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Full Ranking of Decision Making Units by Relaying Hyperplane on the Set of Produce Possibility

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Abstract The exit methods for producing the common weight in DEA are complicated or they can't produce full ranking for Decision Making Units (DMUs). Wang and et al. [10, 11] introduced two methods based on regresion analysis for finding common weights. In these method they fined set of common weight so that the efficeincy which is computed by set of common weight for all units, are always smaller or and equal to obtained optimistic efficeincy from CCR model. In this methods we are trying to make the computed efficeincy by common weight closer to obtained optimistic efficeincy from CCR model, and or in other words the goal is to obtain unique hyperplane as all distance of DMUs from this hyperplane will be minimal. In this paper by using an example, we show that the obtained hyperplane from suggested methods of Wang is passing throug PPS. In other words, in introduced method of proposed hyperplane for ranking, isn't relaying PPS, neccessarly. At the end, we will introduce a new method for ranking the decision makind units that is a correspond hyperplane of relaying common weight set on PPS and finally we can use this techniqe for real data.

Keywords Data envelopment analysis · Ranking efficeincy · Optimistic efficeincy · Set of common weight.

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1 Introduction

Data Envelopment Analysis (DEA) developed by Charnes and et al. [1], is a non-parametric method to evaluate the efficeincy of a group of Decision Making Units (DMUs) that these units produce several outputs by consume several inputs. Since DEA does not have a default for the production function, therefore the type of relation between input and output is not default. On the other hand, DEA separately calculates and evaluates the efficeincy of each DMUs with input and output weights under the best conditions, that are ideal for them. In addition, there is no default on the weights expect being them nonnegative. Therefore, the choice of weights has a great flexibility and freedom. Based on the freedom to choice input and output weight, it is very likely that more than one DMU of efficient DEA Evaluated. Therefore, DMUs are not completely separated. In other words, using variable weights for different DMUs makes it possible to compare their efficeincy and rank the DMUs. In order to reduce flexibility in selecting input and output weights, a set of common weights instead of variable weights is proposed to evaluate DMUs. Using common weights allows us to compare and rank the efficeincy of DMUs on a base. Some proposed methods to obtain a set of common weights for DMUs in DEA are as follow. Ganley and Cubbin [3], were given the common weights by using maximizing total efficeincys of DMUs. Barboy and et al. [8] have developed a two-step linear separate analysis method to produce common weights. Friedman and Sinuany-Stern [2], used a canonical correlation analysis to provide a weight vector for inputs and outputs, for all DMUs. Friedman and Sinuany-Stern [2], provided a nonlinear separate analysis to produce common weights. Liu and Peng [7], proposed a Common Weight Analysis (CWA) method to search for a common set of weights for DMUs. Hashymoto and Wu [4], presented the Compromised Programming (DEA-CP) model to search for a common set of weights by combining DEA and compromise programming for DMUs. Also, Kao and Hung [6], suggested a similar compromise solution to produce common weights under framework of DEA. Lou and et al. [11], provided ranking the DMUs by applying weight restrictions that produce a set of common weights for DMUs under comparison. Wang and et al. [10], presented the methods to obtain a set of common weights based on regression analysis. In this research, it is presented a new approach to search a set of common weights that easily is estimated, as well as be able to completely rank DMUs.

The proposed method, tries to find only a PPs-supported hyperplane, which distance of all DMUs from the hyperplane is the lowest. The distance is accepted as the error. In other words, it finds the set of common weight such that the sum of these errors is minimized, so that efficeincy is obtained by a set of common weights for DMUp ($p = 1, \ldots, n$), always less than or equal to the optimistic efficeincy obtained from CCR model for DMUp. The remainder of this paper is organized as follows: Section 2, briefly introduces CCR model which is presented by Charnes et al. [1] and definition efficeincy with a set of weights. Section 3, presents the proposed models of Wang et al. [10] for ranking

Decision Making Units which are based on regression analysis and the set of common weights, in summary. Section 4 contains an example that shows the hyperplane corresponding set of common weight from Wang's model does not rely on PPS. In Section 5, we present a new model to rank based on the set of common weight and we will show that hyperplane corresponding that relies on the PPS. In Section 6, we compare the proposed method from Wang and new method by providing a practical example. Finally, we give conclusions in Section 7.

2 CCR model and efficiency of DMUs

Let n, DMU have been evaluated with m input and s output. If x_{ij}, $(i = 1,\ldots,m)$ and y_{rj}, $(r = 1,\ldots,s)$ be input and output values and v_i, $(i = 1,\ldots,m)$ and u_r, $(r = 1,\ldots,s)$ be input and output weights for n, DMU and Θ_j be efficeincy of DMU$_j$ that definded by following equality

$$\Theta_j = \frac{\sum_{r=1}^{s} u_r y_{rj}}{\sum_{i=1}^{m} v_j x_{ij}}, \quad j = 1,\ldots,n \tag{1}$$

According to the Charnes and et al model [1], the best relative efficiency of any DMU can be measured.

$$\begin{aligned} &\Theta_p^* = \max\ \Theta_p = \sum_{r=1}^{s} u_r y_{rp} \\ &s.t \\ &\qquad \sum_{i=1}^{m} v_i x_{ip} = 1 \\ &\qquad \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \le 0, \quad j = 1,\ldots,n, \\ &\qquad u_r \ge 0, r = 1,\ldots,s, \\ &\qquad v_i \ge 0, i = 1,\ldots,m, \end{aligned} \tag{2}$$

where DMU$_P$ refers to the under evaluation DMU, v_i, $(i = 1,\ldots,m)$ and u_r, $(r = 1,\ldots,s)$ are decision variables. If optimal objective function value,Θ_p^* equals one, that means $\Theta_p^* = 1$ then is efficency in terms of DEA or in summary is efficiency. Otherwise is inefficiency of DEA or in summary is inefficiency. Now because the considered efficiency be recognizable from the other efficiencies, the efficiencies as defined by CCR model often named CCR efficiency and in this paper considered as lucky efficiencies of DMU. The CCR model is feasible for any (2). Therefore, the optimal weights are different from a DMU to another. On the other hand, for optimal solution $(U^*, V^*)^t$ from product CCR model in order to evaluate of DMU$_P$, alwayes reliance hyperplane $\{(x, y) : U^*y - V^*x = 0\}$ is obtianed. In next section, we introduce the proposed Wang and et al models.

