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Research article

Existence of Common Best Proximity Points in Neutrosophic B-Metric Spaces with Applications to Integral Equations

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Abstract

This paper establishes new existence results for common best proximity points of two non-self mappings within the framework of complete neutrosophic b-metric spaces. Two main theorems are proved under distinct contractive conditions: the first relies on a contraction-type inequality involving the mappings, while the second employs the notion of proximal neutrosophic domination. The obtained theorems generalize and extend several classical and recent results from the literature. In addition, as a direct application of the self-mapping case, the existence of a common fixed point for a system of nonlinear Fredholm integral equations is demonstrated. Concrete examples are provided to illustrate the validity of the main results.

Keywords: Neutrosophic b-metric spaces, Common best proximity points, Nonlinear Fredholm integral equations

Mathematics Subject Classification (2020): 47H10, 54H25

1 Introduction and Preliminaries

Fixed point theory is a fundamental branch of nonlinear analysis, initiated by the Banach Contraction Principle, which guarantees that every contraction on a complete metric space has a unique fixed point. For a self-mapping $G : X \to X$, one seeks $x^* \in X$ such that $Gx^* = x^*$. For non-self operators $G : A \to B$, with $A \neq B$ nonempty subsets of a metric space, a fixed point cannot generally be expected. This motivated best proximity point theory, which seeks a point realizing

$$d(x, Gx) = d(A, B) = \inf\{d(a, b) : a \in A \text{ and } b \in B\},$$

yielding an optimal approximate solution when no fixed point exists. When $A = B$, this reduces to a fixed point. Numerous authors have extended these results to non-self proximal contractions and generalized contractive mappings [1, 8, 26, 40, 43, 50].

The notion was further generalized to common best proximity points for pairs and families of non-self mappings, offering a unified framework connected with optimization, variational inequalities, equilibrium problems, and nonlinear operator equations.

Key developments include Basha et al. [5], extended in [45]; Sadiq Basha [4] for proximally commuting non-self mappings; refinements in [33]; and many further results surveyed in [9, 11, 25, 27, 30–32, 37, 38, 51–53].

Attention has since shifted to generalized metric structures. In particular, b-metric spaces introduced by Czerwik [6] offer a relaxed

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triangle inequality while preserving key topological and analytical properties, enabling results not attainable in ordinary metric spaces. Common best proximity points in b-metric spaces were first established by Lo′lo′ et al. [29], providing a unified setting for non-self mappings in generalized metric environments.

The theory of fuzzy sets (FSs), introduced by Zadeh [54] in 1965, marked a fundamental breakthrough in the mathematical treatment of uncertainty and imprecision. Unlike classical set theory, FSs assign to each element a degree of membership, providing a flexible framework for modeling vague and incomplete information. Owing to their remarkable ability to represent uncertainty arising in real-world phenomena, FSs have become indispensable in numerous areas of mathematics, engineering, computer science, optimization, decision-making, and artificial intelligence. To overcome some intrinsic limitations of classical FSs, Atanassov [2] introduced the concept of intuitionistic fuzzy sets (IFSs) in 1986 by incorporating both membership and non-membership degrees, thereby allowing a more comprehensive representation of uncertainty. Since then, intuitionistic fuzzy theory has attracted considerable attention and has been successfully applied to diverse mathematical disciplines. For instance, Gulzar et al. [13, 14] investigated intuitionistic fuzzy algebraic structures, while Kanwal et al. [20] developed intuitionistic fuzzy mappings and established several fixed point results for specific classes of mappings.

The growing importance of distance-based methods in analysis, approximation theory, and optimization naturally motivated the extension of metric structures to fuzzy environments. Kramosil and Michalek [24] introduced the notion of fuzzy metric spaces (FMSs) in 1975, laying the foundation for the development of fuzzy metric geometry. Subsequently, George and Veeramani [10] proposed a refined formulation based on continuous t-norms, which generated a Hausdorff topology and became the standard framework for FMSs. Since then, various generalizations have been developed to broaden the applicability of fuzzy metrics. Nadaban [34], inspired by the concept of a b-metric space, proposed fuzzy b-metric spaces (FbMSs) in 2016 as a natural generalization of FMSs. Further contributions include the graphical FMSs introduced by Saleem et al. [42], fixed point results in generalized FMSs established by Ishtiaq et al. [15], fixed point theorems in fuzzy strong b-metric spaces obtained by Kanwal et al. [20], and investigations of fuzzy cone metric spaces by Rehman and Aydi [39].

Another significant advancement was achieved by Park [35] in 2004 through the introduction of intuitionistic fuzzy metric spaces (IFMSs), which combine IFSs with continuous t-norms and continuous t-conorms. This framework extends FMSs by simultaneously considering membership and non-membership information. Subsequently, numerous fixed point results were established in IFMSs, including those of Turkoglu et al. [49], who generalized Jungck's common fixed point theorems. Meanwhile, Jungck's pioneering contributions [17, 18] on coincidence points, common fixed points, and compatible mappings, together with the concept of weak compatibility introduced by Jungck and Rhoades [19], significantly enriched modern fixed point theory. Other notable developments include Grabiec's extension of Banach's contraction principle to FMSs [12], Schweizer and Sklar's statistical metric spaces [44], and the intuitionistic fuzzy b-metric fixed point results obtained by Kanwal *et al.* [21].

A further milestone in uncertainty modeling was achieved by Smarandache [47], who introduced neutrosophic sets (NSs) in 2005 as a powerful generalization of IFS. In contrast to fuzzy and intuitionistic fuzzy theories, neutrosophic sets characterize each element through three independent functions representing the degrees of truth, indeterminacy, and falsity, thereby offering greater flexibility in modeling incomplete, inconsistent, and indeterminate information.

Motivated by this framework, Kirici and imek [23] introduced neutrosophic metric spaces (NMSs) in 2019 and investigated their fundamental topological properties. In this setting, three functions corresponding to truth, indeterminacy, and falsity satisfy suitable symmetry and continuity conditions, leading to an effective generalization of fuzzy metric spaces. Subsequently, imek and Kirici [46] established several Banach-type contraction principles in NMSs, while Ishtiaq et al. [16] introduced orthogonal neutrosophic metric spaces and proved new classes of fixed point theorems.

Saeed et al. [41] introduced the notion of a neutrosophic b-metric space (NbMS) as a natural extension of NMS. Within this generalized framework, they established several coincidence point and common fixed point theorems, where the symmetry property of NbMSs plays a fundamental role in the development of the underlying theory. More recently, Pragadeeswarar and Unni [36] investigated the existence of common best proximity points for non-self mappings in complete NMS, thereby extending the scope of fixed point techniques to approximation problems in the neutrosophic setting.

To the best of our knowledge, the existence of common best proximity points for non-self mappings in complete NbMSs has not yet been investigated in the literature. In this paper, we fill this gap by establishing existence results for common best proximity points of non-self mappings under an appropriate neutrosophic framework. More precisely, we prove two new theorems under distinct contractive conditions, and as an application, we derive the existence of common fixed points for a system of nonlinear Fredholm integral equations. The obtained

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results are supported by illustrative examples.

The concept of b-metric space was introduced by I.A. Bakhtin [3] and extensively used by S. Czerwic [6, 7].

Definition 1. *Let X be a nonempty set and $s \geq 1$ be a given real number. A function $d : X \times X \to [0,\infty)$ is a b-metric on X if, for all $\mu, \nu, \xi \in X$, the following conditions hold:*

(b_1) $d(\mu,\nu) = 0$ *if and only if* $\mu = \nu$,

(b_2) $d(\mu,\nu) = d(\nu,\mu)$,

(b_3) $d(\mu,\xi) \leq s[d(\mu,\nu) + d(\nu,\xi)]$.

