Global Analysis and Discrete Mathematics
Volume 6, Issue 2, pp. 179–186
ISSN: 2476-5341 *Research Article*

Pairs of Finite Dimensional Nilpotent and Filiform Lie Algebras

Homayoon Arabyani · Elaheh Khamseh

Received: 27 September 2021 / Accepted: 22 November 2021

Abstract Let (N, L) be a pair of finite dimensional nilpotent Lie algebras. If N admits a complement K in L such that $\dim N = n$ and $\dim K = m$, then $\dim \mathcal{M}(N, L) = \frac{1}{2}n(n + 2m - 1) - t(N, L)$, where $\mathcal{M}(N, L)$ is the Schur multiplier of the pair (N, L) and $t(N, L)$ is a non-negative integer. In this paper, we characterize the pair (N, L) for $t(N, L) = 0, 1, 2, \ldots, 23$, where N is a finite dimensional filiform Lie algebra and N, K are ideals of L such that $L = N \oplus K$. Moreover, we classify the pair (N, L) for $s^{'}(N, L) = 3$, where $s^{'}(N, L) = \frac{1}{2}(n - 1)(n - 2) + 1 + (n - 1)m - \dim \mathcal{M}(N, L)$, L is a finite dimensional nilpotent Lie algebra and N is a non-abelian ideal of L.

Keywords Filiform Lie algebra · nilpotent Lie algebra · pair of Lie algebras · Schur multiplier.

Mathematics Subject Classification (2010) 17B30 · 17B60 · 17B99

1 Introduction

All Lie algebras are considered over a fixed field Λ and $[,]$ denotes the Lie bracket. Let (N, L) be a pair of Lie algebras, in which N is an ideal in L. The *Schur multiplier* of (N, L) is the abelian Lie algebra $\mathcal{M}(N, L)$ appearing in the

H. Arabyani (Corresponding Author)
Department of Mathematics, Neyshabur Branch, Islamic Azad University, Neyshabur, Iran.
Tel.: +123-45-678910
Fax: +123-45-678910
E-mail: arabyani.h@gmail.com, h.arabyani@iau-neyshabur.ac.ir

E. Khamseh
Department of Mathematics, Shahr-e-Qods Branch, Islamic Azad University Tehran, Iran.

following natural exact sequence of Lie algebras

$$H_3(L) \to H_3(L/N) \to \mathcal{M}(N,L) \to \mathcal{M}(L) \to$$
$$\mathcal{M}(L/N) \to \frac{L}{[N,L]} \to \frac{L}{L^2} \to \frac{L}{(L^2+N)} \to 0,$$

where $\mathcal{M}(-)$, $H_3(-)$ and L^2 denote the Schur multiplier, the third homology of a Lie algebra and the derived subalgebra of L, respectively.
Let $0 \to R \to F \to L \to 0$ be a free presentation of L. Then $\mathcal{M}(N,L)$ is defined to be the factor Lie algebra $(R \cap [S,F])/[R,F]$, in which S is an ideal in F such that $N \cong S/R$ (see [1,12], for more information). In particular, if $N = L$, then $\mathcal{M}(L,L) = \mathcal{M}(L)$ is the Schur multiplier of L. Moneyhun [8] proved that if L is a Lie algebra of dimension n, then $\dim \mathcal{M}(L) = \frac{1}{2}n(n-1) - t(L)$, where $t(L)$ is a non-negative integer. In [2,5,6], all nilpotent Lie algebras are characterized, when $t(L) = 0,1,\ldots,8$. Let (N,L) be a pair of finite dimensional nilpotent Lie algebras. Saeedi et al. [12] proved that if N admits a complement K say, in L with $\dim N = n$ and $\dim K = m$, then

$$\dim \mathcal{M}(N,L) = \frac{1}{2}n(n+2m-1) - t(N,L), \tag{1}$$

where $t(N,L) \geq 0$. This gives us the Moneyhun's result, if $m = 0$.
The first author and colleagues [1] characterized the pair (N,L), for which $t(N,L) = 0,1,2,3,4$. Moreover, they determined pairs (N,L) for $t(N,L) = 0,1,\ldots,10$, when L is a filiform Lie algebra. Also, Niroomand and Russo [10] proved that

$$\dim \mathcal{M}(L) \leq \frac{1}{2}(n+m-2)(n-m-1) + 1, \tag{2}$$

where L is a non-abelian nilpotent Lie algebra with $\dim L = n$ and $\dim L^2 = m$. The above upper bound implies that $\dim \mathcal{M}(L) = \frac{1}{2}(n-1)(n-2) + 1 - s(L)$, where $s(L) \geq 0$. Niroomand et al. in [9–11] classified the structure of L, when $s(L) = 0,1,2,3$.