3 Wang and et al's proposed models

Wang, Luo and Lan [11] propose two models (3) and (4) to estimate the common weights.

$$\begin{aligned} \min \quad & z=\sum_{j=1}^{n}\Big(\Theta_j^*-\frac{\sum_{r=1}^{s}u_r y_{rj}}{\sum_{i=1}^{m}v_j x_{ij}}\Big)^2 \\ s.t. \quad & \\ & u_r \geq 0, r=1,\ldots,s, \\ & v_i \geq 0, i=1,\ldots,m, \end{aligned} \tag{3}$$

$$\begin{aligned} \min \quad & j=\sum_{j=1}^{n}\Big(\sum_{r=1}^{s}u_r y_{rj}-\Theta_j^*\sum_{i=1}^{m}v_i x_{ij}\Big)^2 \\ s.t. \quad & \\ & \sum_{r=1}^{s}u_r\Big(\sum_{j=1}^{n}y_{rj}\Big)+\sum_{i=1}^{m}v_i\Big(\sum_{j=1}^{n}x_{ij}\Big)=n, \quad (b), \\ & u_r \geq 0, r=1,\ldots,s, \\ & v_i \geq 0, i=1,\ldots,m. \end{aligned} \tag{4}$$

The set of constraints (b) in model4 is applied for normalization. The aim of these methods is finding the set of the common optimal weight of model3 or 4 that minimizes the difference Θ_j and Θ_j^* for $j \in \{1,\ldots,n\}$ or found the uniqe hyperplane that the distance all DMUs will be minimum from it.

Because in the Wang models, the constraint $U^t y_j - V^t x_j \leq 0, \forall j \in \{1,\ldots,n\}$ doesn't exit, then maybe the achieved hyperplane from these models passes through PPS.

If we use the another distance such as norm one or Chebyshev distance for expanded the regression models 3 or 4 for production the common weights, maybe these models can't produce the perfect ranking for DMUs.

The model (3) is definded by the norm infinity distance in following form

$$\begin{aligned} \min \quad & \max\Big\{|\Theta_j^*-\frac{U^t y_j}{V^t x_j}| : j=1,\ldots,n\Big\} \\ s.t. \quad & \\ & U \geq 0, \\ & V \geq 0. \end{aligned} \tag{5}$$

This model converts the following simple form

$$
\begin{aligned}
\min \quad & N \\
s.t. \quad & \\
& V^t x_j(-\Theta_j^* + N) + U^t y_j \geq 0, j = 1, \ldots, n, \\
& V^t x_j(\Theta_j^* + N) - U^t y_j \geq 0, j = 1, \ldots, n, \\
& U \geq 0, \\
& V \geq 0,
\end{aligned} \tag{6}
$$

and so the model (3) with norm 1 is defined in form

$$
\begin{aligned}
\min \quad & \sum_{j=1}^{n} |\Theta_j^* - \frac{U^t y_j}{V^t x_j}| \\
s.t. \quad & \\
& U \geq 0 \\
& V \geq 0.
\end{aligned} \tag{7}
$$

The nonlinear model (7) with definition $\Theta_j^* - \frac{U^t y_j}{V^t x_j} = f_j - g_j$ where f_j and g_j are nonnegative for all $j \in \{1, \ldots, n\}$ is in following form

$$
\begin{aligned}
\min \quad & \sum_{j=1}^{n} (f_j + g_j) \\
s.t. \quad & \\
& V^t x_j(f_j - g_j - \Theta_j^*) + U^t y_j = 0, \quad j = 1, \ldots, n, \\
& U \geq 0, \\
& V \geq 0, \\
& f_j \geq 0, \quad j = 1, \ldots, n, \\
& g_j \geq 0, \quad j = 1, \ldots, n.
\end{aligned} \tag{8}
$$

In Section 4, we will give a numerical example that in this example the hyperplane corresponding to the models (5) and (7) passed from through PPS.

4 Numerical example

The obtained hyperplane from models (5) and (7) that is corresponding the optimal solution $(U^*, V^*)^t$, is defined in form

$$\{(x, y) : U^* y - V^* x = 0\}.$$

We show in the following example that this hyperplanes corresponding models (5) and (7) passed through PPS and don't be relay on PPS.

Table 1 The set of example data

DMU	U_{01}	U_{02}	U_{03}	U_{04}	U_{05}	U_{06}	U_{07}	U_{08}
I_1	1	2	3	5	12	8	24	30
I_2	10	8	5	1	18	30	15	40
O	1	1	1	1	1	1	1	1

Let Table 1 with data with two inputs and one output corresponding 8 determiners unit.