In this case, (X,d,s) is called a b-metric space with coefficient s. Here, we observe that every metric space is a b-metric space, with $s = 1$.

Definition 2. *[54] Let X be a nonempty set. A mapping $M : X \to [0,1]$ is called a fuzzy set (FS) on X. If M is a fuzzy set and $x \in X$, then the value $M(x)$ is called the grade of membership of x in F.*

Definition 3. *[2] Let X be a nonempty set. An intuitionistic fuzzy set (IFS) A on X is defined as*

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in X\},$$

where $\mu_A : X \to [0,1]$ is the degree of membership and $\nu_A : X \to [0,1]$ is the degree of nonmembership of x in A, satisfying

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1 \quad \text{for all } x \in X.$$

Definition 4. *[10] An operation $\otimes : [0,1] \times [0,1] \to [0,1]$ is called continuous t-norm (CTN) if it satisfies the following conditions:*

(i) $m \otimes 1 = m$ *for all* $m \in [0,1]$,

(ii) $\otimes$ *is commutative and associative,*

(iii) $\otimes$ *is continuous,*

(iv) $m \leq n$ *and* $p \leq q$ *then* $m \otimes p \leq n \otimes q$ *for all* $m,n,p,q \in [0,1]$.

Definition 5. *[10] An operation $\oplus : [0,1] \times [0,1] \to [0,1]$ is called continuous t-conorm (CTCN) if it satisfies the following conditions:*

(i) $m \oplus 0 = m$ *for all* $m \in [0,1]$,

(ii) $\oplus$ *is commutative and associative,*

(iii) $\oplus$ *is continuous,*

(iv) $m \leq n$ *and* $p \leq q$ *then* $m \oplus p \leq n \oplus q$ *for all* $m,n,p,q \in [0,1]$.

Definition 6. *[34] Assume X is a non-empty set. Let $s \in \mathbb{R}$, $s \geq 1$ and $\otimes$ be CTN. A FS M on $X \times X \times [0,\infty)$ is known as fuzzy b-metric if, for all $\mu, \nu, \xi \in X$ and for all positive scalars λ, κ the below conditions are verified:*

$(FBM1)$ $M(\mu,\nu,0) = 0$,

$(FBM2)$ $M(\mu,\nu,\lambda) = 1 \Leftrightarrow \mu = \nu$,

$(FBM3)$ $M(\mu,\nu,\lambda) = M(\nu,\mu,\lambda)$,

$(FBM4)$ $M(\mu,\nu,s(\lambda+\kappa)) \geq M(\mu,\xi,\lambda) \otimes M(\xi,\nu,\kappa)$,

$(FBM5)$ $\lim_{\lambda\to\infty} M(\mu,\nu,\lambda) = 1$ *and* $M(\mu,\nu,\cdot) : [0,\infty) \to [0,1]$ *is left continuous.*

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Then $(X, M, \otimes, s)$ is called a fuzzy b-metric space.

Definition 7. *[21] A six-tuple $(X, M, N, \otimes, \oplus, S)$ is said to be IFBMS if X is a nonempty set, $s \geq 1$ is a given real number, $\otimes$ is a continuous t-norm (CTN), $\oplus$ denotes a continuous t-conorm (CTCN), and M, N are fuzzy sets defined on $X^2 \times [0, \infty)$ satisfying the following conditions for all $\mu, \nu, \xi \in X$ and for all positive scalars λ, κ:*

(*i*) $M(\mu, \nu, \lambda) + N(\mu, \nu, \lambda) \leq 1$,

(*ii*) $0 \leq M(\mu, \nu, \lambda) \leq 1$,

(*iii*) $M(\mu, \nu, \lambda) = 1 \Leftrightarrow \mu = \nu$,

(*iv*) $M(\mu, \nu, \lambda) = M(\nu, \mu, \lambda)$,

(*v*) $M(\mu, \nu, s(\lambda + \kappa)) \geq M(\mu, \xi, \lambda) \otimes M(\xi, \nu, \kappa)$,

(*vi*) *The function* $M(\mu, \nu, \cdot) : [0, \infty) \to [0, 1]$ *is left continuous,*

(*vii*) $\lim_{\lambda \to \infty} M(\mu, \nu, \lambda) = 1$,

(*viii*) $0 \leq N(\mu, \nu, \lambda) \leq 1$,

(*ix*) $N(\mu, \nu, \lambda) = 0 \Leftrightarrow \mu = \nu$,

(*x*) $N(\mu, \nu, \lambda) = N(\nu, \mu, \lambda)$,

(*xi*) $N(\mu, \nu, s(\lambda + \kappa)) \leq N(\mu, \xi, \lambda) \oplus N(\xi, \nu, \kappa)$,

(*xii*) *The function* $N(\mu, \nu, \cdot) : [0, \infty) \to [0, 1]$ *is right continuous,*

(*xiii*) $\lim_{\lambda \to \infty} N(\mu, \nu, \lambda) = 0$.

Definition 8. *[47] Let X be a non-empty fixed set and $x \in X$. A Neutrosophic Set (NS) N in X is categorized by three components*

(*i*) *truth-membership function $M(x)$,*

(*ii*) *indeterminacy-membership function $N(x)$,*

(*iii*) *falsity-membership function $P(x)$.*

The functions $M(x), N(x)$ and $P(x)$ are real standard or non-standard subsets of $]0^-, 1^+[$, that is $M(x) : X \to]0^-, 1^+[, N(x) : X \to]0^-, 1^+[$ and $P(x) : X \to]0^-, 1^+[$ such that

$$0^- \leq \sup M(x) + \sup N(x) + \sup P(x) \leq 3^+.$$

Definition 9. *[41] A 7-tuple $(X, M, N, P, \otimes, \oplus, s)$ is said to be an NbMS if $X \neq \phi$, $s \geq 1$ is a given real number, $\otimes$ and $\oplus$ are a CTN and a CTCN, respectively, and M, N, P are neutrosophic sets on $X^2 \times [0, +\infty)$ satisfying the following conditions for all $\mu, \nu, \xi \in X$ and for all positive scalars λ, κ:*

(*i*) $M(\mu, \nu, \lambda) + N(\mu, \nu, \lambda) + P(\mu, \nu, \lambda) \leq 3$,

(*ii*) $0 \leq M(\mu, \nu, \lambda) \leq 1$,

(*iii*) $M(\mu, \nu, \lambda) = 1 \Leftrightarrow \mu = \nu$,

(*iv*) $M(\mu, \nu, \lambda) = M(\nu, \mu, \lambda)$,

(*v*) $M(\mu, \xi, s(\lambda + \kappa)) \geq M(\mu, \nu, \lambda) \otimes M(\nu, \xi, \kappa)$,

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(*vi*) *The function* $M(\mu,\nu,\cdot):[0,\infty)\to[0,1]$ *is left continuous,*

(*vii*) $\lim_{\lambda\to\infty} M(\mu,\nu,\lambda)=1$,

(*viii*) $0\le N(\mu,\nu,\lambda)\le 1$,

(*ix*) $N(\mu,\nu,\lambda)=0\Leftrightarrow\mu=\nu$,

(*x*) $N(\mu,\nu,\lambda)=N(\nu,\mu,\lambda)$,

(*xi*) $N(\mu,\xi,s(\lambda+\kappa))\le N(\mu,\nu,\lambda)\oplus N(\nu,\xi,\kappa)$,

(*xii*) *The function* $N(\mu,\nu,\cdot):[0,\infty)\to[0,1]$ *is right continuous,*

(*xiii*) $\lim_{\lambda\to\infty} N(\mu,\nu,\lambda)=0$,

(*xiv*) $0\le P(\mu,\nu,\lambda)\le 1$,

(*xv*) $P(\mu,\nu,\lambda)=0\Leftrightarrow\mu=\nu$,

(*xvi*) $P(\mu,\nu,\lambda)=P(\nu,\mu,\lambda)$,

(*xvii*) $P(\mu,\xi,s(\lambda+\kappa))\le P(\mu,\nu,\lambda)\oplus P(\nu,\xi,\kappa)$,

(*xviii*) *The function* $P(\mu,\nu,\cdot):[0,\infty)\to[0,1]$ *is right continuous,*

(*xix*) $\lim_{\lambda\to\infty} P(\mu,\nu,\lambda)=0$.