Also in [7], using (1), all pairs (N,L) are classified, when $t(N,L) = 0,1,\ldots,6$. Moreover, it is proved under some conditions that

$$\dim \mathcal{M}(N,L) = \frac{1}{2}(n-1)(n-2) + 1 + (n-1)m - s'(N,L), \tag{3}$$

where $s'(N,L) \geq 0$, $\dim N = n$ and $\dim K = m$. Furthermore, all pairs (N,L) are also classified for $s'(N,L) = 0,1,2$.

In the present paper, we continue the above works and characterize all pairs (N,L), when L is a finite dimensional filiform Lie algebra and $t(N,L) = 0,1,\ldots,23$. Moreover, using (3), we classify all pairs (N,L) for $s'(N,L) = 3$.

Note that in the proof of main theorems, the upper bound (2) enables us to provide a new technique in our classification which makes the upper bound 3 smaller than the one in (1).

2 Preliminaries

In this section, we discuss some preliminary results which will be used in the main theorems. A filiform Lie algebra is an algebra with maximal nilpotency class. More precisely, an n-dimensional Lie algebra L is called filiform, if $\dim L^i = n - i$, for $2 \leq i \leq n$, where L^i is the ith term of the lower central series of L. In [4], filiform Lie algebras are classified up to dimension 11.

The following theorem is proved by H. Darabi and M. Eshrati. Also, by using [11] and [13] we can prove it.

Theorem 1 *Suppose A is a filiform n-Lie algebra. Then*

(i) $t(A) = 1$ *if and only if A is isomorphic to $H(n, 1)$;*
(ii) $t(A) = 4$ *if and only if A is isomorphic to $L_{4,3}$;*
(iii) $t(A) = 7$ *if and only if A is isomorphic to $L_{5,6}$, $L_{5,7}$ or $A_{3,5,2}$;*
(iv) $t(A) = 11$ *if and only if A is isomorphic to $A_{4,6,2}$;*
(v) $t(A) = 12$ *if and only if A is isomorphic to $L_{6,15}$, $L_{6,17}$ or $L_{6,18}$;*
(vi) $t(A) = 13$ *if and only if A is isomorphic to $L_{6,14}$ or $L_{6,16}$;*
(vii) $t(A) = 15$ *if and only if A is isomorphic to $A_{3,6,6}$;*
(viii) $t(A) = 16$ *if and only if A is isomorphic to $A_{5,7,1}$ or $A_{3,6,7}$;*
(ix) $t(A) = 17$ *if and only if A is isomorphic to F_7^1, F_7^3, F_7^5, F_7^6, F_7^7 or F_7^8;*
(x) $t(A) = 18$ *if and only if A is isomorphic to F_7^2 or F_7^4;*
(xi) $t(A) = 22$ *if and only if A is isomorphic to $A_{6,8,2}$.*

There is no filiform Lie algebra for $t(A) = 2, 3, 5, 6, 8, 9, 10, 14, 19, 20, 21$ *and* 23.

Here $H(m)$ denotes the Heisenberg Lie algebra of dimension $2m + 1$, $A(n)$ is an n-dimensional abelian Lie algebra and $L(a, b, c, d)$ denotes the algebra discovered for the case $t(L) = a$, where $b = \dim L$, $c = \dim Z(L)$ and $d = t(L)$.