We obtain the set of the optimal solutions of this models with solving the models (5) and (7) for all data of Table 1 by the GAMS program. The set of the optimal solutions of the model (5) is in following form

$$v_1^* = 0.0002, \quad v_2^* = 0.0001, \quad u^* = 0.0012.$$

The hyperplane corresponding of this set of optimal soltion is in form

$$\{(x_1, x_2, y) : u^*y - v_1^*x_1 - v_2^*x_2 = 0\} = \{(x_1, x_2, y) : 0.0012y - 0.0002x_1 - 0.0001x_2 = 0\}$$

The intersection of these hyperplanes with line $y = 1$, is

$$l_2 : 0.0002x_1 + 0.0001x_2 = 0.0012.$$

The set of optimal solutions (7) obtain following form

$$v_1^* = 0.0012, \quad v_2^* = 0.0005, \quad u^* = 0.0066.$$

Also the uniqe hyperplane corresponding with the above set of the optimal soution is in form

$$\{(x_1, x_2, y) : u^*y - v_1^*x_1 - v_2^*x_2 = 0\} = \{(x_1, x_2, y) : 0.0066y - 0.0012x_1 - 0.0005x_2 = 0\}$$

The line $l_2 : 0.0012x_1 + 0.0005x_2 = 0.0066$ is obtianed from intersection this hyperplane and the line $y = 1$.

We showed in Figure 1, the Farel bound of the data Table 1 with the lines l_1 and l_2. As seen in Figure 2, both of two hyperlines passed inside PPS. Then in section 5 we will introduce a model that hyperline corresponding with the set of common weights obtained from this model,is reliable on PPS.

5 The new method for full ranking of the decision units by the reliable hyperplane on PPS

The new ranking model DMUs is following form with the set common weights

$$\begin{aligned}
\min \quad & \left(\sum_{j=1}^{n} |\Theta_j^* - U^t y_j|^p \right)^{\frac{1}{p}} \\
s.t. \quad & \\
& V^t x_j \leq 1, \quad j = 1, \ldots, n, \\
& U^t y_j - V^t x_j \leq 0, \quad j = 1, \ldots, n, \\
& U \geq 1\varepsilon, \\
& V \geq 1\varepsilon.
\end{aligned} \tag{9}$$

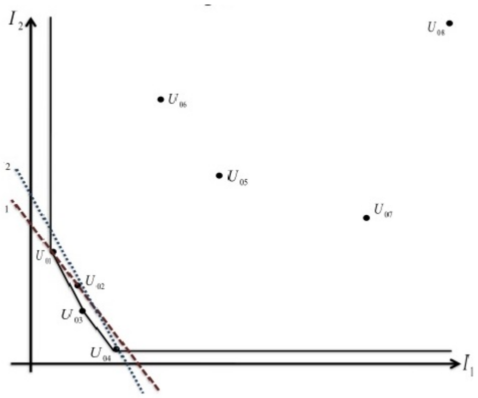

Fig. 1 The hyperplane passing in PPS

Where P is the distance parameter and $P \in \{1, 2, \ldots\}$ and also ε is the non-Archimedes number that $1\varepsilon = (\varepsilon, \ldots, \varepsilon)$. According to $\Theta_j^*(j \in \{1, \ldots, n\})$ is ideal efficiency amount figurated DMU_j, $(j = 1, \ldots, n)$ by CCR model, and is obtained by model (2) independenty, is always $\Theta_j^* \geq U^t y_j$. So model (9) is written in following form

$$
\begin{aligned}
&\min \quad \Big(\sum_{j=1}^{n}(\Theta_j^* - U^t y_j)^p\Big)^{\frac{1}{p}} \\
&s.t. \\
&\qquad V^t x_j \leq 1, \quad j = 1, \ldots, n, \\
&\qquad U^t y_j - V^t x_j \leq 0, \quad j = 1, \ldots, n, \\
&\qquad (U, V) \geq 1\varepsilon.
\end{aligned}
\tag{10}
$$

Model (10) is written in following form for all distance parameters different from norm 1 and Chebisheve norm

$$
\begin{aligned}
&P = 1; \\
&\min \quad w = \sum_{j=1}^{n}(\Theta_j^* - U^t y_j) \\
&s.t. \\
&\qquad V^t x_j \leq 1, \quad j = 1, \ldots, n, \\
&\qquad U^t y_j - V^t x_j \leq 0, \quad j = 1, \ldots, n, \\
&\qquad (U, V) \geq 1\varepsilon.
\end{aligned}
\tag{11}
$$

$$\begin{aligned}
&P = +\infty;\\
&min \quad z = \max\{(\Theta_j^* - U^t y_j) : j = 1, \dots, n\}\\
&s.t.\\
&\qquad V^t x_j \le 1, \quad j = 1, \dots, n,\\
&\qquad U^t y_j - V^t x_j \le 0, \quad j = 1, \dots, n,\\
&\qquad (U, V) \ge 1\varepsilon.
\end{aligned} \tag{12}$$

Theorem 1 *Models (11) and (12) are feasible.*

Proof According to the feasible region of two models (11) and (12) is the same, it is sufficient to prove feasiblity for only one of the two above models. Since $x_j \neq 0$ and $x_j \geq 0$ and $y_j \geq 0$ and $y_j \neq 0$ is established for all of $j \in \{1, \dots, n\}$ then without entering into the totality of the argument, let

$$\begin{aligned}
x_j &= (x_{1j}, \dots, x_{kj}, 0, \dots, 0) \in \mathbb{R}^m, \quad x_{ij} > 0, \quad (i = 1, \dots, k)(j = 1, \dots, n),\\
y_j &= (y_{1j}, \dots, y_{tj}, 0, \dots, 0) \in \mathbb{R}^s, \quad\ \ y_{rj} > 0, \quad (r = 1, \dots, t)(j = 1, \dots, n).
\end{aligned}$$