Then $(X,M,N,P,\otimes,\oplus,s)$ *is called an NbMS.*

Remark 1. If we take $s=1$ in the above definition, then it becomes an NMS. Hence, every NMS is an NbMS, but the converse is not generally true.

Example 1. [41] Suppose (X,ρ,s) be a b-metric space and $m\otimes n=\min\{m,n\}$, $m\oplus n=\max\{m,n\}$, for all $m,n\in[0,1]$, and let M,N and P be NSs on $X^2\times[0,+\infty)$, defined as follows:

$$M(\mu,\nu,\lambda)=\begin{cases}\dfrac{\lambda}{\lambda+\rho(\mu,\nu)}, & \text{if } \lambda>0,\\ 0, & \text{if } \lambda=0,\end{cases}$$

$$N(\mu,\nu,\lambda)=\begin{cases}\dfrac{\rho(\mu,\nu)}{\lambda+\rho(\mu,\nu)}, & \text{if } \lambda>0,\\ 1, & \text{if } \lambda=0,\end{cases}$$

$$P(\mu,\nu,\lambda)=\begin{cases}\dfrac{\rho(\mu,\nu)}{\lambda}, & \text{if } \lambda>0,\\ 1, & \text{if } \lambda=0.\end{cases}$$

Then $(X,M,N,P,\otimes,\oplus,s)$ is an NbMS.

Definition 10. *[48] Let* $(X,M,N,P,\otimes,\oplus,s)$ *be an NbMS. The following set*

$$D(\alpha,r,\lambda)=\{\beta\in X: M(\alpha,\beta,\lambda)>1-r, N(\alpha,\beta,\lambda)<r, P(\alpha,\beta,\lambda)<r\}.$$

is said to be the open ball centered at $\alpha\in X$*, with radius* $0<r<1$*, and parameter* $\lambda>0$.

The topology on X induced by the neutrosophic metric is given by:

$$\tau_{M,N,P}=\{A\subseteq X:\forall\alpha\in A,\ \exists\tau>0,\ r\in(0,1)\ \textit{such that}\ D(\alpha,r,\lambda)\subset A\}.$$

This $\tau_{(M,N,P)}$ *is then designated as the topology on X arising from the neutrosophic metric.*

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Definition 11. *[48] Let $(X,M,N,P,\otimes,\oplus,s)$ be an NbMS.*

- *A sequence $\{\mu_n\}$ in X is deemed to converge to a point $\mu \in X$ if, for every $\lambda > 0$, the following limits hold:*

$$\lim_{n\to\infty} M(\mu_n,\mu,\lambda) = 1; \quad \lim_{n\to\infty} N(\mu_n,\mu,\lambda) = 0; \quad \lim_{n\to\infty} P(\mu_n,\mu,\lambda) = 0.$$

- *A sequence $\{\mu_n\}$ in $(X,M,N,P,\otimes,\oplus,s)$ is declared to be a Cauchy sequence if for every $\varepsilon > 0$ and $\lambda > 0$, there exists a natural number $n_0 \in \mathbb{N}$ such that for all $n,m \geq n_0$, we have:*

$$M(\mu_n,\mu_m,\lambda) > 1-\varepsilon; \quad N(\mu_n,\mu_m,\lambda) < \varepsilon; \quad P(\mu_n,\mu_m,\lambda) < \varepsilon.$$

- *$(X,M,N,P,\otimes,\oplus,s)$ is said to be complete if every Cauchy sequence within X converges to a point in X.*

- *$(X,M,N,P,\otimes,\oplus,s)$ is called compact neutrosophic metric space if every sequence $\{\mu_n\}$ in X admits a convergent subsequence $\{\mu_{n_k}\}$.*

In this paper, we unify and extend the results established in [29], [36], and [41]. In [29], Lo′lo′ et al. proved common best proximity point theorems for four mappings in b-metric spaces, while Saeed et al. [41] introduced the notion of neutrosophic b-metric spaces and subsequently established common fixed point results in this more general framework. More recently, Pragadeeswarar et al. [36] employed the concepts of approximate compactness, neutrosophic proximal commutativity, and neutrosophic proximal interchangeability to prove common best proximity point theorems in neutrosophic metric spaces (without the b-metric structure). In the present work, we extend these notions to neutrosophic b-metric spaces and obtain common best proximity point results for two non-self mappings in this setting. Finally, we provide an application to a system of nonlinear Fredholm integral equations.

2 Main Results

Here we start with some notions:
Given non-empty subsets A and B of an NbMS $(X,M,N,P,\otimes,\oplus,s)$, the subsequent definitions are employed:

$$M(A,B,\lambda) := \sup\{M(\eta,\mu,\lambda) : \eta \in A \text{ and } \mu \in B\}$$

$$N(A,B,\lambda) := \inf\{N(\eta,\mu,\lambda) : \eta \in A \text{ and } \mu \in B\}$$

$$P(A,B,\lambda) := \inf\{P(\eta,\mu,\lambda) : \eta \in A \text{ and } \mu \in B\}$$

$$A_0(\lambda) = \left\{ \eta \in A \;\middle|\; \begin{array}{l} M(\eta,\mu,\lambda) = M(A,B,\lambda),\ N(\eta,\mu,\lambda) = N(A,B,\lambda), \text{ and} \\ P(\eta,\mu,\lambda) = P(A,B,\lambda) \text{ for some } \mu \in B \end{array} \right\}$$

$$B_0(\lambda) = \left\{ \mu \in B \;\middle|\; \begin{array}{l} M(\eta,\mu,\lambda) = M(A,B,\lambda),\ N(\eta,\nu,\lambda) = N(A,B,\lambda), \text{ and} \\ P(\eta,\mu,\lambda) = P(A,B,\lambda) \text{ for some } \eta \in A \end{array} \right\}$$

for all $\lambda > 0$. The concept of a best proximity point for non-self mappings is articulated as follows.

Definition 12. *Let (A,B) be a pair of non-empty subsets of an NbMS $(X,M,N,P,\otimes,\oplus,s)$. An element $\mu \in A$ is defined as a best proximity point for the non-self map $G : A \to B$ if it fulfills the conditions:*

$$M(\mu,G\mu,\lambda) = M(A,B,\lambda), \quad N(\mu,G\mu,\lambda) = N(A,B,\lambda) \quad \textit{and} \quad P(\mu,G\mu,\lambda) = P(A,B,\lambda) \quad \textit{for all } \lambda > 0.$$

If in the above definition $A = B$ then:

$$M(\mu,G\mu,\lambda) = M(A,B,\lambda) = 1, \quad N(\mu,G\mu,\lambda) = N(A,B,\lambda) = 0 \quad \text{and} \quad P(\mu,G\mu,\lambda) = P(A,B,\lambda) = 0 \quad \text{for all } \lambda > 0.$$

Then $\mu = G\mu$ or μ is a fixed point of the self-mapping G.

