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Research article

A Discrete Hardy-Hilbert-Type Inequality with an Exponential-Power Kernel

Aida Azizi[1], Mohammad Taghi Heydari[1,*]

[1] Department of Mathematics, College of Sciences, Yasouj University, Yasouj, Iran

* **Corresponding author(s):** heydari@yu.ac.ir

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Abstract

We establish a discrete Hardy–Hilbert-type inequality associated with the non-homogeneous exponential-power kernel

$$K_{\gamma,\lambda}(m,n)=\frac{e^{-\gamma(m+n)}}{(m+n)^{\lambda}},\qquad \gamma>0,\ \lambda>0.$$

For conjugate exponents $p,q>1$, we prove that the corresponding bilinear form on $\ell^p\times\ell^q$ is bounded by an explicit upper-bound constant represented by a one-dimensional integral. The proof combines the gamma-integral representation of $(m+n)^{-\lambda}$ with Tonelli's theorem and Hölder's inequality. We derive a discrete Hardy-type inequality and, by integrating with respect to the exponential parameter, an inequality involving the kernel

$$\frac{1-e^{-\theta(m+n)}}{(m+n)^{\lambda+1}},\qquad \theta>0.$$

We further establish the strict monotonicity of the resulting constant with respect to the exponential parameter and determine its asymptotic behavior as $\gamma\to 0^{+}$ and as $\gamma\to\infty$. Special cases, including $p=q=2$ and $\lambda=1/2$, are also considered, and numerical examples illustrate how the resulting constants behave. The result gives an explicit, computable bound for the bilinear form generated by an exponentially damped power kernel.

Keywords: Hardy–Hilbert inequality, Discrete inequality, Exponential-power kernel, Sequence spaces, Double series

Mathematics Subject Classification (2020): 26D15, 40, 46

1 Introduction

Hardy–Hilbert-type inequalities have been studied extensively in discrete, continuous, and half-discrete settings. The classical Hilbert inequality is a fundamental result in this theory. For conjugate exponents $p,q>1$, satisfying

$$\frac{1}{p}+\frac{1}{q}=1,$$

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we have

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n} \le \frac{\pi}{\sin(\pi/p)}\left(\sum_{m=1}^{\infty}a_m^p\right)^{1/p}\left(\sum_{n=1}^{\infty}b_n^q\right)^{1/q},$$

for non-negative sequences $\{a_m\}$ and $\{b_n\}$. This inequality has motivated numerous extensions involving weighted, homogeneous, and non-homogeneous kernels [1, 2]. Operator-theoretic formulations and more general kernel structures have also been investigated. Yang [3] studied the relation between Hilbert's inequality and a Hilbert-type inequality, while Yang [4] considered Hilbert-type operators associated with homogeneous kernels. These results provide useful frameworks for estimating bilinear forms generated by generalized kernels. Half-discrete extensions have attracted considerable attention. Yang and Chen [5] established a half-discrete Hilbert-type inequality with a homogeneous kernel and an associated extension. Azar [6] investigated the connection between Hilbert and Hardy inequalities, while Chen and Yang [7] surveyed developments in Hilbert-type inequalities. Hong, He, and Yang [8] subsequently established necessary and sufficient conditions for Hilbert-type integral inequalities involving quasi-homogeneous kernels and considered applications to operator theory. Exponential and other nonclassical kernels have also been treated. You [9] established a halfdiscrete Hilberttype inequality related to hyperbolic functions and obtained parameterdependent equivalent forms. Later [10], he introduced a halfdiscrete kernel covering both homogeneous and nonhomogeneous cases together with the corresponding Hardytype inequalities. Taken together, these results indicate that parameterized exponential kernels extend the classical Hilbert framework in a natural way.

Yimer, Persson, Ruzhansky, Samko, and Ayele [11] took this further, studying Hardy–Hilbert-type inequalities on homogeneous groups and formulating a general kernel framework that encompasses several classical settings; their approach makes clear how much the kernel structure itself controls both the inequality and its constant.

Closer to the present setting, Chesneau [12] treated an exponential-power kernel in the continuous case, obtaining the integral kernel

$$\frac{e^{-\lambda(x+y)}}{(x+y)^{\mu}}, \qquad \lambda>0, \quad 0<\mu<1,$$

and obtaining a Hardy–Hilbert-type integral inequality with an explicit constant expressed via the gamma function; a multidimensional two-parameter extension followed shortly after [13]. No discrete analogue of this exponential-power kernel appears to have been worked out yet, which is the gap this paper addresses.

The discrete problem considered here is associated with

$$K_{\gamma,\lambda}(m,n)=\frac{e^{-\gamma(m+n)}}{(m+n)^{\lambda}}, \qquad \gamma>0, \quad \lambda>0.$$

Although the kernel is formally analogous to its continuous counterpart, the discrete setting is not obtained merely by replacing integrals with sums: the summation produces geometric-series factors in the norm estimate and leads to a distinct parameter-dependent constant. We further characterize this constant through its monotonicity and its endpoint behavior in the regimes $0<\lambda<1$, $\lambda=1$, $\lambda>1$, and as $\gamma\to\infty$.

The main objective of this paper is to establish a discrete Hardy–Hilbert-type inequality for the kernel $K_{\gamma,\lambda}$. For conjugate exponents $p,q>1$, we prove that the associated bilinear form on $\ell^p\times\ell^q$ is bounded by an explicit upper-bound constant given by a one-dimensional integral. The proof combines the gamma-integral representation of the power factor with Tonelli's theorem and Hölder's inequality. We then derive a discrete Hardy-type inequality and an integrated exponential-kernel inequality. We further distinguish this computable bound from the best constant: it is strictly non-optimal for every admissible parameter choice, but becomes asymptotically sharp as the damping parameter tends to infinity.

We further investigate the dependence of the resulting constant on the exponential parameter. In particular, we prove strict monotonicity and determine its asymptotic behavior as $\gamma\to 0^+$ in the three regimes $0<\lambda<1$, $\lambda=1$, and $\lambda>1$. We also obtain the asymptotic relation as $\gamma\to\infty$. Special cases, including $p=q=2$ and $\lambda=1/2$, are considered, and numerical examples are provided for representative parameter choices.

The paper is organized as follows. Section 2 presents the main discrete Hardy-Hilbert-type inequality. The discrete Hardytype inequality and the integrated exponentialkernel inequality are derived in Sections 2.1 and 2.2, respectively. Section 2.3 studies the properties of the resulting constant, including monotonicity and asymptotic behaviour. Numerical examples are given in Section 3, and Section 4 contains the conclusion and some open problems.