Put $x_{lj} = \max\{x_{ij} | i = 1, \dots, k\}$ and $y_{pj} = \max\{y_{rj} | r = 1, \dots, t\}$.
The weight vectors of the output U and input V defined in form

$$V^t = \Big[\frac{1}{kx_{lj}}, \dots, \frac{1}{kx_{lj}}, \varepsilon, \dots, \varepsilon\Big]^t \ge \varepsilon,$$

where the t first component of U is $\frac{1}{kx_{lj}}$.

$$U^t = \Big[\frac{\varepsilon}{ty_{pj}}, \dots, \frac{\varepsilon}{ty_{pj}}, \varepsilon, \dots, \varepsilon\Big]^t \ge \varepsilon,$$

where the t first component of U is $\frac{\varepsilon}{ty_{pj}}$. Now, we show that

$$V^t x_j \le 1, \ (j = 1, \dots, n) \ \text{ and } \ U^t y_j - V^t x_j \le 0, \ (j = 1, \dots, n).$$

$$\begin{aligned}
V^t x_j &= \begin{bmatrix} \frac{1}{kx_{lj}} \\ \vdots \\ \frac{1}{kx_{lj}} \\ \varepsilon \\ \vdots \\ \varepsilon \end{bmatrix} (x_{1j}, \cdots, x_{kj}, 0, \cdots, 0) = \frac{x_{1j}}{kx_{lj}} + \cdots + \frac{x_{kj}}{kx_{lj}}\\
&\le \frac{x_{lj}}{kx_{lj}} + \cdots + \frac{x_{lj}}{kx_{lj}} = \frac{1}{k} + \cdots + \frac{1}{k} = \frac{k}{k} = 1.
\end{aligned}$$

So we prove that $V^t x_j \leq 1$, $(j = 1, \ldots, n)$.

$$U^t y_j - V^t x_j = \begin{bmatrix} \frac{\varepsilon}{ty_{pj}} \\ \vdots \\ \frac{\varepsilon}{ty_{pj}} \\ \varepsilon \\ \vdots \\ \varepsilon \end{bmatrix} (y_{1j}, \cdots, y_{tj}, 0, \cdots, 0) - \begin{bmatrix} \frac{1}{kx_{lj}} \\ \vdots \\ \frac{1}{kx_{lj}} \\ \varepsilon \\ \vdots \\ \varepsilon \end{bmatrix} (x_{1j}, \cdots, x_{kj}, 0, \cdots, 0)$$
$$= \varepsilon \frac{y_{1j}}{ty_{pj}} + \cdots + \varepsilon \frac{y_{tj}}{ty_{pj}} - (\frac{x_{1j}}{kx_{lj}} + \cdots + \frac{x_{kj}}{kx_{lj}})$$
$$\leq \varepsilon \frac{y_{pj}}{ty_{pj}} + \cdots + \varepsilon \frac{y_{pj}}{ty_{pj}} - (\frac{x_{1j}}{kx_{lj}} + \cdots + \frac{x_{kj}}{kx_{lj}})$$
$$= \varepsilon - (\frac{x_{1j}}{kx_{lj}} + \cdots + \frac{x_{kj}}{kx_{lj}}) \leq 0.$$

Because ε is a tiny Archimedes number and $(\frac{x_{1j}}{kx_{lj}} + \cdots + \frac{x_{kj}}{kx_{lj}})$ is a positive expression, so $\varepsilon - (\frac{x_{1j}}{kx_{lj}} + \cdots + \frac{x_{kj}}{kx_{lj}})$ will be smaller and or equals zero.

Theorem 2 *Hyperplane which is corresponding to set of common optimal weight of model (11), is relaying on PPS.*

Proof Let $(U^*, V^*)^t$ is the set of common optimal weight of model (11). So $H = \{(x, y) : U^* y - V^* x = 0\}$ is hyperplane which is corresponding to it. For proving of relaying, it's sufficient to prove these two conditions

1. $\forall \ (x, y), \ \Big((x, y) \in PPS \Longrightarrow U^* y - V^* x \leq 0\Big),$
2. $\exists \ (x, y) \in H \cap PPS.$

For proving of condition 1 we should prove that all DMU$_j$, $(j = 1, \ldots, n)$ with input and output (x_j, y_j): $U^* y_j - V^* x_j \leq 0$. This matter is true with feasible solution $(U^*, V^*)^t$.

For proving condition 2, it is sufficient that we have DMU$_j$, $(j = 1, \ldots, n)$ which is $U^* y_j - V^* x_j = 0$. By contradiction, let hyperplane H isn't passing to none of decision units. Whereas $(U^*, V^*)^t$ is the optimal solution of model (11), it is also the feasible solution. So

$$\begin{cases} U^* y_1 - V^* x_1 \leq 0, \\ U^* y_2 - V^* x_2 \leq 0, \\ \vdots \\ U^* y_n - V^* x_n \leq 0. \end{cases} \tag{13}$$

Since $y_1 \geq 0$ and $y_1 \neq 0$. Without interrupting the whole argument, we assume $y_{11} > 0$. We determine Δ that $(u_1^* + \Delta, u_2^*, \ldots, u_s^*, V^*)^t$ is the fisible solution of madel (11). In other words,

$$\begin{cases} (\Delta + u_1^*)y_{11} + u_2^*y_{21} + \cdots + u_s^*y_{s1} - V^*x_1 \leq 0, \\ (\Delta + u_1^*)y_{12} + u_2^*y_{22} + \cdots + u_s^*y_{s2} - V^*x_2 \leq 0, \\ \vdots \\ (\Delta + u_1^*)y_{1n} + u_2^*y_{2n} + \cdots + u_s^*y_{sn} - V^*x_n \leq 0, \end{cases} \tag{14}$$