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Definition 13. *Let (A,B) be a pair of non-empty subsets of an NbMS $(X,M,N,P,\otimes,\oplus,s)$. An element $\mu \in A$ is defined as a common best proximity point for two non-self mappings $G_1 : A \to B$ and $G_2 : A \to B$ if it fulfills the conditions:*

$$\begin{aligned} &M(\mu, G_1\mu, \lambda) = M(\mu, G_2\mu, \lambda) = M(A,B,\lambda), \\ &N(\mu, G_1\mu, \lambda) = N(\mu, G_2\mu, \lambda) = N(A,B,\lambda), \quad \textit{and} \\ &P(\mu, G_1\mu, \lambda) = P(\mu, G_2\mu, \lambda) = P(A,B,\lambda) \quad \textit{for all} \quad \lambda > 0. \end{aligned}$$

Remark 2. It is noteworthy that in Definition 13, if the sets A and B are identical (i.e., $A = B$), the stipulated conditions simplify considerably to:

$$\begin{aligned} &M(\mu, G_1\mu, \lambda) = M(\mu, G_2\mu, \lambda) = M(A,B,\lambda) = 1 \\ &N(\mu, G_1\mu, \lambda) = N(\mu, G_2\mu, \lambda) = N(A,B,\lambda) = 0, \quad \text{and} \\ &P(\mu, G_1\mu, \lambda) = P(\mu, G_2\mu, \lambda) = P(A,B,\lambda) = 0 \quad \text{for all} \quad \lambda > 0. \end{aligned}$$

These simplifications directly imply that $\mu = G_1\mu$ and $\mu = G_2\mu$. Consequently, μ serves as a common fixed point for the self-maps G_1 and G_2.

Given non-empty subset A and B of an NbMS $(X,M,N,P,\otimes,\oplus,s)$ the subsequent definitions defined in [36] are employed.

Definition 14. *A set A is said to be approximatively compact with respect to B if, whenever a sequence $\{\mu_n\}$ in A satisfies*

$$M(\nu,\mu_n,\lambda) \to M(\nu,A,\lambda), \quad N(\nu,\mu_n,\lambda) \to N(\nu,A,\lambda), \quad P(\nu,\mu_n,\lambda) \to P(\nu,A,\lambda)$$

for some $\nu \in B$ and for all $\lambda > 0$, it follows that $\{\mu_n\}$ has a convergent subsequence.

Definition 15. *The mappings $G_1 : A \to B$ and $G_2 : A \to B$ are said to neutrosophic proximally commute if:*

$$\left.\begin{aligned} &M(\eta, G_2\mu, \lambda) = M(\nu, G_1\mu, \lambda) = M(A,B,\lambda) \\ &N(\eta, G_2\mu, \lambda) = N(\nu, G_1\mu, \lambda) = N(A,B,\lambda) \\ &P(\eta, G_2\mu, \lambda) = P(\nu, G_1\mu, \lambda) = P(A,B,\lambda) \end{aligned}\right\} \Rightarrow G_2\nu = G_1\eta,$$

for all η, ν, μ in A and $\lambda > 0$.

Definition 16. *A mapping $G_1 : A \to B$ is said to proximally neutrosophic dominate a mapping $G_2 : A \to B$ if there exists a non-negative real number $k \in [0,1)$ such that, for all $\eta_1, \eta_2, \mu_1, \mu_2, \nu_1, \nu_2$ in A, the following implications hold:*

$$\left.\begin{aligned} &M(\eta_1, G_2\mu_1, \lambda) = M(\eta_2, G_2\mu_2, \lambda) = M(A,B,\lambda) \\ &M(\nu_1, G_1\mu_1, \lambda) = M(\nu_2, G_1\mu_2, \lambda) = M(A,B,\lambda) \end{aligned}\right\} \Rightarrow M(\eta_1,\eta_2,k\lambda) \geq M(\nu_1,\nu_2,\lambda)$$

$$\left.\begin{aligned} &N(\eta_1, G_2\mu_1, \lambda) = N(\eta_2, G_2\mu_2, \lambda) = N(A,B,\lambda) \\ &N(\nu_1, G_1\mu_1, \lambda) = N(\nu_2, G_1\mu_2, \lambda) = N(A,B,\lambda) \end{aligned}\right\} \Rightarrow N(\eta_1,\eta_2,k\lambda) \leq N(\nu_1,\nu_2,\lambda)$$

$$\left.\begin{aligned} &P(\eta_1, G_2\mu_1, \lambda) = P(\eta_2, G_2\mu_2, \lambda) = P(A,B,\lambda) \\ &P(\nu_1, G_1\zeta_1, \lambda) = P(\nu_2, G_1\zeta_2, \lambda) = P(A,B,\lambda) \end{aligned}\right\} \Rightarrow P(\eta_1,\eta_2,k\lambda) \leq P(\nu_1,\nu_2,\lambda)$$

for all $\lambda > 0$.

Definition 17. *The mappings $G_2 : A \to B$ and $G_1 : A \to B$ are said to be neutrosophic proximally interchanged if for all $\mu, \nu \in A$, $\eta \in B$, and $\lambda > 0$,*

$$\left.\begin{aligned} &M(\eta, \mu, \lambda) = M(\eta, \nu, \lambda) = M(A,B,\lambda) \\ &N(\eta, \mu, \lambda) = N(\eta, \nu, \lambda) = N(A,B,\lambda) \\ &P(\eta, \mu, \lambda) = P(\eta, \nu, \lambda) = P(A,B,\lambda) \\ &G_2\mu = G_1\nu \end{aligned}\right\} \Rightarrow G_2\nu = G_1\mu.$$

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Theorem 1. *Let $(X, M_\phi, N_\phi, P_\phi, \otimes, \oplus, s)$ be a complete NbMS with coefficient $s \geq 1$. Let $A, B \subseteq X$ be non-empty closed subsets such that $A_0(\lambda) \neq \emptyset$ and A is approximately compact with respect to B. Suppose $G_1, G_2 : A \to B$ satisfy:*

1) *There exists $k \in [0,1)$ such that for all $\eta_1, \eta_2 \in A$:*

$$M(G_2\eta_1, G_2\eta_2, k\lambda) \geq M(G_1\eta_1, G_1\eta_2, \lambda),$$
$$N(G_2\eta_1, G_2\eta_2, k\lambda) \leq N(G_1\eta_1, G_1\eta_2, \lambda),$$
$$P(G_2\eta_1, G_2\eta_2, k\lambda) \leq P(G_1\eta_1, G_1\eta_2, \lambda),$$

   for every $\lambda > 0$.

2) *G_1 and G_2 are continuous.*

3) *G_1, G_2 are neutrosophic proximally commuting.*

4) *G_1, G_2 are neutrosophic proximally interchangeable.*

5) *$G_2(A_0(\lambda)) \subseteq B_0(\lambda)$ and $G_2(A_0(\lambda)) \subseteq G_1(A_0(\lambda))$.*

Then there exists $\eta \in A$ such that:

$$M(\eta, G_1\eta, \lambda) = M(\eta, G_2\eta, \lambda) = M(A, B, \lambda),$$

and similarly for N and P. That is, η is a common best proximity point.

