Lie algebra, the c-nilpotent multiplier of L is defined as

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$$\mathcal{M}^{(c)}(L) = \frac{R \cap \gamma_{c+1}(F)}{\gamma_{c+1}(R, F)},$$

where $\gamma_{c+1}(F)$ is the $(c+1)$-st term of the lower central series of F, $\gamma_1(R, F) = R$ and $\gamma_{c+1}(R, F) = [\gamma_c(R, F), F]$. In particular, if $c = 1$, then $\mathcal{M}^{(1)}(L)$ is the Schur multiplier of L. By the Hopf type formula, $\mathcal{M}(L)$ is isomorphic to the second homology of L with coefficients in Λ.

Let (N, L) be a pair of Lie algebras, in which N is an ideal in L. The Schur multiplier of (N, L) to be the abelian Lie algebra $\mathcal{M}(N, L)$ appearing in the following natural exact sequence of Lie algebras

$$H_3(L) \to H_3(L/N) \to \mathcal{M}(N, L) \to \mathcal{M}(L) \to \mathcal{M}(L/N) \to \frac{L}{[N, L]} \to \frac{L}{L^2} \to \frac{L}{(L^2 + N)} \to 0,$$

where $\mathcal{M}(-)$ and $H_3(-)$ denote the Schur multiplier and the third homology of a Lie algebra, respectively.

Let $0 \to R \to F \to L \to 0$ be a free presentation of L. If the ideal N possesses complement in L, then

$$\mathcal{M}(N, L) = \frac{R \cap [S, F]}{[R, F]},$$

in which S is an ideal in F, such that $N \cong S/R$ (see [4,9,16] for further details). Similarly, we can define the c-nilpotent multiplier of a pair (N, L) as

$$\mathcal{M}^{(c)}(N, L) = \frac{R \cap [S, {}_cF]}{[R, {}_cF]}.$$

In particular, if $N = L$, then $\mathcal{M}^{(c)}(N, L) = \mathcal{M}^{(c)}(L)$ is the c-nilpotent multiplier of L. (See [1,2,5,15,17,19] for more information).

2 Main Results

In this section, we prove some properties of the c-nilpotent multiplier of a pair of Lie algebras. Let (N, L) be a pair of Lie algebras, we first recall that the subalgebras $Z_c(N, L)$ and $[N, {}_cL]$ for all $c \geq 1$ as follows:

$$Z_c(N, L) = \{n \in N \mid [n, l_1, \dots, l_c] = 0, \forall l_1, \dots, l_c \in L\},$$

$$[N, {}_cL] = \langle [n, l_1, \dots, l_c] \mid n \in N, l_1, \dots, l_c \in L \rangle,$$

where,

$$[n, l_1, \dots, l_c] = [\dots, [[n, l_1], l_2], \dots, l_c], \ \ (c \geq 1),$$

(see [16,17] for more information). Let (N, L) be a pair of Lie algebras. We recall that the cth precise center of the pair (N, L) is defined to be

$$Z_c^*(N, L) = \cap\{\varphi(Z_c(M, L))\},$$

where, $\varphi : M \to L$ is a relative c-central extension of (N, L). It is easy to see that $Z_c^*(L, L) = Z_c^*(L)$ (See [10,18]). Let (N, L) and (H, K) be two pairs of Lie algebras. A homomorphism from (N, L) to (H, K) is a homomorphism $f : L \to K$ such that $f(N) \subseteq H$. We say that (N, L) and (H, K) are isomorphic if f is an isomorphism and $f(N) = H$.

Moreover, a pair (N, L) is called nilpotent of class c, if $[N, {}_cL] = 0$ and $[N, {}_{c-1}L] \neq 0$ for some positive integer c (see [9] for more information). The following Lemmas are useful in the proof of the next results.

Lemma 1 *([14], Theorem 3.3) Let $(f, f|) : (N, L) \to (K, H)$ be a homomorphism of pairs of Lie algebras. Suppose that f induces isomorphism $f_0 : L/N \to H/K$ and $f_1 : N/\gamma_{c+1}(N, L) \to K/\gamma_{c+1}(K, H)$. Also, we assume that $\bar{f} : \mathcal{M}^{(c)}(N, L) \to \mathcal{M}^{(c)}(K, H)$ is an epimorphism. Then f induces the following isomorphism*

$$(f_n, f_n|) : \left(\frac{N}{\gamma_{nc+1}(N, L)}, \frac{L}{\gamma_{nc+1}(N, L)}\right) \to \left(\frac{K}{\gamma_{nc+1}(K, H)}, \frac{H}{\gamma_{nc+1}(K, H)}\right),$$

for $n \geq 0$.