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2 Main Discrete Inequality

Theorem 1. *Let $p > 1$ and let $q = p/(p-1)$ be its conjugate exponent. Let $\gamma > 0$ and $\lambda > 0$. Suppose that $a = (a_m)_{m\in\mathbb{N}} \in \ell^p$ and $b = (b_n)_{n\in\mathbb{N}} \in \ell^q$ are non-negative sequences. Then*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{e^{-\gamma(m+n)}}{(m+n)^{\lambda}}a_m b_n \le \Theta_{\gamma,\lambda,p}\,\|a\|_p\|b\|_q, \tag{1}$$

where

$$\Theta_{\gamma,\lambda,p} = \frac{1}{\Gamma(\lambda)}\int_0^{\infty} t^{\lambda-1}\left(\frac{e^{-q(\gamma+t)}}{1-e^{-q(\gamma+t)}}\right)^{1/q}\left(\frac{e^{-p(\gamma+t)}}{1-e^{-p(\gamma+t)}}\right)^{1/p} dt. \tag{2}$$

Proof. By the gamma-integral representation, for $m,n \ge 1$,

$$\frac{e^{-\gamma(m+n)}}{(m+n)^{\lambda}} = \frac{1}{\Gamma(\lambda)}\int_0^{\infty} t^{\lambda-1}e^{-(\gamma+t)(m+n)}\,dt.$$

Since all terms are non-negative, Tonelli's theorem gives

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{e^{-\gamma(m+n)}}{(m+n)^{\lambda}}a_m b_n = \frac{1}{\Gamma(\lambda)}\int_0^{\infty} t^{\lambda-1}\left(\sum_{m=1}^{\infty}a_m e^{-(\gamma+t)m}\right)\left(\sum_{n=1}^{\infty}b_n e^{-(\gamma+t)n}\right)dt. \tag{3}$$

Applying Hölder's inequality to each sum and evaluating the resulting geometric series, we obtain

$$\sum_{m=1}^{\infty}a_m e^{-(\gamma+t)m} \le \|a\|_p\left(\frac{e^{-q(\gamma+t)}}{1-e^{-q(\gamma+t)}}\right)^{1/q},$$

and

$$\sum_{n=1}^{\infty}b_n e^{-(\gamma+t)n} \le \|b\|_q\left(\frac{e^{-p(\gamma+t)}}{1-e^{-p(\gamma+t)}}\right)^{1/p}.$$

Substitution into the preceding integral yields

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{e^{-\gamma(m+n)}}{(m+n)^{\lambda}}a_m b_n \le \Theta_{\gamma,\lambda,p}\,\|a\|_p\|b\|_q,$$

where $\Theta_{\gamma,\lambda,p}$ is the constant defined in (2), as required. □

Remark 1. The integral defining $\Theta_{\gamma,\lambda,p}$ is finite for every $\gamma > 0$, $\lambda > 0$, and $p > 1$. Indeed, as $t \to 0^+$, the two factors involving the exponential functions remain bounded because $\gamma > 0$, so the integrand is of order $t^{\lambda-1}$. As $t \to \infty$, the integrand decays exponentially. Hence the integral converges.

Corollary 1. Let $p > 1$, let $q = p/(p-1)$, and let $\gamma > 0$. If $\lambda = 1/2$, then the constant $\Theta_{\gamma,\lambda,p}$ in Theorem 1 is given by

$$\Theta_{\gamma,1/2,p} = \frac{1}{\sqrt{\pi}}\int_0^{\infty} t^{-1/2}\left(\frac{e^{-q(\gamma+t)}}{1-e^{-q(\gamma+t)}}\right)^{1/q}\left(\frac{e^{-p(\gamma+t)}}{1-e^{-p(\gamma+t)}}\right)^{1/p} dt. \tag{4}$$

In particular, for $p = q = 2$, Theorem 1 yields

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{e^{-\gamma(m+n)}}{\sqrt{m+n}}a_m b_n \le \Theta_2(\gamma)\,\|a\|_2\|b\|_2, \tag{5}$$

where

$$\Theta_2(\gamma) = \frac{1}{\sqrt{\pi}}\int_0^{\infty} t^{-1/2}\frac{e^{-2(\gamma+t)}}{1-e^{-2(\gamma+t)}}\,dt. \tag{6}$$

Moreover, $\Theta_2(\gamma)$ admits the series representation

$$\Theta_2(\gamma) = \frac{1}{\sqrt{2}}\sum_{k=1}^{\infty}\frac{e^{-2k\gamma}}{\sqrt{k}}. \tag{7}$$

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Proof. Since $\Gamma(1/2)=\sqrt{\pi}$, substituting $\lambda=1/2$ into (2) gives (4). For $p=q=2$, the two factors in the integrand coincide, and hence

$$\left(\frac{e^{-2(\gamma+t)}}{1-e^{-2(\gamma+t)}}\right)^{1/2}\left(\frac{e^{-2(\gamma+t)}}{1-e^{-2(\gamma+t)}}\right)^{1/2}=\frac{e^{-2(\gamma+t)}}{1-e^{-2(\gamma+t)}}.$$

Thus (6) follows directly from (4).

For $t\geq 0$ and $\gamma>0$,

$$0<e^{-2(\gamma+t)}<1,$$

and therefore the geometric-series expansion

$$\frac{e^{-2(\gamma+t)}}{1-e^{-2(\gamma+t)}}=\sum_{k=1}^{\infty}e^{-2k(\gamma+t)}$$

holds. Since all terms are non-negative, Tonelli's theorem gives

$$\begin{aligned}\Theta_2(\gamma)&=\frac{1}{\sqrt{\pi}}\int_0^{\infty}t^{-1/2}\sum_{k=1}^{\infty}e^{-2k(\gamma+t)}\,dt\\&=\frac{1}{\sqrt{\pi}}\sum_{k=1}^{\infty}e^{-2k\gamma}\int_0^{\infty}t^{-1/2}e^{-2kt}\,dt.\end{aligned}\tag{8}$$

Using the standard gamma-integral identity

$$\int_0^{\infty}t^{-1/2}e^{-2kt}\,dt=\frac{\Gamma(1/2)}{(2k)^{1/2}}=\frac{\sqrt{\pi}}{\sqrt{2k}},$$

we obtain

$$\Theta_2(\gamma)=\frac{1}{\sqrt{2}}\sum_{k=1}^{\infty}\frac{e^{-2k\gamma}}{\sqrt{k}},$$

which proves (7). □

Remark 2. For $p=q=2$ and general $\lambda>0$, the same geometric-series argument used in the proof of Corollary 1 gives

$$\Theta_{\gamma,\lambda,2}=\sum_{k=1}^{\infty}\frac{e^{-2k\gamma}}{(2k)^{\lambda}},\qquad\gamma,\lambda>0.\tag{9}$$

This reduces to (7) at $\lambda=1/2$.