Proof. Step 1: Construction of sequences
Choose $\eta_0 \in A_0(\lambda)$. Since $G_2(A_0(\lambda)) \subseteq G_1(A_0(\lambda))$, there exists $\eta_1 \in A_0(\lambda)$ such that:

$$G_2\eta_0 = G_1\eta_1.$$

By induction, we obtain $\{\eta_n\} \subseteq A_0(\lambda)$ with:

$$G_2\eta_n = G_1\eta_{n+1}, \quad \forall n \in \mathbb{N}.$$

Step 2: Cauchy sequence in B
From the contraction condition (1), we have:

$$M(G_2\eta_n, G_2\eta_{n+1}, k\lambda) \geq M(G_1\eta_n, G_1\eta_{n+1}, \lambda) = M(G_2\eta_{n-1}, G_2\eta_n, \lambda),$$

and similarly for N, P, for all $n \in \mathbb{N}$, $\lambda > 0$ and $k \in (0,1)$.

By Mathematical induction, we obtain:

$$M(G_2\eta_{n+1}, G_2\eta_n, \lambda) \geq M\left(G_2\eta_1, G_2\eta_0, \frac{\lambda}{k^n}\right),$$

therefore

$$1 \geq M(G_2\eta_{n+1}, G_2\eta_n, \lambda) \geq M\left(G_2\eta_1, G_2\eta_0, \frac{\lambda}{k^n}\right) \to 1, \quad \text{when} \quad n \to \infty$$

and similarly for N, P.
That is

$$\lim_{n\to\infty} M(G_2\eta_{n+1}, G_2\eta_n, \lambda) = 1, \tag{1}$$

and similarly

$$\lim_{n\to\infty} N(G_2\eta_{n+1}, G_2\eta_n, \lambda) = \lim_{n\to\infty} P(G_2\eta_{n+1}, G_2\eta_n, \lambda) = 0.$$

To show that $\{G_2\eta_n\}$ is a Cauchy sequence, assume it is not, then there exists $0 < \varepsilon < 1$ and two sequences of positive integers $\{s(\tau)\}$ and $\{t(\tau)\}$ with $s(\tau) > t(\tau) \geq \tau$ such that

$$M(G_2\eta_{s(\tau)}, G_2\eta_{t(\tau)}, \lambda) \leq 1 - \varepsilon$$

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and

$$M(G_2\eta_{s(\tau)-1}, G_2\eta_{t(\tau)-1}, \lambda) > 1-\varepsilon, \qquad M(G_2\eta_{s(\tau)-1}, G_2\eta_{t(\tau)}, \lambda) > 1-\varepsilon.$$

Now, using the b-metric property:

$$\begin{aligned} 1-\varepsilon \geq M(G_2\eta_{s(\tau)}, G_2\eta_{t(\tau)}, \lambda) &\geq M(G_2\eta_{s(\tau)}, G_2\eta_{s(\tau)-1}, \frac{\lambda}{2s}) \otimes M(G_2\eta_{s(\tau)-1}, G_2\eta_{t(\tau)}, \frac{\lambda}{2s}) \\ &> M(G_2\eta_{s(\tau)}, G_2\eta_{s(\tau)-1}, \frac{\lambda}{2s}) \otimes 1-\varepsilon. \end{aligned}$$

As $\tau \to \infty$ and by 1 we have:

$$1-\varepsilon \geq M(G_2\eta_{s(\tau)}, G_2\eta_{t(\tau)}, \lambda) > 1-\varepsilon,$$

which, it is a contradiction. Thus $\{G_2\eta_n\}$ is Cauchy in B. Since X is complete and B is closed it converges to some $\mu \in B$.

Step 3: Convergence of $G_1\eta_n$ and related sequences

Since $G_1\eta_n = G_2\eta_{n-1}$, we have $G_1\eta_n \to \mu$.

Because $G_2(A_0(\lambda)) \subseteq B_0(\lambda)$, there exists $\mu_n \in A$ such that:

$$M(G_2\eta_n, \mu_n, \lambda) = M(A,B,\lambda), \text{ for all } n \in \mathbb{N} \quad \text{and similarly for } N,P. \tag{2}$$

Now, it follows from the construction of η_n that

$$M(G_1\eta_n, \mu_{n-1}, \lambda) = M(A,B,\lambda), \text{ for all } n \in \mathbb{N} \quad \text{and similarly for } N,P. \tag{3}$$

From the neutrosophic proximal commutativity of G_1 and G_2 and by using (2) and (3), we have:

$$G_1\mu_n = G_2\mu_{n-1}, \quad \text{for all } n \in \mathbb{N}.$$

Step 4: Approximative compactness and convergence in A

Using the b-metric inequality:

$$\begin{aligned} M(\mu,A,\lambda) &\geq M(\mu,\mu_n,\lambda) \\ &\geq M\left(\mu, G_2\eta_n, \frac{\lambda}{2s}\right) \otimes M\left(G_2\eta_n, \mu_n, \frac{\lambda}{2s}\right) \\ &= M\left(\mu, G_2\eta_n, \frac{\lambda}{2s}\right) \otimes M(A,B,\lambda) \\ &\geq M\left(\mu, G_2\eta_n, \frac{\lambda}{2s}\right) \otimes M(\mu,A,\lambda) \end{aligned}$$

so $M(\mu,\mu_n,\lambda) \to M(\mu,A,\lambda)$, and similarly for N,P. Since A is approximatively compact, $\{\mu_n\}$ has a convergent subsequence $\mu_{n_j} \to \mu' \in A$.

Step 5: Common best proximity point

By continuity of G_1, G_2:

$$G_1\mu' = \lim_{j\to\infty} G_1\mu_{n_j} = \lim_{j\to\infty} G_2\mu_{n_j-1} = G_2\mu'.$$

Finally, since $G_2(A_0(\tau)) \subseteq B_0(\tau)$, there exists $\eta \in A$ with:

$$M(\eta, G_1\mu', \lambda) = M(\eta, G_2\mu', \lambda) = M(A,B,\lambda) \quad \text{and similarly for } N,P, \tag{4}$$

and by proximal commutativity:

$$G_1\eta = G_2\eta.$$

Using the contraction again, we have

$$M(G_2\mu', G_2\eta, k\lambda) \geq M(G_1\mu', G_1\eta, \lambda) = M(G_2\mu', G_2\eta, \lambda), \quad \text{and similarly for } N,P,$$

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then we have

$$1 \geq M(G_2\mu', G_2\eta, \lambda) \geq M(G_2\mu', G_2\eta, \frac{\lambda}{k}) \geq M(G_2\mu', G_2\eta, \frac{\lambda}{k^2}) \geq \cdots \geq M(G_2\mu', G_2\eta, \frac{\lambda}{k^n}),$$

and similarly for N and P.

As $n \to \infty$, by the definition of a neutrosophic metric space, we conclude that $G_2\mu' = G_2\eta$ and therefore $G_1\mu' = G_1\eta$. By (4)

$$M(\eta, G_1\eta, \lambda) = M(\eta, G_2\eta, \lambda) = M(A, B, \lambda), \quad \text{and similarly for } N, P.$$

Hence η is a common best proximity point. □

Example 2. Let $X = \mathbb{R}^2$ and define a b-metric $\rho : X \times X \to [0, \infty)$ by

$$\rho((x_1, y_1), (x_2, y_2)) = |x_1 - x_2|^2 + |y_1 - y_2|^2,$$

with coefficient $s = 2$. Let $A = \{(-1, y) : y \in \mathbb{R}\}$ and $B = \{(1, y) : y \in \mathbb{R}\}$. Then clearly, $A_0 = A$ and $B_0 = B$. Define $M, N, P : X^2 \times (0, \infty) \to [0, 1]$ by

$$M(\mu, \nu, \lambda) = \frac{\lambda}{\lambda + \rho(\mu, \nu)}, \quad N(\mu, \nu, \lambda) = \frac{\rho(\mu, \nu)}{\lambda + \rho(\mu, \nu)}, \quad P(\mu, \nu, \lambda) = \frac{\rho(\mu, \nu)}{\lambda},$$

with $a \otimes b = \min\{a, b\}$ and $a \oplus b = \max\{a, b\}$. Then $(X, M, N, P, \otimes, \oplus, s = 2)$ is a complete NbMS (see Example 1).