Lemma 1 *([1], Lemma 3.1) Let N and L be finite filiform Lie algebras. Then $L \ncong N \oplus A(n)$.*

Theorem 2 *([10], Theorem 3.1) Let L be a non-abelian nilpotent Lie algebra such that* $\dim(L) = n$ *and* $\dim(L^2) = m \geq 1$. *Then*

$$\dim \mathcal{M}(L) \leq \frac{1}{2}(n + m - 2)(n - m - 1) + 1.$$

Moreover, if $m = 1$, then the equality holds if and only if $L \cong H(1) \oplus A(n-3)$.

The above upper bound implies that $\dim \mathcal{M}(L) \leq \frac{1}{2}(n-1)(n-2) + 1 - s(L)$, where $s(L) \geq 0$. Niroomand et al. in [9–11] classified the structure of L, when $s(L) = 0, 1, 2, 3$ as follows:

Lemma 2 *([11, Lemma 3.3]) Let L be an n-dimensional nilpotent Lie algebra and $dimL^2 = 1$. Then for some $m \geq 1$,*

$$L \cong H(m) \oplus A(n - 2m - 1).$$

Note that in the statement of the above result in [9], it is not mistakenly stated that L is nilpotent.

Theorem 3 *([9–11]) Let L be a non-abelian n-dimensional nilpotent Lie algebra. Then*

(i) $s(L) = 0$ *if and only if* $L \cong H(1) \oplus A(n-3)$;
(ii) $s(L) = 1$ *if and only if* $L \cong L(4,5,2,4)$;
(iii) $s(L) = 2$ *if and only if* $L \cong L(3,4,1,4), L(4,5,2,4) \oplus A(1)$ *or* $H(m) \oplus A(n-2m-1)$ $(m \geq 2)$;
(iv) $s(L) = 3$ *if and only if* $L \cong L(4,5,1,6)$, $L(5,6,2,7)$, $L(3,4,1,4) \oplus A(1)$, $L(4,5,2,4) \oplus A(2)$ *or* $L_{6,26}$.

3 Main Results

In this section, we classify finite dimensional pairs of filiform Lie algebras with $t(N,L) = 0,1,2,\ldots,23$. Moreover, we characterize the structure of finite dimensional pairs (N,L) with $s'(N,L) = 3$.

Theorem 4 *Let (N,L) be a pair of finite dimensional nilpotent Lie algebras and K be an ideal of L such that $L = N \oplus K$, $\dim N = n$, $\dim K = m$, N be a filiform Lie algebra and $\dim \mathcal{M}(N,L) = \frac{1}{2}n(n+2m-1) - t(N,L)$, where $t(N,L) \geq 0$. Then*

(a) If N is an abelian Lie algebra, then $(N,L) \cong (A(2), A(2) \oplus K)$ where $\dim K^2 = \frac{1}{2}t(N,L)$;
(b) If N is a non-abelian Lie algebra, then:
(1) $t(N,L) = 1$ *if and only if* $(N,L) \cong (H(1), H(1))$;
(2) $t(N,L) = 4$ *if and only if* $(N,L) \cong (L(3,4,1,4), L(3,4,1,4))$;
(3) $t(N,L) = 7$ *if and only if* $(N,L) \cong (L_{5,6}, L_{5,6}), (L_{5,7}, L_{5,7})$ *or* $(A_{3,5,2}, A_{3,5,2})$;
(4) $t(N,L) = 11$ *if and only if* $(N,L) \cong (A_{4,6,2}, A_{4,6,2})$;
(5) $t(N,L) = 12$ *if and only if* $(N,L) \cong (L_{6,15}, L_{6,15}), (L_{6,17}, L_{6,17})$ *or* $(L_{6,18}, L_{6,18})$;
(6) $t(N,L) = 13$ *if and only if* $(N,L) \cong (L_{6,14}, L_{6,14}), (L_{6,16}, L_{6,16})$;
(7) $t(N,L) = 15$ *if and only if* $(N,L) \cong (A_{3,6,6}, A_{3,6,6})$;
(8) $t(N,L) = 16$ *if and only if* $(N,L) \cong (A_{5,7,2}, A_{5,7,2})$ *or* $(A_{3,6,7}, A_{3,6,7})$;
(9) $t(N,L) = 17$ *if and only if* $(N,L) \cong (L_1, L_1), (L_2, L_2), (L_4, L_4), (L_5, L_5)$ *or* (L_8, L_8) *for* $\lambda = 3$;
(10) $t(N,L) = 18$ *if and only if* $(N,L) \cong (L_3, L_3), (L_6, L_6), (L_7, L_7)$ *or* (L_8, L_8) *for* $\lambda \neq 3$;
(11) $t(N,L) = 22$ *if and only if* $(N,L) \cong (A_{6,8,2}, A_{6,8,2})$;
(12) There is no pair for $t(N,L) = 2, 3, 5, 6, 8, 9, 10, 14, 19, 20, 21$ *and* 23.