Lemma 2 *([1], Proposition 2.3) Let L be a Lie algebra and K be an ideal in L contained in N, then the following sequences are exact*

$$0 \to \mathcal{M}^{(c)}(K, L) \to \mathcal{M}^{(c)}(N, L) \xrightarrow{\alpha} \mathcal{M}^{(c)}(\frac{N}{K}, \frac{L}{K}) \to \frac{K \cap [N, {}_cL]}{[K, {}_cL]} \to 0, \quad (1)$$

$$\mathcal{M}^{(c)}(N, L) \to \mathcal{M}^{(c)}(\frac{N}{K}, \frac{L}{K}) \to N \to \frac{L}{[N, {}_cL]} \to \frac{L}{[N, {}_cL] + K} \to 0. \quad (2)$$

Lemma 3 *([6], Theorem 3.4) Let N be a c-central ideal of Lie algebra L. Then the following conditions are equivalent.*

(i) $N \cap \gamma_{c+1}(L) \cong (\mathcal{M}^{(c)}(L/N)/((\mathcal{M}^{(c)}(L))$,
(ii) $N \subseteq Z_c^*(L)$,
(iii) *the homomorphism $\mathcal{M}^{(c)}(L) \to \mathcal{M}^{(c)}(L/N)$ is injective.*

Theorem 1 *Let (N, L) be a pair of Lie algebras and $0 \to R \to F \xrightarrow{\pi} L \to 0$ be a free presentation of L such that $N \cong S/R$ for an ideal S in F. If $K \subseteq Z_c^*(N, L)$, then*

(i) *the natural homomorphism $\mathcal{M}^{(c)}(L) \to \mathcal{M}^{(c)}(L/K)$ is injective,*
(ii) $K \subseteq Z_c^*(L) \cap N$,
(iii) $\gamma_{c+1}^*(N, L) = \gamma_{c+1}^*(N/K, L/K)$,

where, $\gamma_{c+1}^*(N, L) = [S, {}_cF]/[R, {}_cF]$.

Proof We define the following homomorphism:

$$\delta : S/[R, {}_cF] \to L$$

$$s + [R, {}_cF] \xrightarrow{\delta} \pi(s).$$

We can see that δ is a relative c-central extension by an action of L on $S/[R, {}_cF]$, defined by

$$^{\ell}(s + [R, {}_cF]) = [s, f] + [R, {}_cF],$$

where $\pi(f) = \ell$. Thus,

$$Z^*(N, L) \subseteq \delta(Z(S/[R, {}_cF], L)).$$

Let $0 \to R \to T \to K \to 0$ be a free presentation of K. If $K \subseteq Z^*(N, L)$, then

$$\delta(T/[R, {}_cF]) \subseteq \delta(Z(S/[R, {}_cF], L).$$

Also, we have

$$Ker(\mathcal{M}^{(c)}(L) \to \mathcal{M}^{(c)}(L/K)) = [T, {}_cF]/[R, {}_cF] = Ker([S, {}_cF]/[R, {}_cF] \to [S, {}_cF]/[T, {}_cF]).$$

Hence, (i) and (iii) hold. By Lemma 3, $K \subseteq Z_c^*(L)$ if and only if the homomorphism $\mathcal{M}^{(c)}(L) \to \mathcal{M}^{(c)}(L/K)$ is injective and so, the result is held.

By Theorem 1, we obtain the following result.

Corollary 1 *Let (N, L) be a pair of Lie algebras such that $Z_c^*(N, L) = N$. Then $\gamma_{c+1}^*(N, L) = 0$.*

The next lemma is useful in the proof of Theorem 2.

Lemma 4 *Let (N, L) be a pair of Lie algebras and $0 \to R \to F \xrightarrow{\pi} L \to 0$, be a free presentation of L such that $N \cong S/R$ for an ideal S in F, then for all $c \geq 1$*

(i) $\gamma_{c+1}^*(N, L) = 0$ if and only if (N, L) is nilpotent and $\mathcal{M}^{(c)}(N, L) = 0$.
(ii) If $\gamma_{c+1}^*(N, L) = 0$, then $\gamma_{c+1}^*(N/K, L/K) = 0$, where K is an ideal of L such that $K \subseteq Z_c^*(N, L)$.