Now, define $G_1, G_2 : A \to B$ by

$$G_1(-1, y) = \left(1, \frac{y}{4}\right), \qquad G_2(-1, y) = \left(1, \frac{y}{5}\right).$$

We verify the conditions of Theorem 1:

(1) For any $k \in [\frac{4}{5}, 1)$, and for all $\eta_1 = (-1, y_1), \eta_2 = (-1, y_2) \in A$, we have

$$M(G_2\eta_1, G_2\eta_2, k\lambda) = \frac{k\lambda}{k\lambda + \left|\frac{y_1}{5} - \frac{y_2}{5}\right|^2} \geq \frac{\lambda}{\lambda + \left|\frac{y_1}{4} - \frac{y_2}{4}\right|^2} = M(G_1\eta_1, G_1\eta_2, \lambda),$$

and similarly for N and P. Thus condition (1) holds.

(2) Both G_1 and G_2 are continuous on A (they are affine functions).

(3) Neutrosophic proximal commutativity: If

$$M(\eta, G_2\mu, \lambda) = M(\nu, G_1\mu, \lambda) = M(A, B, \lambda) = \frac{\lambda}{\lambda + 4},$$

because,we know $\rho(A, B) = |(-1) - (1)|^2 = 4$ and therefore,

$$M(A, B, \lambda) = \sup_{\eta \in A, \mu \in B} M(\eta, \mu, \lambda) = \frac{\lambda}{\lambda + \rho(A, B)} = \frac{\lambda}{\lambda + 4},$$

then the only possible values are

$$\mu = (-1, y), \quad \eta = \left(-1, \frac{y}{5}\right), \quad \nu = \left(-1, \frac{y}{4}\right).$$

Therefore,

$$G_1\eta = \left(1, \frac{y}{20}\right) = G_2\nu,$$

so G_1 and G_2 neutrosophically proximally commute.

(4) Neutrosophic proximal interchangeability: If

$$M(\eta, \mu, \lambda) = M(\eta, \nu, \lambda) = M(A, B, \lambda)$$

and $G_2\mu = G_1\nu$, then $\mu = \nu = (-1, 0)$, and $G_2\nu = G_1\mu$. Hence condition (4) holds.

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(5) Clearly,

$$G_2(A_0(\lambda)) = \left\{\left(1, \frac{y}{5}\right) : y \in \mathbb{R}\right\} \subseteq B_0(\lambda) = B,$$

and

$$G_2(A_0(\lambda)) \subseteq G_1(A_0(\lambda)) = \left\{\left(1, \frac{y}{4}\right) : y \in \mathbb{R}\right\},$$

since for every y, $\frac{y}{5} = \frac{1}{4}(4y/5) \in G_1(A_0(\lambda))$.

Thus all hypotheses of Theorem 1 are satisfied. The common best proximity point is $\eta = (-1,0) \in A$, since

$$G_1(-1,0) = (1,0) = G_2(-1,0),$$

and

$$M((-1,0),(1,0),\lambda) = \frac{\lambda}{\lambda+4} = M(A,B,\lambda),$$

with analogous equalities for N and P.

The following corollary is a consequence of the last theorem with $A = B = X$.

Corollary 1. Let $(X, M, N, P, \otimes, \oplus, s)$ be a complete NbMS with coefficient $s \geq 1$, and let $G_1, G_2 : X \to X$ be two self-mappings satisfying:

1. There exists $k \in [0,1)$ such that, for all $x, y \in X$,

$$M(G_2x, G_2y, k\lambda) \geq M(G_1x, G_1y, \lambda),$$
$$N(G_2x, G_2y, k\lambda) \leq N(G_1x, G_1y, \lambda),$$
$$P(G_2x, G_2y, k\lambda) \leq P(G_1x, G_1y, \lambda),$$

   for every $\lambda > 0$.

2. G_1 and G_2 are continuous.
3. G_1 and G_2 are commuting.
4. G_1 and G_2 are neutrosophically proximally interchangeable.
5. $G_2(X) \subseteq G_1(X)$.

Then there exists $x^* \in X$ such that

$$G_1x^* = G_2x^* = x^*.$$

That is, G_1 and G_2 possess a common fixed point.

The next theorem provides a common best proximity point result for non-self mappings satisfying a proximal neutrosophic domination condition, without requiring the explicit contraction inequality of Theorem 1.

Theorem 2. *Let $(X, M_\phi, N_\phi, P_\phi, \otimes, \oplus, s)$ be a complete NbMS with coefficient $s \geq 1$. Let $A, B \subseteq X$ be non-empty closed subsets such that $A_0(\lambda) \neq \emptyset$ and closed and A is approximately compact with respect to B. Let $G_1, G_2 : A \to B$ be non-self mappings satisfying:*

1) G_1 proximally neutrosophic dominates G_2;

2) G_1, G_2 are neutrosophic proximally commuting;

3) G_1 and G_2 are continuous;

4) $G_2(A_0(\lambda)) \subseteq B_0(\lambda)$ and $G_2(A_0(\lambda)) \subseteq G_1(A_0(\lambda))$.

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Then there exists $\eta \in A$ such that:

$$M(\eta, G_1\eta, \lambda) = M(\eta, G_2\eta, \lambda) = M(A, B, \lambda),$$

and similarly for N and P. That is, η is a common best proximity point for G_1 and G_2.

Proof. By Theorem 1, there is a sequence $\eta_n \in A_0(\lambda)$ such that $G_2\eta_n = G_1\eta_{n+1}$, for all $n \in \mathbb{N}$.
Because $G_2(A_0(\lambda)) \subseteq B_0(\lambda)$, there exists $\mu_n \in A_0(\lambda)$ such that:

$$M(G_2\eta_n, \mu_n, \lambda) = M(A, B, \lambda), \text{ for all } n \in \mathbb{N} \quad \text{and similarly for } N, P. \tag{5}$$

Now, it follows from the construction of η_n and μ_n we have

$$M(G_1\eta_{n+1}, \mu_n, \lambda) = M(G_2\eta_{n+1}, \mu_{n+1}, \lambda) = M(G_1\eta_n, \mu_{n-1}, \lambda) = M(A, B, \lambda), \text{ for all } n \in \mathbb{N}, \tag{6}$$

and similarly for N, P.
Since $G_1 : A \to B$ proximally neutrosophic dominates G_2, by using (5) and (6), we have:

$$M(\mu_{n+1}, \mu_n, k\lambda) \geq M(\mu_n, \mu_{n-1}, \lambda), \quad \text{for all } n \in \mathbb{N},$$

and similarly for N, P.
By mathematical induction, we have

$$M(\mu_{n+1}, \mu_n, \lambda) \geq M(\mu_0, \mu_1, \frac{\lambda}{k^n}), \quad \text{for all } n \in \mathbb{N},$$

therefore

$$1 \geq M(\mu_{n+1}, \mu_n, \lambda) \geq M(\mu_0, \mu_1, \frac{\lambda}{k^n}) \to 1, \quad \text{when} \quad n \to \infty$$

and similarly for N, P.
That is

$$\lim_{n\to\infty} M(\mu_{n+1}, \mu_n, \lambda) = 1, \tag{7}$$

and similarly

$$\lim_{n\to\infty} N(\mu_{n+1}, \mu_n, \lambda) = \lim_{n\to\infty} P(\mu_{n+1}, \mu_n, \lambda) = 0$$