Proof Put $l = l(N) = \frac{1}{2}n(n-1) - \dim \mathcal{M}(N)$ and

$$t = t(N,L) = \frac{1}{2}n(n+2m-1) - \dim \mathcal{M}(N,L),$$

where l and t are non-negative integers. A well-known result in [7] states that $\dim \mathcal{M}(N,L) = \dim N + \dim(N/N^2 \otimes K/K^2)$. Thus one may easily obtain that

$$mn = (t-l) + (\dim N/N^2)(\dim K/K^2). \qquad (4)$$

Now since $\dim K/K^2 \leq m$ and $\dim N/N^2 = n - \dim N^2$, then

$$m. \dim N^2 \leq t - l, \qquad (5)$$

which implies that $l \leq t$.

(a) Let N be an abelian Lie algebra. Then by Theorem 1, $N \cong A(2)$. Therefore, $(N,L) \cong (A(2), A(2) \oplus K)$, where $\dim K^2 = \frac{1}{2}t$, by (4).

(b) If N is a non-abelian Lie algebra, then we have

Case $t = 1$. By (5), $l = 1$ and $m = 0$. So, using Theorem 1 we get $(N,L) \cong (H(1), H(1))$.

Case $t = 2$. If $l = 1$, then by Theorem 1 and (5) we have $N \cong H(1)$ and $m \leq 1$. If $m = 0$, then $(N,L) \cong (H(1), H(1))$, which contradicts the case $t = 1$. Assume that $m = 1$, then $K \cong A(1)$, which is a contradiction by Lemma 1. If $l = 2$, then there is no pair by Theorem 1.

Case $t = 3$. One can easily check that Theorem 1 and Lemma 1 imply that there is no pair.

Case $t = 4$. If $l = 1$, then by (5), $m \leq 3$. Now if $m = 0$, then $(N,L) \cong (H(1), H(1))$ which contradicts the case $t = 1$. If $m = 1, 2$, then there does not exist any pair by Lemma 1. Suppose that $m = 3$, then $K \cong H(1)$ or $A(3)$. If $K \cong A(3)$, then there is not any pair by Lemma 1. If $K \cong H(1)$, there is not any pair by (3.4). Assume that $l = 4$, then by (5), we get $m = 0$ and so Theorem 1 implies that

$$(N,L) \cong (L(3,4,1,4), L(3,4,1,4)).$$

Case $t = 5$. In this case $l = 1$ or 4. If $l = 1$, then $m \leq 4$. Now if $m = 3$, then by (4) we obtain a contradiction. If $m = 4$, then there is not any pair by (4) and Lemma 1. Suppose that $l = 4$, then using Theorem 1 and (5) we have $t = 4$, which is impossible.

Case $t = 6$. If $l = 1$, then $m \leq 5$. If $m = 3$, then (4) implies that $\dim K^2 = 1$. Hence, $\dim L/L^2 = 4$, which is a contradiction. If $m = 5$, then by Lemma 1 and (4), there is not any pair. Suppose that $l = 4$, then there is no pair by Lemma 1.