2.1 A Discrete Hardy-Type Inequality

Theorem 2. *Let $p>1$ and let $q=p/(p-1)$ be its conjugate exponent. Let $\gamma>0$ and $\lambda>0$. Suppose that $a=(a_m)_{m\in\mathbb{N}}\in\ell^p$ is a non-negative sequence. Then*

$$\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}\frac{e^{-\gamma(m+n)}}{(m+n)^{\lambda}}a_m\right)^p\leq\Theta_{\gamma,\lambda,p}^{p}\,\|a\|_p^p,\tag{10}$$

where $\Theta_{\gamma,\lambda,p}$ is the constant defined in Theorem 1.

Proof. For $a\in\ell^p$, define

$$(Ta)_n:=\sum_{m=1}^{\infty}\frac{e^{-\gamma(m+n)}}{(m+n)^{\lambda}}a_m,\qquad n\in\mathbb{N}.$$

Since $a_m\geq 0$, all terms in this series are non-negative.

We first assume that a has finite support. For $N\in\mathbb{N}$, set

$$K_N:=\sum_{n=1}^{N}(Ta)_n^p$$

and define the finitely supported sequence

$$b_n^{(N)}:=\begin{cases}(Ta)_n^{p-1}, & 1\leq n\leq N,\\ 0, & n>N.\end{cases}$$

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Since $b^{(N)} \in \ell^q$, Theorem 1 can be applied to a and $b^{(N)}$. Moreover, using $q(p-1) = p$, we obtain

$$\begin{aligned} K_N &= \sum_{n=1}^{N} (Ta)_n^p \\ &= \sum_{n=1}^{N} \left(\sum_{m=1}^{\infty} \frac{e^{-\gamma(m+n)}}{(m+n)^{\lambda}} a_m \right)^{p-1} \left(\sum_{m=1}^{\infty} \frac{e^{-\gamma(m+n)}}{(m+n)^{\lambda}} a_m \right) \\ &= \sum_{m=1}^{\infty} \sum_{n=1}^{N} \frac{e^{-\gamma(m+n)}}{(m+n)^{\lambda}} a_m b_n^{(N)}. \end{aligned} \tag{11}$$

Hence, by Theorem 1,

$$K_N \le \Theta_{\gamma,\lambda,p} \|a\|_p \|b^{(N)}\|_q. \tag{12}$$

On the other hand,

$$\begin{aligned} \|b^{(N)}\|_q^q &= \sum_{n=1}^{N} \left((Ta)_n^{p-1} \right)^q \\ &= \sum_{n=1}^{N} (Ta)_n^{q(p-1)} = \sum_{n=1}^{N} (Ta)_n^p = K_N. \end{aligned} \tag{13}$$

Therefore,

$$\|b^{(N)}\|_q = K_N^{1/q}.$$

Substituting this identity into (12) gives

$$K_N \le \Theta_{\gamma,\lambda,p} \|a\|_p K_N^{1/q}.$$

If $K_N = 0$, the desired estimate is immediate. Otherwise, dividing by $K_N^{1/q}$ and using $1 - 1/q = 1/p$, we obtain

$$K_N^{1/p} \le \Theta_{\gamma,\lambda,p} \|a\|_p.$$

Consequently,

$$K_N \le \Theta_{\gamma,\lambda,p}^p \|a\|_p^p \tag{14}$$

for every $N \in \mathbb{N}$.

Since K_N is non-decreasing in N, the monotone convergence theorem and (14) yield

$$\sum_{n=1}^{\infty} (Ta)_n^p = \lim_{N\to\infty} K_N \le \Theta_{\gamma,\lambda,p}^p \|a\|_p^p.$$

It remains to remove the finite-support assumption. For $M \in \mathbb{N}$, define the truncation

$$a^{(M)} := (a_1, \dots, a_M, 0, 0, \dots).$$

Then $a^{(M)} \in \ell^p$ has finite support, and

$$\|a^{(M)}\|_p^p = \sum_{m=1}^{M} |a_m|^p \le \sum_{m=1}^{\infty} |a_m|^p = \|a\|_p^p.$$

Hence $\|a^{(M)}\|_p \le \|a\|_p$. Moreover, since $a_m \ge 0$,

$$(Ta^{(M)})_n \uparrow (Ta)_n \qquad \text{as } M \to \infty$$

for every $n \in \mathbb{N}$. Applying the estimate already proved to $a^{(M)}$ and then using the monotone convergence theorem gives

$$\begin{aligned} \sum_{n=1}^{\infty} (Ta)_n^p &= \lim_{M\to\infty} \sum_{n=1}^{\infty} (Ta^{(M)})_n^p \\ &\le \Theta_{\gamma,\lambda,p}^p \limsup_{M\to\infty} \|a^{(M)}\|_p^p \\ &\le \Theta_{\gamma,\lambda,p}^p \|a\|_p^p. \end{aligned}$$

This proves (10). □

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Remark 3. For $\lambda = 1/2$ and $p = q = 2$, Theorem 2 reduces to

$$\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}\frac{e^{-\gamma(m+n)}}{\sqrt{m+n}}\,a_m\right)^2 \le \Theta_2(\gamma)^2\|a\|_2^2, \tag{15}$$

where $\Theta_2(\gamma)$ is the constant defined in (6).

2.2 An Integrated Exponential-Kernel Inequality

We first establish the integrability property required for integrating the main inequality with respect to the exponential parameter.

Lemma 1. *Let $p > 1$, let $q = p/(p-1)$, let $\lambda > 0$, and let $\theta > 0$. Then*

$$\int_0^{\theta}\Theta_{\gamma,\lambda,p}\,d\gamma < \infty. \tag{16}$$

Proof. For $x > 0$, the inequality

$$e^x - 1 \ge x$$

implies

$$\frac{e^{-x}}{1-e^{-x}} = \frac{1}{e^x-1} \le \frac{1}{x}.$$

Hence, for $0 < t \le 1$ and $0 < \gamma \le \theta$,

$$\left(\frac{e^{-q(\gamma+t)}}{1-e^{-q(\gamma+t)}}\right)^{1/q}\left(\frac{e^{-p(\gamma+t)}}{1-e^{-p(\gamma+t)}}\right)^{1/p} \le q^{-1/q}p^{-1/p}\frac{1}{\gamma+t}.$$

Therefore,

$$\int_0^{\theta}\int_0^1 t^{\lambda-1}\left(\frac{e^{-q(\gamma+t)}}{1-e^{-q(\gamma+t)}}\right)^{1/q}\left(\frac{e^{-p(\gamma+t)}}{1-e^{-p(\gamma+t)}}\right)^{1/p} dt\,d\gamma \le q^{-1/q}p^{-1/p}\int_0^{\theta}\int_0^1 \frac{t^{\lambda-1}}{\gamma+t}\,dt\,d\gamma.$$

Since the integrand is non-negative, Tonelli's theorem gives

$$\int_0^{\theta}\int_0^1 \frac{t^{\lambda-1}}{\gamma+t}\,dt\,d\gamma = \int_0^1 t^{\lambda-1}\log\left(\frac{\theta+t}{t}\right)dt.$$

The last integral is finite for every $\lambda > 0$. Indeed,

$$\log\left(\frac{\theta+t}{t}\right) = \log(\theta+t) - \log t \le C_\theta + |\log t|, \qquad 0 < t \le 1,$$

and

$$\int_0^1 t^{\lambda-1}(1+|\log t|)\,dt < \infty$$

for every $\lambda > 0$.