To show that μ_n is a Cauchy sequence, assume it is not, then there exists $0 < \varepsilon < 1$ and two sequences of positive integers $\{s(\tau)\}$ and $\{t(\tau)\}$ with $s(\tau) > t(\tau) \geq \tau$ such that

$$M(\mu_{s(\tau)}, \mu_{t(\tau)}, \lambda) \leq 1 - \varepsilon$$

and

$$M(\mu_{s(\tau)-1}, \mu_{t(\tau)-1}, \lambda) > 1 - \varepsilon, \qquad M(\mu_{s(\tau)-1}, \mu_{t(\tau)}, \lambda) > 1 - \varepsilon.$$

Now, using the b-metric property

$$1 - \varepsilon \geq M(\mu_{s(\tau)}, \mu_{t(\tau)}, \lambda) \geq M(\mu_{s(\tau)}, \mu_{s(\tau)-1}, \frac{\lambda}{2s}) \otimes M(\mu_{s(\tau)-1}, \mu_{t(\tau)}, \frac{\lambda}{2s}) > M(\mu_{s(\tau)}, \mu_{s(\tau)-1}, \frac{\lambda}{2s}) \otimes (1 - \varepsilon)$$

As $\tau \to \infty$ and by (7), each term tends to 1, we have:

$$1 - \varepsilon \geq M(\mu_{s(\tau)}, \mu_{t(\tau)}, \lambda) > 1 - \varepsilon,$$

which, it is a contradiction. Thus, $\{\mu_n\}$ is a Cauchy sequence in $A_0(\lambda)$ and therefore converges to some element $\mu^* \in A_0(\lambda)$. Since the mappings G_2 and G_1 are neutrosophically proximally commuting, it follows from (6) that

$$G_1\mu_n = G_2\mu_{n-1}, \quad \text{for every } n \in \mathbb{N}.$$

Hence, by the continuity of G_2 and G_1, we obtain that $G_2\mu^*$ and $G_1\mu^*$ coincide, and moreover,

$$G_2\mu^* = G_1\mu^* \in G_2(A_0(\lambda)).$$

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Since $G_2(A_0(\lambda)) \subseteq B_0(\lambda)$, there exists an element $\eta \in A_0(\lambda)$ such that

$$M(\eta, G_1\mu^*, \lambda) = M(\eta, G_2\mu^*, \lambda) = M(A, B, \lambda) \quad \text{and similarly for } N, P, \tag{8}$$

and by proximal commutativity:

$$G_1\eta = G_2\eta.$$

Again, since $G_2(A_0(\lambda)) \subseteq B_0(\lambda)$,there exists $\mu \in A_0(\lambda)$ such that

$$M(\mu, G_1\eta) = M(\mu, G_2\eta) = M(A, B, \lambda) \quad \text{and similarly for } N, P. \tag{9}$$

Since $G_1 : A \to B$ proximally neutrosophic dominates G_2, by using (8) and (9), we have:

$$M(\mu, \eta, k\lambda) \geq (\mu, \eta, \lambda),$$

and similarly for N, P.
Then, we have

$$1 \geq M(\mu, \eta, \lambda) \geq M(\mu, \eta, \frac{\lambda}{k}) \geq M(\mu, \eta, \frac{\lambda}{k^2}) \geq \cdots \geq M(\mu, \eta, \frac{\lambda}{k^n}).$$

As $n \to \infty$, by the definition of a neutrosophic metric space, we conclude that $\mu = \eta$. Hence, by (9), η is a common best proximity point. □

The following corollary is a consequence of the last theorem with $A = B = X$.

Corollary 2. Let $(X, M, N, P, \otimes, \oplus, s)$ be a complete NbMS with coefficient $s \geq 1$, and let $G_1, G_2 : X \to X$ be two self-mappings satisfying:

1. G_1 proximally neutrosophically dominates G_2.
2. G_1 and G_2 are neutrosophically proximally commuting.
3. G_1 and G_2 are continuous.
4. $G_2(X) \subseteq G_1(X)$.

Then there exists $x^* \in X$ such that

$$G_1x^* = G_2x^* = x^*.$$

Hence, the mappings G_1 and G_2 admit a common fixed point.

3 Application: Existence of Common Fixed Points for a System of Nonlinear Fredholm Integral Equations

In this section, we present a rigorous application of Corollary 1 (the self-mapping case of Theorem 1) to establish the existence of a common fixed point for a system of nonlinear Fredholm integral equations. This corresponds to the case where the sets A and B coincide, i.e., $A = B = X$, so the best proximity point reduces to a common fixed point.

Problem Setting.
Consider the following system of nonlinear Fredholm integral equations:

$$\begin{aligned} u(t) - \int_a^b K_1(t, s, u(s))\, ds &= f_1(t), \\ u(t) - \int_a^b K_2(t, s, u(s))\, ds &= f_2(t), \end{aligned} \tag{10}$$

where $t \in [a, b]$, $f_1, f_2 \in C[a, b]$ are known continuous functions, and $K_1, K_2 : [a, b] \times [a, b] \times \mathbb{R} \to \mathbb{R}$ are continuous kernels. We seek an exact common solution $u^* \in C[a, b]$ to both equations simultaneously.

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Theorem 3. ***(Existence of Common Fixed Points for Integral Equations)***
Let $X = C([a,b],\mathbb{R})$ be the space of real-valued continuous functions on $[a,b]$, equipped with the supremum norm $\|u\|_\infty = \max_{t\in[a,b]} |u(t)|$. Define two mappings $G_1, G_2 : X \to X$ by

$$(G_i u)(t) = \int_a^b K_i(t,s,u(s))\,ds + f_i(t), \quad t \in [a,b], \quad i = 1,2.$$

Assume the following hypotheses hold:

(H1) There exist constants $L_1, L_2 \geq 0$ such that for all $t,s \in [a,b]$ and $x,y \in \mathbb{R}$,

$$|K_i(t,s,x) - K_i(t,s,y)| \leq L_i\,|x-y|, \quad i = 1,2.$$

(H2) There exists a point $t_0 \in [a,b]$ and a subinterval $J \subset [a,b]$ with positive length $\delta_1 > 0$ such that for all $s \in J$ and $x,y \in \mathbb{R}$,

$$|K_1(t,s,x) - K_1(t,s,y)| \geq m_1\,|x-y|,$$

where t_0 is a point at which $|u(t)-v(t)|$ attains its maximum, i.e.,

$$|u(t_0) - v(t_0)| = \|u-v\|_\infty = \rho(u,v).$$

(H3) The constants satisfy $L_2 < m_1$.

(H4) The mappings G_1 and G_2 are continuous and commuting.

(H5) The inclusion $G_2(X) \subseteq G_1(X)$ holds.

Then there exists $u^* \in X$ such that

$$G_1 u^* = G_2 u^* = u^*.$$

That is, u^* is a common fixed point of G_1 and G_2, and hence an exact common solution to the system 10.

Proof. We verify that all hypotheses of Corollary 1 are satisfied.
Step 1: Construction of the neutrosophic b-metric space.

Define a b-metric ρ on X by $\rho(u,v) = \|u-v\|_\infty$ with coefficient $s = 1$. Taking the metric ρ into account, let us assume that $(X, M, N, P, \otimes, \oplus, s = 1)$ is the complete NbMS defined in Example 1.
Step 2: Verification of the contraction condition.