Case $t = 7$. If $l = 1, 4$, then by a similar manner, there does not exist any pair. If $l = 7$, then using Theorem 1, we get $N \cong L_{5,6}, L_{5,7}$ or $A_{3,5,2}$. Hence

$$(N,L) \cong (L_{5,6}, L_{5,6}), (L_{5,7}, L_{5,7}) \ \ \textit{or} \ \ (A_{3,5,2}, A_{3,5,2}).$$

Cases $t = 11, 12, 13, 15, 16, 17, 18, 22$. Similar to the previous cases, we have

$$(N, L) \cong (A_{4,6,2}, A_{4,6,2}), (A_{6,8,2}, A_{6,8,2}), (L_{6,15}, L_{6,15}), (L_{6,17}, L_{6,17}),$$
$$(L_{6,18}, L_{6,18}), (L_{6,14}, L_{6,14}), (L_{6,16}, L_{6,16}), (A_{3,6,6}, A_{3,6,6}),$$
$$(A_{5,7,2}, A_{5,7,2}), (A_{3,6,7}, A_{3,6,7}), (L_1, L_1), (L_2, L_2), (L_4, L_4),$$
$$(L_5, L_5), (L_8, L_8) \ for \ \lambda = 3, (L_3, L_3), (L_7, L_7), (L_6, L_6),$$
$$or (L_8, L_8) \quad for \quad \lambda \neq 3.$$

Finally, in the cases $t = 8, 9, 10, 19, 20, 21, 23$, by a similar manner one can easily see that there is no any pair.

Theorem 5 *Let L be a finite dimensional nilpotent Lie algebra, N and K be ideals of L, $\dim N = n$, $\dim K = m$, $\dim N^2 \geq 1$ and*

$$s' = s'(N, L) = \frac{1}{2}(n-1)(n-2) + 1 + (n-1)m - \dim \mathcal{M}(N, L).$$

Then $s' = 3$ if and only if (N, L) is isomorphic to one of the following pairs:

(1) $(H(1) \oplus A(1), H(1) \oplus H(r) \oplus A(m-2r))$ $(r \geq 1)$,
(2) $(L_{5,8} \oplus A(i), L_{5,8} \oplus A(2))$ $(0 \leq i \leq 2)$,
(3) $(L_{4,3} \oplus A(i), L_{4,3} \oplus A(1))$ $(i = 0, 1)$,
(4) $(L_{6,22}(\epsilon), L_{6,22})$,
(5) $(L_{6,26}, L_{6,26})$,
(6) $(L_{5,5}, L_{5,5})$.

Proof Put $s = s(N) = \frac{1}{2}(n-1)(n-2) + 1 - \dim \mathcal{M}(N)$. Using again the equality $\dim \mathcal{M}(N, L) = \dim N + \dim(N/N^2 \otimes K/K^2)$, we get

$$mn - m = (s' - s) + (\dim N/N^2)(\dim K/K^2). \tag{6}$$

It is not difficult to show that $s \leq s'$, since $\dim K/K^2 \leq m$, $\dim N/N^2 = n - \dim N^2$, and $\dim N^2 \geq 1$. Now, suppose that $s' = 3$, then $s = 0, 1, 2$ or 3. By using Lemma 2, Theorem 3 and similar to Theorem 4 we can prove these results.

Table 1

t(A)	Filiform n-Lie algebra	t(A)	Filiform n-Lie Algebra
0	Abelian	12	$L_{6,15}, L_{6,17}, L_{6,18}$
1	$H(n,1)$	13	$L_{6,14}, L_{6,16}$
2	None	14	None
3	None	15	$A_{3,6,6}$
4	$L_{4,3}$	16	$A_{5,7,2}, A_{3,6,7}$
5	None	17	$L_1, L_2, L_4, L_5, L_8 \ for \ \lambda = 3$
6	None	18	$L_3, L_6, L_7, L_8 \ for \ \lambda \neq 3$
7	$L_{5,6}, L_{5,7}, A_{3,5,2}$	19	None
8	None	20	None
9	None	20	None
10	None	22	$A_{6,8,2}$
11	$A_{4,6,2}$	23	None