For $t \ge 1$ and $r > 0$,

$$\frac{e^{-r(\gamma+t)}}{1-e^{-r(\gamma+t)}} \le \frac{e^{-rt}}{1-e^{-r}},$$

because $\gamma \ge 0$. Consequently,

$$\left(\frac{e^{-q(\gamma+t)}}{1-e^{-q(\gamma+t)}}\right)^{1/q}\left(\frac{e^{-p(\gamma+t)}}{1-e^{-p(\gamma+t)}}\right)^{1/p} \le C_p e^{-2t},$$

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where

$$C_p = \left(1-e^{-q}\right)^{-1/q}\left(1-e^{-p}\right)^{-1/p}.$$

Thus,

$$\int_0^\theta \int_1^\infty t^{\lambda-1}\left(\frac{e^{-q(\gamma+t)}}{1-e^{-q(\gamma+t)}}\right)^{1/q}\left(\frac{e^{-p(\gamma+t)}}{1-e^{-p(\gamma+t)}}\right)^{1/p} dt\,d\gamma \le \theta C_p \int_1^\infty t^{\lambda-1}e^{-2t}\,dt < \infty.$$

Combining the estimates on $(0,1)$ and $[1,\infty)$, and using the definition of $\Theta_{\gamma,\lambda,p}$, proves (16). □

Theorem 3. *Let $p>1$ and let $q=p/(p-1)$ be its conjugate exponent. Let $\theta>0$ and $\lambda>0$. Suppose that $a=(a_m)_{m\in\mathbb{N}}\in\ell^p$ and $b=(b_n)_{n\in\mathbb{N}}\in\ell^q$ are non-negative sequences. Then*

$$\sum_{m=1}^\infty\sum_{n=1}^\infty \frac{1-e^{-\theta(m+n)}}{(m+n)^{\lambda+1}}a_m b_n \le \Xi_{\theta,\lambda,p}\,\|a\|_p\|b\|_q, \tag{17}$$

where

$$\Xi_{\theta,\lambda,p} := \int_0^\theta \Theta_{\gamma,\lambda,p}\,d\gamma, \tag{18}$$

and $\Theta_{\gamma,\lambda,p}$ is the constant defined in Theorem 1.

Proof. For every $\gamma>0$, Theorem 1 gives

$$\sum_{m=1}^\infty\sum_{n=1}^\infty \frac{e^{-\gamma(m+n)}}{(m+n)^{\lambda}}a_m b_n \le \Theta_{\gamma,\lambda,p}\,\|a\|_p\|b\|_q. \tag{19}$$

By Lemma 1, $\Theta_{\gamma,\lambda,p}$ is integrable on $(0,\theta)$. Therefore, integrating (19) with respect to γ over $(0,\theta)$ yields

$$\int_0^\theta \sum_{m=1}^\infty\sum_{n=1}^\infty \frac{e^{-\gamma(m+n)}}{(m+n)^{\lambda}}a_m b_n\,d\gamma \le \left(\int_0^\theta \Theta_{\gamma,\lambda,p}\,d\gamma\right)\|a\|_p\|b\|_q. \tag{20}$$

Since all terms in the double series are non-negative, Tonelli's theorem permits the interchange of the integral and the double sum. Hence

$$\int_0^\theta \sum_{m=1}^\infty\sum_{n=1}^\infty \frac{e^{-\gamma(m+n)}}{(m+n)^{\lambda}}a_m b_n\,d\gamma = \sum_{m=1}^\infty\sum_{n=1}^\infty \frac{a_m b_n}{(m+n)^{\lambda}}\int_0^\theta e^{-\gamma(m+n)}\,d\gamma. \tag{21}$$

For $m,n\ge 1$,

$$\int_0^\theta e^{-\gamma(m+n)}\,d\gamma = \frac{1-e^{-\theta(m+n)}}{m+n}.$$

Therefore,

$$\int_0^\theta \sum_{m=1}^\infty\sum_{n=1}^\infty \frac{e^{-\gamma(m+n)}}{(m+n)^{\lambda}}a_m b_n\,d\gamma = \sum_{m=1}^\infty\sum_{n=1}^\infty \frac{1-e^{-\theta(m+n)}}{(m+n)^{\lambda+1}}a_m b_n. \tag{22}$$

Combining (20), (22), and the definition (18), we obtain (17). □

Remark 4. Substituting the representation of $\Theta_{\gamma,\lambda,p}$ from Theorem 1 into (18), we obtain

$$\Xi_{\theta,\lambda,p} = \frac{1}{\Gamma(\lambda)}\int_0^\theta\int_0^\infty t^{\lambda-1}\left(\frac{e^{-q(\gamma+t)}}{1-e^{-q(\gamma+t)}}\right)^{1/q}\left(\frac{e^{-p(\gamma+t)}}{1-e^{-p(\gamma+t)}}\right)^{1/p} dt\,d\gamma. \tag{23}$$

The finiteness of this double integral follows from Lemma 1.

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Remark 5. For $\lambda = 1/2$ and $p = q = 2$, Theorem 3 reduces to

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1-e^{-\theta(m+n)}}{(m+n)^{3/2}}a_m b_n \le \Xi_2(\theta)\,\|a\|_2\|b\|_2, \tag{24}$$

where

$$\Xi_2(\theta) = \frac{1}{\sqrt{\pi}}\int_0^{\theta}\int_0^{\infty} t^{-1/2}\frac{e^{-2(\gamma+t)}}{1-e^{-2(\gamma+t)}}\,dt\,d\gamma. \tag{25}$$

2.3 Properties of the Constant

The integral representation of $\Theta_{\gamma,\lambda,p}$ allows us to describe its dependence on the exponential parameter. We first analyze its behavior as $\gamma \to 0^+$, and then establish its strict monotonicity and asymptotic decay as $\gamma \to \infty$.