After making simple the (14) relations we have

$$\begin{cases} \Delta y_{11} \leq V^*x_1 - (u_2^*y_{21} + \cdots + u_s^*y_{s1}) = V^*x_1 - U^*y_1, \\ \Delta y_{12} \leq V^*x_2 - (u_2^*y_{22} + \cdots + u_s^*y_{s2}) = V^*x_2 - U^*y_2, \\ \vdots \\ \Delta y_{1n} \leq V^*x_n - (u_2^*y_{2n} + \cdots + u_s^*y_{sn}) = V^*x_n - U^*y_n, \end{cases} \tag{15}$$

with displacing $U^*y_j - V^*x_j = \alpha_j$, $(j = 1, \ldots, n)$ and with regard to relation (13) always $\alpha_j > 0$, $(j = 1, \ldots, n)$, and relations (15) will convert to below relation

$$\Delta y_{1j} \leq \alpha_j, \quad (j = 1, \ldots, n).$$

For above relatios will happen two cases:
First case: If $y_{1j} = 0$ then for every real number Δ, $(u_1^* + \Delta, u_2^*, \ldots, u_s^*, V^*)^t$ will be the feasible solution of (11).
Second case: If $y_{1j} > 0$, then $\Delta \leq \frac{\alpha_j}{y_{1j}}$.

So $0 \leq \Delta \leq min\{\frac{\alpha_j}{y_{1j}} : y_{1j} > 0\}$. At the first of proof it is assumed that $y_{11} > 0$. Therefor set of $\{\frac{\alpha_j}{y_{1j}} : y_{f1j} > 0\}$ has at least one member $\frac{\alpha_1}{y_{11}}$, so $min\{\frac{\alpha_j}{y_{1j}} : y_{1j} > 0\}$ will be bigger than zero. If $\delta = min\{\frac{\alpha_j}{y_{1j}} : y_{1j} > 0\}$ with increasing u_1^* to $u_1^* + \delta$ always $(u_1^* + \delta, u_2^*, \ldots, u_s^*, V^*)^t$ will be the feasible solution of model (11). Now, the amount of the targetfunction will be computed for this feasible solution like as below

$$\begin{aligned} w &= (\Theta_1^* - ((u_1^* + \delta)y_{11} + \cdots + u_s^*y_{s1})) + \cdots + (\Theta_n^* - ((u_1^* + \delta)y_{1n} + \cdots + u_s^*y_{sn})) \\ &= (\Theta_1^* - U^*y_1 - \delta y_{11}) + \cdots + (\Theta_n^* - U^*y_n - \delta y_{1n}). \end{aligned}$$

Because δ is bigger than zero emphasisly, and at least one positive component of $y_{1j} \quad (j = 1, \ldots, n)$, so it is for at least one $\delta y_{1j} > 0, j \in \{1, \ldots, n\}$. Then above solution will be smaller that below relation.

$$(\Theta_1^* - U^*y_1) + \cdots + (\Theta_*^n - U^*y_n) = \sum_{j=1}^{n} (\Theta_j^* - U^*y_j).$$

The fiseable solution $(u_1^* + \delta, u_2^*, \ldots, u_s^*, V)^t$ has amount of target function smaller than amount of function of optimal solution $(U^*, V^*)^t$ that this subject is in contrast with optimazing of $(U^*, V^*)^t$. So contradiction assumption is fault, and the said hyperplane should passing on just one DMU. then this hyperplane will be reliance on PPS.

Theorem 3 *The correspondent hyperplane of common optimal weight set of model (12) will be relaying on PPS.*

Proof Let $(U^*, V^*)^t$ is set of common optimal weight of model (12). So $H = \{(x, y) : U^*y - V^*x = 0\}$ is correspondent hyperplane of it. As we said in proof of Theorem 2, for proving relaying, it is sufficient to prove these two conditions.

First condition: For all $j \in \{1, \ldots, n\}$, we have $U^*y_j - V^*x_j \leq 0$. Second condition: At least exit a $j \in \{1, \ldots, n\}$ that the hyperplane be passing on DMU_j. In other words

$$\exists\, j \in \{1, \ldots, n\} \;\; s.t. \;\; U^*y_j - V^*x_j = 0.$$

According to the $(U^*, V^*)^t$ is the optimal feasible solution of model (12) so applies to the constraints of this model. In other words

$$\begin{aligned} &V^*x_j \leq 1, && (j = 1, \ldots, n),\\ &U^*y_j - V^*x_j \leq 0, && (j = 1, \ldots, n). \end{aligned}$$

So the first condition is established.

For proving the second condition we should show that at least one of the constraints $U^*y_j - V^*x_j \leq 0$, $(j = 1, \ldots, n)$ is equal. For this, according to theorem of complementary slackness is sufficient the daul variable corresponding to one of these constraints to be non-zero. At the first model (12) write following

$$\begin{aligned} \min \quad & z \\ s.t. \quad & \\ & -V^t x_j \geq -1, \quad j = 1, \ldots, n, \\ & -U^t y_j + V^t x_j \geq 0, \quad j = 1, \ldots, n, \\ & z + U^t y_j \geq \Theta_j^*, \quad j = 1, \ldots, n, \\ & U \geq 1\varepsilon, \\ & V \geq 1\varepsilon. \end{aligned} \tag{16}$$