Table 2

n-Lie algebra	None-zero multiplications
$L_{4,3}$	$[e_1, e_2] = e_3, [e_1, e_3] = e_4$
$A_{n,n+2,2}$	$[e_1, ..., e_n] = e_{n+1}, [e_2, ..., e_{n+1} = e_{n+2}$
$A_{n,n+3,6}$	$[e_1, ..., e_n] = e_{n+1}, [e_2, ..., e_{n+1}] = e_{n+2},$ $[e_2, ..., e_n, e_{n+2}] = e_{n+3}$
$A_{n,n+3,7}$	$[e_1, ..., e_n] = e_{n+1}, [e_2, ..., e_{n+1}] = e_{n+2},$ $[e_2, ..., e_n, e_{n+2}] = [e_1, e_3, ..., e_{n+1}] = e_{n+3}$
$L_{6,14}$	$[e_1, e_2] = e_3, [e_1, e_3] = e_4, [e_1, e_4] = [e_2, e_3] = e_5,$ $[e_2, e_5] = -[e_3, e_4] = e_6$
$L_{6,15}$	$[e_1, e_2] = e_3, [e_1, e_3] = e_4, [e_1, e_4] = [e_2, e_3] = e_5$ $[e_1, e_5] = [e_2, e_4] = e_6$
$L_{6,16}$	$[e_1, e_2] = e_3, [e_1, e_3] = e_4, [e_1, e_4] = e_5, [e_2, e_5] = -[e_3, e_4] = e_6$
$L_{6,17}$	$[e_1, e_2] = e_3, [e_1, e_3] = e_4$ $[e_1, e_4] = e_5, [e_1, e_5] = [e_2, e_3] = e_6$
$L_{6,18}$	$[e_1, e_2] = e_3, [e_1, e_3] = e_4, [e_1, e_4] = e_5, [e_1, e_5] = e_6$
$L_1 = (123457A)$	$[e_1, e_i] = e_{i+1}\ (2 \leq i \leq 6)$
$L_2 = (123457B)$	$[e_1, e_i] = e_{i+1}\ (2 \leq i \leq 6), [e_2, e_3] = e_7$
$L_3 = (123457C)$	$[e_1, e_i] = e_{i+1}\ (2 \leq i \leq 6), [e_2, e_5] = e_7, [e_3, e_4] = -e_7$
$L_4(123457D)$	$[e_1, e_i] = e_{i+1}\ (2 \leq i \leq 6), [e_2, e_4] = e_7, [e_2, e_3] = e_6$
$L_5(123457E)$	$[e_1, e_i] = e_{i+1}\ (2 \leq i \leq 6), [e_2, e_4] = e_7, [e_2, e_3] = e_6 + e_7$
$L_6(123457F)$	$[e_1, e_i] = e_{i+1}\ (i = 2, 3, 4, 5, 6), [e_3, e_4] = -e_7,$ $[e_2, e_3] = e_6, [e_2, e_4] = [e_2, e_5] = e_7$
$L_7(123457H)$	$[e_1, e_i] = e_{i+1}\ (i = 2, 3, 4, 5, 6), [e_2, e_4] = e_6,$ $[e_2, e_5] = e_7, [e_2, e_3] = e_5 + e_7$
$L_8(123457I)$	$[e_1, e_i] = e_{i+1}\ (i = 2, 3, 4, 5, 6), [e_2, e_5] = \lambda e_7,$ $[e_3, e_4] = (1 - \lambda)e_7, [e_2, e_3] = e_5, [e_2, e_4] = e_6$