Proposition 1. *Let $p > 1$, let $q = p/(p-1)$, and let $\lambda > 0$. Then the following asymptotic relations hold as $\gamma \to 0^+$.*

1. *If $0 < \lambda < 1$, then*

$$\Theta_{\gamma,\lambda,p} \sim \frac{p^{-1/p}q^{-1/q}\pi}{\Gamma(\lambda)\sin(\pi\lambda)}\gamma^{\lambda-1}. \tag{26}$$

2. *If $\lambda = 1$, then*

$$\Theta_{\gamma,1,p} \sim p^{-1/p}q^{-1/q}\log\frac{1}{\gamma}. \tag{27}$$

3. *If $\lambda > 1$, then $\Theta_{\gamma,\lambda,p}$ has a finite limit as $\gamma \to 0^+$, namely*

$$\lim_{\gamma\to 0^+}\Theta_{\gamma,\lambda,p} = \Theta_{0,\lambda,p}, \tag{28}$$

where

$$\Theta_{0,\lambda,p} := \frac{1}{\Gamma(\lambda)}\int_0^{\infty} t^{\lambda-1}\left(\frac{e^{-qt}}{1-e^{-qt}}\right)^{1/q}\left(\frac{e^{-pt}}{1-e^{-pt}}\right)^{1/p}dt < \infty. \tag{29}$$

Proof. For $r > 0$, define

$$H_r(x) := \left(\frac{e^{-rx}}{1-e^{-rx}}\right)^{1/r}, \qquad x > 0.$$

Since

$$\frac{e^{-rx}}{1-e^{-rx}} = \frac{1}{e^{rx}-1} = \frac{1}{rx} + O(1), \qquad x \to 0^+,$$

we have

$$H_q(x)H_p(x) = C_p x^{-1} + O(1), \qquad x \to 0^+,$$

where

$$C_p := p^{-1/p}q^{-1/q}.$$

Suppose first that $0 < \lambda < 1$. Choose $\delta \in (0,1)$. Since

$$H_q(x)H_p(x) = \frac{C_p}{x} + O(1), \qquad x \to 0^+,$$

where

$$C_p = p^{-1/p}q^{-1/q},$$

the function

$$R(x) := H_q(x)H_p(x) - \frac{C_p}{x}$$

extends continuously to $x = 0$. Hence there exists a constant $M_\delta > 0$ such that

$$|R(x)| \le M_\delta, \qquad 0 \le x \le \delta.$$

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Split the integral defining $\Theta_{\gamma,\lambda,p}$ as

$$\Theta_{\gamma,\lambda,p} = \frac{1}{\Gamma(\lambda)} \left(\int_0^{\delta} + \int_{\delta}^{\infty} \right) t^{\lambda-1} H_q(\gamma+t) H_p(\gamma+t)\, dt.$$

For $0 < \gamma < \delta$ and $0 < t < \delta$,

$$H_q(\gamma+t)H_p(\gamma+t) = \frac{C_p}{\gamma+t} + R(\gamma+t),$$

and therefore

$$\int_0^{\delta} t^{\lambda-1} H_q(\gamma+t)H_p(\gamma+t)\, dt = C_p \int_0^{\delta} \frac{t^{\lambda-1}}{\gamma+t}\, dt + E_{\gamma},$$

where

$$|E_{\gamma}| \le M_{\delta} \int_0^{\delta} t^{\lambda-1}\, dt = \frac{M_{\delta}}{\lambda} \delta^{\lambda}.$$

Thus $E_{\gamma} = O(1)$ as $\gamma \to 0^{+}$.

On the interval $[\delta, \infty)$, the integrand is bounded uniformly for sufficiently small γ, and hence

$$\int_{\delta}^{\infty} t^{\lambda-1} H_q(\gamma+t)H_p(\gamma+t)\, dt = O(1).$$

Consequently,

$$\Theta_{\gamma,\lambda,p} = \frac{C_p}{\Gamma(\lambda)} \int_0^{\delta} \frac{t^{\lambda-1}}{\gamma+t}\, dt + O(1).$$

With the substitution $t = \gamma u$,

$$\int_0^{\delta} \frac{t^{\lambda-1}}{\gamma+t}\, dt = \gamma^{\lambda-1} \int_0^{\delta/\gamma} \frac{u^{\lambda-1}}{1+u}\, du.$$

Since $0 < \lambda < 1$,

$$\int_0^{\infty} \frac{u^{\lambda-1}}{1+u}\, du = \frac{\pi}{\sin(\pi\lambda)},$$

and therefore

$$\int_0^{\delta} \frac{t^{\lambda-1}}{\gamma+t}\, dt \sim \frac{\pi}{\sin(\pi\lambda)} \gamma^{\lambda-1}.$$

Because $\lambda - 1 < 0$, the $O(1)$ remainder is negligible, and (26) follows.

For $\lambda = 1$, using the same δ-decomposition as above, we obtain

$$\Theta_{\gamma,1,p} = C_p \int_0^{\delta} \frac{dt}{\gamma+t} + O(1) = C_p \log\left(\frac{\gamma+\delta}{\gamma}\right) + O(1).$$

Since $\delta > 0$ is fixed,

$$\log\left(\frac{\gamma+\delta}{\gamma}\right) = \log\frac{1}{\gamma} + O(1), \qquad \gamma \to 0^{+}.$$

Hence

$$\Theta_{\gamma,1,p} = C_p \log\frac{1}{\gamma} + O(1),$$

which proves (27).

Finally, let $\lambda > 1$. In this case, the integrand corresponding to $\gamma = 0$ behaves near $t = 0$ like

$$C_p t^{\lambda-2},$$

which is integrable because $\lambda > 1$. For $t \ge 1$, the integrand is bounded by a constant multiple of

$$t^{\lambda-1} e^{-2t}.$$

More precisely, for $0 < \gamma \le 1$,

$$t^{\lambda-1} H_q(\gamma+t)H_p(\gamma+t) \le C t^{\lambda-2}, \qquad 0 < t \le 1,$$

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and

$$t^{\lambda-1}H_q(\gamma+t)H_p(\gamma+t) \le Ct^{\lambda-1}e^{-2t}, \qquad t \ge 1,$$

where $C > 0$ is independent of γ. Hence the integrand is dominated by

$$C\left(t^{\lambda-2}\mathbf{1}_{(0,1)}(t) + t^{\lambda-1}e^{-2t}\mathbf{1}_{[1,\infty)}(t)\right).$$

Since $\lambda > 1$,

$$\int_0^1 t^{\lambda-2}\,dt < \infty, \qquad \int_1^\infty t^{\lambda-1}e^{-2t}\,dt < \infty.$$

Thus the dominating function is integrable and independent of γ. The dominated convergence theorem therefore yields

$$\lim_{\gamma\to 0^+}\Theta_{\gamma,\lambda,p} = \frac{1}{\Gamma(\lambda)}\int_0^\infty t^{\lambda-1}\left(\frac{e^{-qt}}{1-e^{-qt}}\right)^{1/q}\left(\frac{e^{-pt}}{1-e^{-pt}}\right)^{1/p}dt,$$

which proves (28). □

Remark 6. The three regimes in Proposition 1 describe the behavior of $\Theta_{\gamma,\lambda,p}$ near the endpoint $\gamma = 0$. For $0 < \lambda < 1$, the constant has a power-law singularity of order $\gamma^{\lambda-1}$; for $\lambda = 1$, the growth is logarithmic; and for $\lambda > 1$, the constant remains bounded as $\gamma \to 0^+$. In all three cases,

$$\int_0^\theta \Theta_{\gamma,\lambda,p}\,d\gamma < \infty$$

for every $\lambda > 0$, consistently with Lemma 1.