Daul of model (16) is following form

$$
\begin{aligned}
\max \quad & \Big\{ -\sum_{j=1}^{n} \lambda_j + \sum_{j=1}^{n} \gamma_j \Theta_j^* + \varepsilon(1.S^+ + 1.S^-) \Big\} \\
s.t. \quad & \\
& \sum_{j=1}^{n} \gamma_j = 1, \\
& \sum_{j=1}^{n} \lambda_j x_j + \sum_{j=1}^{n} \mu_j x_j + S^- = 0, \\
& -\sum_{j=1}^{n} \mu_j y_j + \sum_{j=1}^{n} \gamma_j y_j + S^+ = 0, \\
& \lambda_j \geq 0, j = 1, \dots, n, \\
& \mu_j \geq 0, j = 1, \dots, n, \\
& \gamma_j \geq 0, j = 1, \dots, n, \\
& S^- \geq 0, \\
& S^+ \geq 0.
\end{aligned}
\tag{17}
$$

Model (17) has the following feasible solution

$$
\begin{aligned}
& \gamma_1 = 1, \gamma_j = 0, (j = 2, \dots, n) \\
& \mu_1 = 1, \mu_j = 0, (j = 2, \dots, n) \\
& \lambda_1, \lambda_j = 0, (j = 2, \dots, n) \\
& S^- = 0, S^+ = 0
\end{aligned}
\tag{18}
$$

So this model always is feasible. Suppose $(\gamma^*, \mu^*, \lambda^*, S^{-^*}, S^{+^*})$ is the optimal feasible solution for model (17) where $\gamma^* = (\gamma_1^*, \dots, \gamma_n^*)$ and $\mu^* = (\mu_1^*, \dots, \mu_n^*)$ and $\lambda^* = (\lambda_1^*, \dots, \lambda_n^*)$. We will be show that exit at least that $\mu_j^* \neq 0$. By contradiction suppose for all $j \in \{1, \dots, n\}$, $\mu_j^* = 0$. Because $(\gamma^*, \mu^*, \lambda^*, S^{-^*}, S^{+^*})$ is a optimal feasible solution for model (17), is lied in this model. In other word

$$
\begin{cases}
\sum_{j=1}^{n} \gamma_j^* = 1, \\
-\sum_{j=1}^{n} \lambda_j^* x_j + \sum_{j=1}^{n} \mu_j^* x_j + S^{-^*} = 0, \\
-\sum_{j=1}^{n} \mu_j^* y_j + \sum_{j=1}^{n} \gamma^* y_j + S^{+^*} = 0.
\end{cases}
\tag{19}
$$

According to the contradiction assumption $\mu_j^* = 0$ for all $j \in \{1, \dots, n\}$. So the relations (18) are in form

$$
\begin{cases}
\sum_{j=1}^{n} \gamma_j^* = 1, \\
-\sum_{j=1}^{n} \lambda_j^* x_j + S^{-^*} = 0, \\
\sum_{j=1}^{n} \gamma^* y_j + S^{+^*} = 0.
\end{cases}
\tag{20}
$$

Since $\sum_{j=1}^{n} \gamma_j^* y_j + S^{+^*} = 0$ then the sum of two positive terms is zero so both of two term should be zero that means

$$\sum_{j=1}^{n} \gamma_j^* y_j = 0$$
$$S^{+^*} = 0.$$

And since $\gamma_j^* \geq 0$ and $y_j \geq 0$ then $\gamma_j^* y_j \geq 0$. So

$$\gamma_j^* y_j = 0, \forall \{j \in 1, \ldots, n\},$$

or

$$\begin{cases} \gamma_1^*(y_{11}, \ldots, y_{s1}) = 0 \\ \vdots \\ \gamma_n^*(y_{1n}, \ldots, y_{sn}) = 0 \end{cases}$$

Since $y_j \neq 0$, $(j \in \{1, \ldots, n\})$ then we have γ_j^* for all $j \in \{1, \ldots, n\}$. Finally $\sum_{j=1}^{n} \gamma_j^* = 0$ and this has contradiction with $\sum_{j=1}^{n} \gamma_j^* = 1$. so the contradiction assumption is fault and there is at least a $j \in \{1, \ldots, n\}$ that $\mu_j^* \neq 0$. So we have $U^* y_j - V^* x_j = 0$ with theorem of complementary slackness.

As proved in Theorems 2 and 3 the uniqe hyperplane of models (11) and (12) will be ralaing on the set of production possibilities.

For full ranking of the decision units by the new method, it is sufficeint to compute the efficeincy amount of all DMUs by using the set of common optimal weight obtained from solving model and any units that has more efficeincy amount, will have the more great rank than other decision units.

If the efficeincy amount of more than one DMU will be equal one and or other words introduces more than one DMU we can rank efficeincy units by using CSW2 by JahanShahloo and et al (2005). It is sufficeint to run this matter on one of two models (11) and (12). for this purpose, Let A is the set of all effoceinty units from model (12). In other words, $A = \{j | \Theta_j = 1\}$.

For ranking of the efficeincy units model (11), the following model is implemented.

$$\begin{aligned} \min \quad & z = \max\{(\Theta_j^* - U^t y_j) : j \in A\} \\ s.t. \quad & \\ & V^t x_j \leq 1, \quad j = 1, \ldots, n, j \notin A, \\ & (U, V) \geq 1\varepsilon. \end{aligned} \tag{21}$$

After the reduction model (19) the following model achived

$$
\begin{aligned}
\min \quad & z \\
s.t. \quad & \\
& V^t x_j \leq 1, \quad j = 1, \cdots, n, j \notin A, \\
& U^t y_j - V^t x_j \leq 0, \quad j = 1, \ldots, n, j \notin A, \\
& z + U^t y_j \geq \Theta_j^*, \quad j = 1, \ldots, n, j \in A, \\
& (U, V) \geq 1\varepsilon.
\end{aligned}
\tag{22}
$$

With solving model (20) and computing the efficeincy units of set A by using the common optimal weight of this model, we can rank all units of set A.

We can apply a method like this method for ranking the efficeincy units from model (11).