Proof (i) It is clear.
(ii) Let $0 \to R \to F \xrightarrow{\pi} L \to 0$ be a free presentation of L and $N \cong S/R$ for an ideal S in F. Using the assumption, we have $[R, {}_cF] = [S, {}_cF]$. Thus, the pair $(S/[R, {}_cF], F/[R, {}_cF])$ is nilpotent of class c. Hence, $N = \bar{\pi}(Z_c(S/[R, {}_cF], F/[R, {}_cF]))$, where $\bar{\pi}$ is the natural epimorphism induced by π. Therefore, the result follows from Theorem 1.

Now, we prove the following theorem.

Theorem 2 *Let (N, L) be a pair of finite dimensional nilpotent Lie algebras of nilpotency class $c \geq 2$ such that $Z_c^*(N, L) \subseteq Z(N, L)$. Then $\mathcal{M}(N, L) \neq 0$.*

Proof Let $\mathcal{M}(N, L) = 0$. By Lemma 1, we can see that there is a free presentation $0 \to R \to F \to L \to 0$ of L with $N \cong S/R$ for an ideal S in F such that $R \subseteq [S, {}_nF]$ for all $n \geq 0$. Moreover, if $\mathcal{M}^{(c)}(N, L) = 0$, for some $c \geq 1$ then $\mathcal{M}^{(d)}(N, L) = 0$ for all $d \geq 1$. Hence, By Lemma 4 (i), we have $\gamma_{c+1}^*(N, L) = 0$. Also, using Lemma 4 (ii), $\gamma_{c+1}^*(N/M, L/M) = 0$, where, M is an ideal in L such that $M \subseteq Z_c^*(N, L)$. Thus,

$$\mathcal{M}(N/M, L/M) = \mathcal{M}^{(c)}(N/M, L/M) = 0.$$

In particular, we obtain

$$\mathcal{M}(N/Z(N,L), L/Z(N,L)) = 0.$$

On the other hand, using Lemma 2, the following sequence is exact:

$$M \otimes L \to \mathcal{M}(N,L) \to \mathcal{M}(N/M, L/M) \to M \cap [N,L] \to 0,$$

Thus, $[N,L] \cap Z(N,L) = 0$ *that implies* $[N,L] \cong [N/Z(N,L), L/Z(N,L)]$*. Hence, we have*

$$[N/Z(N,L), L/Z(N,L)] = N/Z(N,L),$$

which is a contradiction.

By Theorem 2, we obtain the following result. Note that in Corollary 2, we extend a result of Stitzinger and Bosko (2011).

Corollary 2 *Under assumptions of Theorem 2,* $\mathcal{M}^{(c)}(N,L) \neq 0$ *for all* $c \geq 1$*.*

In the next result, we give a sufficient condition under which the Schur multiplier of a pair of Lie algebras is trivial.

Theorem 3 *Let* (N,L) *be a pair of finite dimensional nilpotent Lie algebras and* $f : (N,L) \to (H,K)$ *be an epimorphism. If* $Ker\, f \subseteq N^2$ *and* $\mathcal{M}(H,K)$ *is trivial, then* f *is an isomorphism.*

Proof *Set* $M = Ker\, f$*, then* $\mathcal{M}(N/M, L/M) = 0$*. By Lemma 2,* $(M \cap [N,L])/[M,L]$ *is trivial. Since* $M \subseteq N^2 \subseteq [N,L]$*, so* $M = [M,L]$*. Set*

$$M_1 = M, \text{ and } M_{n+1} = [M, {}_nL].$$

Thus,

$$M = M_n \subseteq \gamma_{n+1}(N) \subseteq [N, {}_nL] = [[N, {}_{n-1}L], L].$$

Now, since (N,L) *is nilpotent,* $[N, {}_nL] = 0$ *for some positive integer* n*. Therefore,* $M = 0$ *and so,* f *is an isomorphism.*

Corollary 3 *Let* (N,L) *be a pair of finite dimensional nilpotent Lie algebras. If* $\mathcal{M}(N/[N,L], L/[N,L]) = 0$*, then* $\mathcal{M}(N,L) = 0$*.*

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