Proposition 2. *Let $p > 1$, let $q = p/(p-1)$, and let $\lambda > 0$. Then the function*

$$\gamma \mapsto \Theta_{\gamma,\lambda,p}, \qquad \gamma > 0,$$

is strictly decreasing. Moreover,

$$\lim_{\gamma\to\infty} e^{2\gamma}\Theta_{\gamma,\lambda,p} = 2^{-\lambda}. \tag{30}$$

Equivalently,

$$\Theta_{\gamma,\lambda,p} \sim 2^{-\lambda}e^{-2\gamma}, \qquad \gamma \to \infty.$$

Proof. For $r > 0$, define

$$\phi_r(x) = \left(\frac{e^{-rx}}{1-e^{-rx}}\right)^{1/r} = (e^{rx}-1)^{-1/r}, \qquad x > 0.$$

Since $x \mapsto e^{rx} - 1$ is strictly increasing, ϕ_r is strictly decreasing on $(0,\infty)$. Hence, for every $t > 0$ and $0 < \gamma_1 < \gamma_2$,

$$\phi_q(\gamma_2+t)\phi_p(\gamma_2+t) < \phi_q(\gamma_1+t)\phi_p(\gamma_1+t).$$

Multiplying by the positive function $t^{\lambda-1}/\Gamma(\lambda)$ and integrating over $(0,\infty)$, we obtain

$$\Theta_{\gamma_2,\lambda,p} < \Theta_{\gamma_1,\lambda,p}.$$

Thus, $\Theta_{\gamma,\lambda,p}$ is strictly decreasing.

For the asymptotic behaviour, define

$$F_\gamma(t) = e^{2\gamma}\left(\frac{e^{-q(\gamma+t)}}{1-e^{-q(\gamma+t)}}\right)^{1/q}\left(\frac{e^{-p(\gamma+t)}}{1-e^{-p(\gamma+t)}}\right)^{1/p}.$$

For every fixed $t > 0$,

$$\lim_{\gamma\to\infty} F_\gamma(t) = e^{-2t},$$

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because

$$e^{\gamma}\left(\frac{e^{-q(\gamma+t)}}{1-e^{-q(\gamma+t)}}\right)^{1/q}=\frac{e^{-t}}{\left(1-e^{-q(\gamma+t)}\right)^{1/q}}\longrightarrow e^{-t},$$

and the analogous relation holds with p in place of q.

Fix $\gamma_0>0$. For $\gamma\geq\gamma_0$,

$$F_\gamma(t)\leq\frac{e^{-2t}}{\left(1-e^{-q\gamma_0}\right)^{1/q}\left(1-e^{-p\gamma_0}\right)^{1/p}}.$$

Note that the right-hand side depends on γ_0, but γ_0 is fixed throughout the limit $\gamma\to\infty$; this therefore provides a valid dominating function for the dominated convergence theorem. Hence

$$\frac{t^{\lambda-1}}{\Gamma(\lambda)}F_\gamma(t)\leq C_{\gamma_0,\lambda,p}t^{\lambda-1}e^{-2t},$$

where

$$C_{\gamma_0,\lambda,p}=\frac{1}{\Gamma(\lambda)\left(1-e^{-q\gamma_0}\right)^{1/q}\left(1-e^{-p\gamma_0}\right)^{1/p}}.$$

Since $\lambda>0$,

$$\int_0^\infty t^{\lambda-1}e^{-2t}\,dt=2^{-\lambda}\Gamma(\lambda)<\infty.$$

The dominated convergence theorem therefore gives

$$\lim_{\gamma\to\infty}e^{2\gamma}\Theta_{\gamma,\lambda,p}=\frac{1}{\Gamma(\lambda)}\int_0^\infty t^{\lambda-1}e^{-2t}\,dt=2^{-\lambda}. \tag{31}$$

This proves (30). □

Remark 7. Proposition 2 shows that increasing the exponential damping parameter strictly decreases $\Theta_{\gamma,\lambda,p}$. Moreover, the leading asymptotic term as $\gamma\to\infty$ is $2^{-\lambda}e^{-2\gamma}$, whose coefficient is independent of p and q.

Remark 8 (The best constant)**.** Let $C^*_{\gamma,\lambda,p}$ denote the sharp (smallest) constant in Theorem 1, defined by

$$C^*_{\gamma,\lambda,p}:=\sup_{\substack{a,b\geq0\\ \|a\|_p=\|b\|_q=1}}\sum_{m=1}^\infty\sum_{n=1}^\infty\frac{e^{-\gamma(m+n)}}{(m+n)^\lambda}a_mb_n.$$

Equivalently, $C^*_{\gamma,\lambda,p}=\|T_{\gamma,\lambda}\|_{\ell^q\to\ell^q}$, where the operator $T_{\gamma,\lambda}$ on ℓ^q is defined by

$$(T_{\gamma,\lambda}b)_m:=\sum_{n=1}^\infty\frac{e^{-\gamma(m+n)}}{(m+n)^\lambda}b_n,\qquad m\geq1. \tag{32}$$

By Theorem 1, $C^*_{\gamma,\lambda,p}\leq\Theta_{\gamma,\lambda,p}$, and we shall prove that the inequality is strict:

$$C^*_{\gamma,\lambda,p}<\Theta_{\gamma,\lambda,p}\qquad\text{for all } p>1,\ \gamma,\lambda>0. \tag{33}$$

Since $(m+n)^{-\lambda}\leq1$, the coordinatewise estimate

$$|(T_{\gamma,\lambda}b)_m|\leq e^{-\gamma m}\left(\sum_{n=1}^\infty e^{-p\gamma n}\right)^{1/p}\|b\|_q$$

shows that the finite-row truncations of $T_{\gamma,\lambda}$ converge to $T_{\gamma,\lambda}$ in operator norm. Indeed, denoting by P_N the projection onto the first N coordinates, we have

$$\|(I-P_N)T_{\gamma,\lambda}b\|_q\leq\left(\sum_{n\geq1}e^{-p\gamma n}\right)^{1/p}\left(\sum_{m>N}e^{-q\gamma m}\right)^{1/q}\|b\|_q\longrightarrow0$$

as $N\to\infty$. Since $P_NT_{\gamma,\lambda}$ has finite rank, $T_{\gamma,\lambda}$ is compact. As ℓ^q is reflexive, the norm of this nonzero compact operator is attained; consequently the supremum defining $C^*_{\gamma,\lambda,p}$ is attained by a pair of non-negative sequences of unit norm.