6 Applied example

In this section, a numerical example is tested with the proposed models of Sections 3 and 5 and a comparison will be made between these methods. It is noteworthy that the data in this example are for 30 bank branches and also these numbers are approximated to 6 decimal places. We use of GAMS program for solving this example.

We consider 30 bank branches with 3 input and 5 output that these inputs and outputs are shown in Table 2. The data of inputs and outputs for the 30 branches are added in Table 3. For this example, we solve only models (7) and (12).

Table 2 Inputs and outputs using in bank evaluation

Outputs	Inputs
1) Facilities	1) Personnel Score
2) Added of four deposits	2) Profit paid
3) Earnings	3) Deferred Claims
4) Fees received	
5) Other sorces	

The set of obtained common optimal weigth from solution of these two models was shown in Table 4.

We show the set of common optimal weight in form of vector

$$(v_1^*, v_2^*, v_3^*, u_1^*, u_2^*, u_3^*, u_4^*, u_5^*)^t.$$

At the end, Table 6 shows the efficeinct amount of decision units for every set of common optimal weight in Table 5 and also the obtained optimistic efficeincy amount from CCR for every decision units. According to the efficeincy amount

Table 3 The input and output date of Table 2

	l_1	l_2	l_3	O_1	O_2	O_3	O_4	O_5
DMUs	$P.S$	$P.p$	$D.C$	Facilities	A. of four d.	Earnings	F. received	Other sorces
1	15.51	27461619631	975137870	$2.78118E \div 11$	$2.26602E11$	29915130646	164933857	532526000
2	9.91	991505052	561929018	40696009129	38767599315	324591770	106283190	74517002
3	13.48	4332302200	5725775229	53992117745	44371043380	3314233292	80642072	356818442
4	56.23	15713640424	94370216509	$3.73786E \div 11$	$3.29971E \div 11$	22330933059	$2.065E \div 09$	$9.224E \div 09$
5	20.14	11969476389	4122612602	85817742385	$1.67941E \div 11$	5412530694	$1.057E \div 09$	$5.72E \div 09$
6	19.73	4878145054	3658861942	$1.3492E \div 11$	$1.08216E \div 11$	10902382836	$1.489E \div 09$	$7.9E \div 09$
7	14.48	3925850598	6129057630	23630844303	58755027036	1595556628	62920765	209412589
8	16.31	15105382313	1425177261	$3.11973E \div 11$	$2.11467E \div 11$	35304269319	$2.688E \div 09$	$1.114E \div 10$
9	12.66	10211488400	117557801	$1.05433E \div 11$	$1.24552E \div 11$	5207072313	566729654	$2.707E \div 09$
10	6.95	4908421028	10360395201	$1.09277E \div 11$	71630484373	6515767382	570261939	$6.056E \div 09$
11	25.65	7344812371	4446919429	939645532370	$1.30915E \div 11$	8384901552	517953877	$1.841E \div 09$
12	16.29	6243794816	50371.37033	88215799891	$1.49623E \div 11$	7274414478	$1.061E \div 09$	$5.651E \div 09$
13	12.8	3995882436	2128015084	34778741074	5405238357	2583300403	710702662	$4.625E \div 09$
14	11.94	2335288900	1876765625	34154537295	37022523029	2041272350	59773518	122336681
15	8.9	2214698810	973271944	50625097597	68961263022	2723235288	60126084	317804434
16	12.47	3055503158	1059749782	24842289674	37535320501	625261710	110547176	146641359
17	15.94	3220265878	2682027091	79141589566	61208648272	5575875859	848644088	$4.797E \div 09$
18	12.95	1609044431	17843797482	18656488036	97819134599	1332095197	89855841	780188538
19	5.64	2248340462	129899484	31047346676	34789289289382	2879984210	53313979	39594661
20	19.5	7892494826	2038441141	$1.33237E \div 11$	90700401785	8235994330	370917381	$1.792E \div 09$
21	12.3	4676629110	5919986612	50132710423	71153048274	4495220591	626688651	$3.104E \div 09$
22	10.16	1829524338	369545850	35451082406	29604992659	3126282331	46631579	414595000
23	5.73	18296524338	369545850	35451082406	29604992659	3126282331	46631579	63900000
24	4.49	2402638260	1158305400	16087737307	40274801.313	1694074928	216434968	$1.868E \div 09$
25	6.2	11346458917	2471933000	54175423616	$1.15507E \div 11$	5369824111	491853212	$1.126E \div 09$
26	9.02	3372219100	432122546	57211751112	51661106637	5331207756	131135618	39493998
27	21.88	2293325158	12812985696	$1.24935E \div 11$	55229459032	4611761025	159005139	187786696
28	17.02	46422885003	296722816	74990268085	62595062008	421579993	2473844354	112706012
29	10.47	3162276806	1022023448	6901981.2605	83991224363	33751.52301	113511469	61406000
30	8.02	5681.800755	643082987	74824470248	75679843437	5844327520	1738322979	78330276

in first and second coloumns of Table 6, the ranking was done between all units. With carefull study in obtained efficeincy amount for DMUs under set of optimal common weight of model (7), it is seem that U_{06} and U_{07} have the efficeincy amount 1, so the ranking couldn't be compeleted.