From the upper Lipschitz condition in (H1), for $i = 1,2$ we have:

$$\rho(G_i u, G_i v) \leq L_i (b-a)\,\rho(u,v). \tag{11}$$

Now, we derive a lower bound for $\rho(G_1 u, G_1 v)$. By definition,

$$\rho(G_1 u, G_1 v) = \max_{t\in[a,b]} \left| \int_a^b \big(K_1(t,s,u(s)) - K_1(t,s,v(s))\big)\,ds \right|.$$

Let $t_0 \in [a,b]$ be a point where the maximum of $|u(t)-v(t)|$ is attained, i.e.,

$$|u(t_0) - v(t_0)| = \rho(u,v). \tag{12}$$

If $\rho(u,v) = 0$, then $u = v$ and the inequalities are trivial. So assume $\rho(u,v) > 0$.

From the local lower Lipschitz condition (H2), for all $s \in J$,

$$|K_1(t_0,s,u(s)) - K_1(t_0,s,v(s))| \geq m_1 |u(s) - v(s)|. \tag{13}$$

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Since u and v are continuous on the compact interval $[a,b]$, the function $s \mapsto |u(s)-v(s)|$ is continuous. At $s=t_0$, by (12), it attains the positive value $\rho(u,v)>0$. Therefore, by continuity, there exists a subinterval $J \subset [a,b]$ with positive length $\delta_1 > 0$ such that for all $s \in J$,

$$|u(s)-v(s)| \geq \frac{1}{2}\rho(u,v). \tag{14}$$

Now, using (13) and (14), for all $s \in J$:

$$|K_1(t_0,s,u(s)) - K_1(t_0,s,v(s))| \geq m_1|u(s)-v(s)| \geq \frac{m_1}{2}\rho(u,v). \tag{15}$$

Then,

$$\begin{aligned}
\rho(G_1u,G_1v) &= \max_{t\in[a,b]}\left|\int_a^b \big(K_1(t,s,u(s)) - K_1(t,s,v(s))\big)\,ds\right| \\
&\geq \left|\int_a^b \big(K_1(t_0,s,u(s)) - K_1(t_0,s,v(s))\big)\,ds\right| \\
&\geq \int_a^b |K_1(t_0,s,u(s)) - K_1(t_0,s,v(s))|\,ds \\
&\geq \int_J |K_1(t_0,s,u(s)) - K_1(t_0,s,v(s))|\,ds \\
&\geq \int_J \frac{m_1}{2}\rho(u,v)\,ds \\
&= \frac{m_1\delta_1}{2}\rho(u,v).
\end{aligned}$$

Thus,

$$\rho(G_1u,G_1v) \geq \frac{m_1\delta_1}{2}\rho(u,v).$$

Let $m^* = \frac{m_1\delta_1}{2(b-a)} > 0$. Then,

$$\rho(G_1u,G_1v) \geq m^*(b-a)\rho(u,v).$$

For simplicity of notation, we relabel m^* as m_1 and write:

$$\rho(G_1u,G_1v) \geq m_1(b-a)\,\rho(u,v). \tag{16}$$

Now, from (11) with $i=2$ and (16)

$$\rho(G_2u,G_2v) \leq L_2(b-a)\,\rho(u,v) \leq \frac{L_2}{m_1}\,\rho(G_1u,G_1v).$$

Define $k = \frac{L_2}{m_1}$. By (H3), we have $L_2 < m_1$, hence $0 \leq k < 1$. Therefore,

$$\rho(G_2u,G_2v) \leq k\,\rho(G_1u,G_1v). \tag{17}$$

Now, using the definitions of M,N,P and (17), for all $\lambda > 0$:

$$\begin{aligned}
M(G_2u,G_2v,k\lambda) &= \frac{k\lambda}{k\lambda+\rho(G_2u,G_2v)} \geq \frac{k\lambda}{k\lambda+k\rho(G_1u,G_1v)} = \frac{\lambda}{\lambda+\rho(G_1u,G_1v)} = M(G_1u,G_1v,\lambda), \\
N(G_2u,G_2v,k\lambda) &= \frac{\rho(G_2u,G_2v)}{k\lambda+\rho(G_2u,G_2v)} \leq \frac{k\rho(G_1u,G_1v)}{k\lambda+k\rho(G_1u,G_1v)} = \frac{\rho(G_1u,G_1v)}{\lambda+\rho(G_1u,G_1v)} = N(G_1u,G_1v,\lambda), \\
P(G_2u,G_2v,k\lambda) &= \frac{\rho(G_2u,G_2v)}{k\lambda} \leq \frac{k\rho(G_1u,G_1v)}{k\lambda} = \frac{\rho(G_1u,G_1v)}{\lambda} = P(G_1u,G_1v,\lambda).
\end{aligned}$$

Thus condition (1) of Corollary 1 is satisfied with $k = \frac{L_2}{m_1} < 1$.

Step 3: Verification of the remaining hypotheses.

By (H4), G_1 and G_2 are continuous and commuting. By (H5), $G_2(X) \subseteq G_1(X)$, which is condition (b) of Corollary 1.

Step 4: Application of Corollary 2.10.

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Since all hypotheses of Corollary 2.10 are satisfied, there exists $u^* \in X$ such that

$$G_1 u^* = G_2 u^* = u^*.$$

Therefore, u^* is a common fixed point of G_1 and G_2, which means it is an exact common solution to the system (10). This completes the proof. □

Example 3. As a concrete example, let $[a,b] = [0,1]$, $K_1(t,s,u(s)) = 2u(s)$, $K_2(t,s,u(s)) = u(s)$, $f_1(t) = 0$, $f_2(t) = 0$. Then $G_1 u = 2\int_0^1 u(s)ds$ and $G_2 u = \int_0^1 u(s)ds$. Here $L_1 = 2$, $L_2 = 1$, $m_1 = 2$, so $k = \frac{1}{2} < 1$. The inclusion $G_2(X) \subseteq G_1(X)$ holds because $G_2 u = \frac{1}{2} G_1 u$. The mappings are continuous and compatible. Hence, by the theorem, there exists a common fixed point u^*, which in this case is the zero function $u^* \equiv 0$.

4 Conclusion

In this paper, we established new existence results for common best proximity points of two non-self mappings in complete neutrosophic b-metric spaces. Two main theorems were proved under distinct contractive conditions: Theorem 1 uses a contraction-type inequality together with neutrosophic proximal commutativity and interchangeability, while Theorem 2 employs the notion of proximal neutrosophic domination. The corresponding common fixed point results for self-mappings were derived as Corollaries 1 and 2, and an application to a system of nonlinear Fredholm integral equations was presented in Theorem 3 Examples were also provided to support the main results.

The present study has some limitations. The results require the approximate compactness of A with respect to B, and the contraction condition in Theorem 1 may be difficult to verify in some practical situations. Moreover, the application is restricted to a class of Fredholm integral equations under Lipschitz-type assumptions.

Future work may extend these results to other generalized structures, such as neutrosophic fuzzy b-metric and neutrosophic soft metric spaces, to families of mappings rather than pairs, and to other types of integral equations. Developing numerical algorithms for approximating common best proximity points in neutrosophic b-metric spaces also remains an interesting direction.

Authors' Contributions

All authors contributed to the study and approved the final version of the manuscript.

Data Availability

The data that support the findings of this study are available from the corresponding author upon reasonable request.

Code Availability

The code used to generate the results in this study is available from the corresponding author upon reasonable request.

Conflicts of Interest

The authors declare that there is no conflict of interest.

Ethical Considerations

This study did not involve human participants or animals, and no ethical approval was required.

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