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Suppose, for contradiction, that $C^*_{\gamma,\lambda,p} = \Theta_{\gamma,\lambda,p}$. Since the supremum is attained by a pair (a^*, b^*) of non-negative unit vectors, equality must hold throughout the chain of inequalities in the proof of Theorem 1. Write the integrand in (3) as $t^{\lambda-1}A(t)B(t)$, where $A(t)$ and $B(t)$ denote the two inner sums. Since a^*, b^* are non-negative and not identically zero, both $A(t)$ and $B(t)$ are strictly positive for every $t > 0$. The integrated inequality reads

$$\int_0^\infty t^{\lambda-1}A(t)B(t)\,dt = \int_0^\infty t^{\lambda-1}X(t)Y(t)\,dt,$$

where $X(t)$ and $Y(t)$ are the corresponding upper bounds given by Hölder's inequality. Since $A(t) \le X(t)$ and $B(t) \le Y(t)$ for every $t > 0$, the equality of the two integrals forces $A(t) = X(t)$ and $B(t) = Y(t)$ for almost every $t > 0$. The equality condition in Hölder's inequality therefore requires

$$a_m = c_t\, e^{-q(\gamma+t)m/p}, \qquad m \ge 1,$$

for some $c_t > 0$ and almost every $t > 0$. Taking the ratio of the first two terms gives

$$\frac{a_2}{a_1} = e^{-q(\gamma+t)/p}.$$

The left-hand side is independent of t, while the right-hand side is strictly decreasing in t; this contradiction proves (33).

The explicit upper-bound constant $\Theta_{\gamma,\lambda,p}$ is therefore not sharp in general, and should be regarded as a computable upper bound for the sharp constant rather than the optimal constant itself.

In the symmetric case $p = q = 2$, the matrix $A = (A_{mn})$ with $A_{mn} = e^{-\gamma(m+n)}/(m+n)^\lambda$ admits an additional spectral interpretation. For finitely supported sequences $x = (x_n)$, the gamma-integral representation gives

$$\sum_{m,n\ge1} A_{mn}x_n\overline{x_m} = \frac{1}{\Gamma(\lambda)}\int_0^\infty t^{\lambda-1}\left|\sum_{n\ge1} x_n e^{-(\gamma+t)n}\right|^2 dt \ge 0,$$

so A is positive semi-definite; symmetry of A then makes it self-adjoint. Moreover, A is Hilbert–Schmidt, since

$$\sum_{m,n\ge1} |A_{mn}|^2 \le \sum_{m,n\ge1} e^{-2\gamma(m+n)} < \infty.$$

Hence A is compact and self-adjoint on ℓ^2. Since A is positive and

$$\sum_{m=1}^\infty A_{mm} = \sum_{m=1}^\infty \frac{e^{-2\gamma m}}{(2m)^\lambda} < \infty,$$

it is also trace class. The geometric-series expansion in the integral defining $\Theta_{\gamma,\lambda,2}$, followed by Tonelli's theorem, gives

$$\operatorname{tr}(A) = \sum_{m=1}^\infty \frac{e^{-2\gamma m}}{(2m)^\lambda} = \Theta_{\gamma,\lambda,2}.$$

Moreover, its leading 2×2 principal minor satisfies

$$A_{11}A_{22} - A_{12}^2 = e^{-6\gamma}\left(\frac{1}{8^\lambda} - \frac{1}{9^\lambda}\right) > 0.$$

Thus A has at least two positive eigenvalues. Since $C^*_{\gamma,\lambda,2} = \lambda_{\max}(A)$, we conclude that

$$C^*_{\gamma,\lambda,2} = \lambda_{\max}(A) < \operatorname{tr}(A) = \Theta_{\gamma,\lambda,2}.$$

Determining $C^*_{\gamma,\lambda,p}$ in closed form for general p remains open.

Proposition 3 (Asymptotic sharpness as $\gamma \to \infty$)**.** *Let $p > 1$, let $q = p/(p-1)$ be its conjugate exponent, and let $\lambda > 0$. Then, as $\gamma \to \infty$,*

$$\frac{C^*_{\gamma,\lambda,p}}{\Theta_{\gamma,\lambda,p}} = 1 + O\left(e^{-\gamma\min(p,q)}\right). \tag{34}$$

In particular,

$$\lim_{\gamma\to\infty} \frac{C^*_{\gamma,\lambda,p}}{\Theta_{\gamma,\lambda,p}} = 1.$$

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Proof. Throughout, set

$$A_{mn} := \frac{e^{-\gamma(m+n)}}{(m+n)^{\lambda}}, \qquad m,n \geq 1.$$

We estimate $C^{*}_{\gamma,\lambda,p}$ from below and from above, and then compare with the asymptotic behaviour of $\Theta_{\gamma,\lambda,p}$.

Lower bound. Since A has non-negative entries, testing the bilinear form on the unit vectors $a = e_1 \in \ell^p$ and $b = e_1 \in \ell^q$ gives

$$C^{*}_{\gamma,\lambda,p} \geq A_{11} = \frac{e^{-2\gamma}}{2^{\lambda}}. \tag{35}$$

Upper bound. For arbitrary $b \in \ell^q$, Hölder's inequality applied to the n-sum yields

$$|(T_{\gamma,\lambda} b)_m| = \left| \sum_{n=1}^{\infty} A_{mn} b_n \right| \leq \left(\sum_{n=1}^{\infty} A_{mn}^{p} \right)^{1/p} \|b\|_q,$$

where $T_{\gamma,\lambda}$ denotes the operator introduced in Remark 8. Taking the ℓ^q norm of the preceding coordinatewise estimate, we obtain

$$\|T_{\gamma,\lambda} b\|_q \leq \left(\sum_{m=1}^{\infty} \left(\sum_{n=1}^{\infty} A_{mn}^{p} \right)^{q/p} \right)^{1/q} \|b\|_q.$$

For fixed m, the inner sum has the form

$$\sum_{n=1}^{\infty} A_{mn}^{p} = \sum_{j=m+1}^{\infty} \frac{e^{-\gamma p j}}{j^{\lambda p}}.$$