Table 4 the set of the optimal models (7) and (12) for data of Table 3

Model	v_1^*	v_2^*	v_3^*	u_1^*	u_2^*	u_3^*	u_4^*	u_5^*
$M(7)$	1.45318	2.11866	0.341571	0.94911	1.598585	0.009145	20.061839	0.000001
$M(8)$	0.507969	0.859886	0.000001	0.294783	10.509082	30.051283	0.000001	0.000001

Now, with regarding to obtained efficeincy amount of Table 6 for every set of the optimal common weight of model (12), DMUs of U_{08} and U_{15} and U_{18} was obtained the efficeincy amount (1). In the other words, these three DMUs are efficeint. For ranking these efficeint units, as we said in section 5, with placing $A = \{8, 15, 18\}$ we can solve model (19). In Table 6, all units are ranked. The other remarkable point is that as we seen in this example, the computed efficeincy amount with using set of optimal common weight of model (12), are always smaller and or equal 1. But the computed efficeincy amount by the set of common optimal weight model (7) for some decision units will be strictly bigger than 1.

Table 5 The amount of efficeincy units for the set of the weights of Table 5 and amount of these efficeincy acheived from CCR

DMU	Model (7)	Model (12)	Θ_j^*
U_{01}	0.715899	0.609498	1
U_{02}	0.87729	0.799105	1
U_{03}	0.519001	0.464606	0.51
U_{04}	0.86496	0.836305	0.92
U_{05}	0.722773	0.599481	079
U_{06}	1	0.869513	1
U_{07}	0.494476	0.438864	0.49
U_{08}	1.179109	1	1
U_{09}	0.790943	0.65243	1
U_{10}	1.07188	0.951038	1
U_{11}	0.709558	0.62223	0.68
U_{12}	1.055775	0.904841	1
U_{13}	0.500225	0.505511	0.91
U_{14}	0538539	0.478843	10.52
U_{15}	1.145049	1	1
U_{16}	0.439282	0.375139	0.45
U_{17}	0.7698	0.670352	0.92
U_{18}	1	1	1
U_{19}	0.776567	0.673485	1
U_{20}	0.69981	0.603808	0.7
U_{21}	0.69416	0.603066	0.69
U_{22}	0.728441	0.698207	1
U_{23}	0.805764	0.713461	0.88
U_{24}	0.788178	0.665989	1
U_{25}	0.677868	0.554755	0.88
U_{26}	0.804375	0.70544	1
U_{27}	0.743368	0.702589	0.61
U_{28}	0.61589	0.538624	1
U_{29}	1.110668	0.960004	1
U_{30}	0.863751	0.732859	0.86

Table 6 The full ranking all of units by the amount of efficeincy achieved from model (12) and (19)

DMU	ranking.Θ_{Θ}
1	0.609488()20
2	0.733105(9)
3	0.46486(28)
4	0.466505(8)
5	0.599181(23)
6	0.833513(6)
7	0.458864(29)
8	0.837770(7)
9	0.65243(18)
10	0.953058(4)
11	0.62228(19)
12	0.904841(5)
13	0.305511(26)
14	0.478843(27)
15	0.57349(2)
16	0.375159(30)
17	0.670442(16)
18	0.294463(1)
19	0.670485(15)
20	0.600803(21)
21	0.603065(22)
22	0.638107(14)
23	0.711461(11)
24	0.655901(17)
25	0.554755(24)
26	0.70540(12)
27	0.700881(13)
28	0.51124(25)
29	0.96500(3)
30	0.732855(10)

7 Conclusion

In this paper, after referring to data envelopment analysis, we introduced the suggesting methods of wang and et al for finding common weight. In these methods, they find the set of common weight as the computed efficeincy with common weight set for all units are always smaller and or equal to the obtained optimistic efficeincy from CCR. And in other words, the goal was obtaining the unique hyperplane, so that the distance between all DMUs from this hyperplane will be minimal. In this paper, we can show that the obtained hyperplane from suggesting methods of Wang passed from PPS. In other words, in the presented method, the desired hyperplane order to ranking is not necessarily reliable on PPS. Finally we presented the new method for the ranking of the decision units that the corresponding hyperplane of the set common weights is reliable on PPS and then this technique applied for the real data.

References

1. A. Charnes, W. W. Cooper, E. Rhodes, Measuring the efficiency of decision making units. European Journal of Operational Research, 2, 429–444 (1978)
2. L. Friedman, Z. Sinuany-Stern, Scaling units via the canonical correlation analysis in the DEA context. European Journal of Operational Research, 100, 629–637 (1997).
3. J. A. Ganley, S. A. Cubbin, Public sector efficiency measurement: Applications of data envelopment analysis. Amsterdam: North-Holland (1992).
4. A. Hashimoto, D. A. Wu, A DEA-compromise programming model for comprehensive ranking. Journal of the Operations Research Society of Japan, 47, 73–81 (2004).
5. G. R. Jahanshahloo, A. Memariani, F. Hosseinzadeh Lotfi, H. Z. Rezai, A note on some DEA models and finding efficiency and complete ranking using common set of weights. Applied Mathematics and Computation, 166, 265–281 (2005).
6. C. Kao, H. T. Hung, Data envelopment analysis with common weights: The compromise solution approach. Journal of the Operational Research Society, 56, 1196– 1203 (2005).
7. F. H. F. Liu, H. H. Peng, Ranking of units on the DEA frontier with common weights. Computers and Operations Research, 35, 1624–1637 (2008).
8. Z. Sinuany-Stern, A. Mehrez, A. Barboy, Academic departments' efficiency in DEA. Computers and Operations Research, 21, 543–556 (1994).
9. Z. Sinuany-Stern, L. Friedman, DEA and the discriminant analysis of ratios for ranking units. European Journal of Operational Research, 111, 470–478 (1998).
10. Y. M. Wang, Y. Luo, Y. X. Lan, Common weights for fully ranking decision making units by regression analysis. Expert System with Applications 38, 9122–9128 (2011).
11. Y. M. Wang, Y. Luo, L. Liang, Ranking decision making units by imposing a minimum weight restriction in the data envelopment analysis. Journal of Computational and Applied Mathematics, 223, 469–484 (2009).