Table 3

Lie algebra	Non-zero Multiplications
$L_{4,3}$	$[e_1, e_2] = e_3, [e_1, e_3] = e_4$
$L_{5,5}$	$[e_1, e_2] = e_3, [e_1, e_3] = [e_2, e_4] = e_5$
$L_{5,6}$	$[e_1, e_2] = e_3, [e_1, e_3] = e_4, [e_1, e_4] = [e_2, e_3] = e_5$
$L_{5,7}$	$[e_1, e_2] = e_3, [e_1, e_3] = e_4, [e_1, e_4] = e_5$
$L_{5,8}$	$[e_1, e_2] = e_4, [e_1, e_3] = e_5$
$L_{5,9}$	$[e_1, e_2] = e_3, [e_1, e_3] = e_4, [e_2, e_3] = e_5$
$L_{6,10}$	$[e_1, e_2] = e_3, [e_1, e_3] = [e_4, e_5] = e_6$
$L_{6,11}$	$[e_1, e_2] = e_3, [e_1, e_3] = e_4, [e_1, e_4] = [e_2, e_3] = [e_2, e_5] = e_6$
$L_{6,13}$	$[e_1, e_2] = e_3, [e_1, e_3] = [e_2, e_4] = e_5, [e_1, e_5] = [e_3, e_4] = e_6$
$L_{6,20}$	$[e_1, e_2] = e_4, [e_1, e_3] = e_5, [e_1, e_5] = [e_2, e_4] = e_6$
$L_{6,23}$	$[e_1, e_2] = e_3, [e_1, e_3] = [e_2, e_4] = e_5, [e_1, e_4] = e_6$
$L_{6,25}$	$[e_1, e_2] = e_3, [e_1, e_3] = e_5, [e_1, e_4] = e_6$
$L_{6,26}$	$[e_1, e_2] = e_4, [e_1, e_3] = e_5, [e_2, e_4] = e_6$
$L_{6,27}$	$[e_1, e_2] = e_3, [e_1, e_3] = e_5, [e_2, e_4] = e_6$
$L_{6,19}(\epsilon)$	$[e_1, e_2] = e_4, [e_1, e_3] = e_5, [e_1, e_5] = [e_2, e_4] = e_6, [e_3, e_5] = \epsilon e_6$
$L_{6,22}(\epsilon)$	$[e_1, e_2] = [e_3, e_4] = e_5, [e_1, e_3] = e_6, [e_2, e_4] = \epsilon e_6$
$L_{6,24}(\epsilon)$	$[e_1, e_2] = e_3, [e_1, e_3] = [e_2, e_4] = e_5, [e_2, e_3] = e_6, [e_1, e_4] = \epsilon e_6$
$37A$	$[e_1, e_2] = e_5, [e_2, e_3] = e_6, [e_2, e_4] = e_7$
$37B$	$[e_1, e_2] = e_5, [e_2, e_3] = e_6, [e_3, e_4] = e_7$
$37D$	$[e_1, e_2] = [e_3, e_4] = e_5, [e_1, e_3] = e_6, [e_2, e_4] = e_7$

References

1. H. Arabyani, F. Saeedi, M. R. R. Moghaddam and E. Khamseh, Characterization of nilpotent Lie algebras pair by their Schur multipliers, Commun. Algebra, 42, 5474–5483 (2014).
2. P. Batten, K. Moneyhun, E. Stitzinger, On characterizing nilpotent Lie algebras by their multipliers, Commun. Algebra, 24, 4319–4330 (1996).
3. G. Ellis, The Schur multiplier of a pair of groups, Appl. Categ. Struct., 6, 355–371 (1998).
4. J. R. Gomez, A. Jimenez-Merchan, Y. Khakimdjanov, Low-dimensional filiform Lie algebras, J. Pure Appl. Algebra, 130, 133–158 (1998).
5. P. Hardy, On characterizing nilpotent Lie algebras by their multipliers III, Commun. Algebra, 33, 4205–4210 (2005).
6. P. Hardy, E. Stitzinger, On characterizing nilpotent Lie algebras by their multipliers, $t(L) = 3, 4, 5, 6$, Commun. Algebra, 26, 3527–3539 (1998).
7. E. Khamseh, S. A. Niri, Classification of pair of nilpotent Lie algebras by their Schur multipliers, Math. Reports, 20, 177–185(2018).
8. K. Moneyhun, Isoclinisms in Lie algebras, Algebras Groups Geom., 11, 9–22 (1994).
9. P. Niroomand, On dimension of the Schur multiplier of nilpotent Lie algebras, Cent. Eur. J. Math, 9, 57–64 (2011).
10. P. Niroomand, F. Russo, A note on the Schur multiplier of a nilpotent Lie algebra, Commun. Algebra, 39, 1293–1297 (2011).
11. F. Saeedi, H. Arabyani, P. Niroomand, On dimension of Schur multiplier of nilpotent Lie algebras II, Asian-Eur. J. Math, 10, 1750076 (8 pages) (2017).
12. F. Saeedi, A. R. Salemkar, B. Edalatzadeh, The commutator subalgebra and Schur multiplier of a pair of nilpotent Lie algebras, J. Lie Theory, 21, 491–498 (2011).
13. A. Shamsaki, P. Niroomand, The Schur multipliers of Lie algebras of maximal class, Internat. J. Algebra. Comput, 29, 795–801 (2019).