Since the ratio of two consecutive terms is at most $e^{-\gamma p}$, we obtain the uniform two-sided estimate

$$\frac{e^{-\gamma p(m+1)}}{(m+1)^{\lambda p}} \leq \sum_{n=1}^{\infty} A_{mn}^{p} \leq \frac{e^{-\gamma p(m+1)}}{(m+1)^{\lambda p}} \left(1 + C_p\, e^{-\gamma p}\right),$$

where $C_p > 0$ depends only on p, and the estimate is uniform in $m \geq 1$. Raising to the power q/p and using $(1+x)^{q/p} \leq 1 + C_{p,q}\, x$ for $x \in [0,1]$, we obtain

$$\left(\sum_{n=1}^{\infty} A_{mn}^{p} \right)^{q/p} = \frac{e^{-\gamma q(m+1)}}{(m+1)^{\lambda q}} \left(1 + O(e^{-\gamma p})\right),$$

uniformly in $m \geq 1$. Summing over m and separating the leading term $m = 1$ from the remainder, we get

$$\sum_{m=1}^{\infty} \left(\sum_{n=1}^{\infty} A_{mn}^{p} \right)^{q/p} = \frac{e^{-2\gamma q}}{2^{\lambda q}} \left(1 + O(e^{-\gamma p}) + O(e^{-\gamma q})\right).$$

Indeed, after factoring out the leading term corresponding to $m = 1$, the remaining tail $\sum_{m \geq 2} e^{-\gamma q(m+1)} (m+1)^{-\lambda q}$ is bounded by a constant multiple of $e^{-\gamma q}$, relative to that leading term; all constants are independent of γ. Taking the $1/q$ power, we conclude that

$$\|T_{\gamma,\lambda}\|_{\ell^q \to \ell^q} \leq \frac{e^{-2\gamma}}{2^{\lambda}} \left(1 + O\left(e^{-\gamma \min(p,q)}\right)\right), \qquad \gamma \to \infty,$$

where p, q, λ are fixed. The preceding upper and lower bounds imply

$$C^{*}_{\gamma,\lambda,p} = \|T_{\gamma,\lambda}\|_{\ell^q \to \ell^q} = \frac{e^{-2\gamma}}{2^{\lambda}} \left(1 + O(e^{-\gamma \min(p,q)})\right), \qquad \gamma \to \infty. \tag{36}$$

Asymptotic of $\Theta_{\gamma,\lambda,p}$. For $r > 0$ and $x > 0$, the identity

$$\frac{e^{-rx}}{1 - e^{-rx}} = e^{-rx} \left(1 + O(e^{-rx})\right)$$

combined with $(1+u)^{1/r} = 1 + O(u)$ as $u \to 0^{+}$ gives

$$\left(\frac{e^{-rx}}{1 - e^{-rx}} \right)^{1/r} = e^{-x} \left(1 + O(e^{-rx})\right).$$

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Applying this with $r = q$ and $r = p$ in the integrand of (2) and using the positivity of the integrand, we obtain

$$\Theta_{\gamma,\lambda,p} = \frac{1}{\Gamma(\lambda)} \int_0^\infty t^{\lambda-1} e^{-2(\gamma+t)} \big(1 + O(e^{-q(\gamma+t)}) + O(e^{-p(\gamma+t)})\big)\, dt.$$

The main term is $\frac{1}{\Gamma(\lambda)} \int_0^\infty t^{\lambda-1} e^{-2(\gamma+t)}\, dt = e^{-2\gamma}/2^\lambda$, while the two correction terms are $O(e^{-(2+q)\gamma})$ and $O(e^{-(2+p)\gamma})$ respectively. Hence

$$\Theta_{\gamma,\lambda,p} = \frac{e^{-2\gamma}}{2^\lambda} \big(1 + O(e^{-\gamma \min(p,q)})\big), \qquad \gamma \to \infty. \tag{37}$$

Conclusion. Dividing (36) by (37) yields

$$\frac{C^*_{\gamma,\lambda,p}}{\Theta_{\gamma,\lambda,p}} = 1 + O\Big(e^{-\gamma \min(p,q)}\Big),$$

which is (34). □

Remark 9. Proposition 3 shows that, for large γ, the computable bound $\Theta_{\gamma,\lambda,p}$ is asymptotically sharp, with a relative error that decays exponentially in γ.

3 Numerical Examples

We illustrate the size and asymptotic behaviour of the explicit upper-bound constant. Numerical values in the table are rounded to six decimal places. Rounded values are reported only as approximations; the inequalities below retain the exact constant $\Theta_{\gamma,\lambda,p}$.

Example 1 (The symmetric case)**.** Let $p = q = 2$ and $\lambda = 1/2$. The series representation (7) gives

$$\Theta_{\gamma,1/2,2} = \frac{1}{\sqrt{2}} \sum_{k=1}^{\infty} \frac{e^{-2k\gamma}}{\sqrt{k}}.$$

The values in Table 1 show the decrease of $\Theta_{\gamma,1/2,2}$. Its normalized values approach $2^{-1/2}$, as proved in Proposition 2.

Table 1. The upper-bound constant and its normalized value for $p = q = 2$ and $\lambda = 1/2$.

γ	$\Theta_{\gamma,1/2,2}$	$e^{2\gamma}\Theta_{\gamma,1/2,2}$
1	0.106001	0.783249
2	0.013121	0.716404
3	0.001756	0.708349
4	0.000237	0.707275

For example, at $\gamma = 1$, Theorem 1 yields the exact bound

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{e^{-(m+n)}}{\sqrt{m+n}}\, a_m b_n \le \Theta_{1,1/2,2}\, \|a\|_2 \|b\|_2,$$

where $\Theta_{1,1/2,2} \approx 0.106001$.

Example 2 (A non-symmetric pair of exponents)**.** Let $p = 4$, $q = 4/3$, $\lambda = 1/2$, and $\gamma = 1$. Numerical evaluation of the defining integral (2) gives $\Theta_{1,1/2,4} \approx 0.114307$. Theorem 1 therefore gives the exact inequality

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{e^{-(m+n)}}{\sqrt{m+n}}\, a_m b_n \le \Theta_{1,1/2,4}\, \|a\|_4 \|b\|_{4/3}.$$

Remark 10. For $p = q = 2$, $\lambda = 1/2$, and $\gamma = 1$, take $a_m = b_m = 1/m$ and set

$$S_N := \sum_{m=1}^{N} \sum_{n=1}^{N} \frac{e^{-(m+n)}}{\sqrt{m+n}\, mn}.$$

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Direct summation gives

$$S_{20} = 0.136472166755\ldots, \qquad S_{50} = 0.136472166766\ldots, \qquad S_{100} = 0.136472166766\ldots,$$

where the ellipses indicate that the expansions continue beyond the displayed digits. The difference $S_{50} - S_{20}$ first appears at the eleventh decimal place, and the difference $S_{100} - S_{50}$ is smaller than 2×10^{-21}, as expected from the rapid exponential decay of the kernel. In particular, S_N is strictly increasing for every N, although the increase is not visible at the displayed precision. Theorem 1 gives the exact upper bound

$$S_N \le \Theta_{1,1/2,2}\,\zeta(2) \approx 0.174365.$$

For the symmetric case, the normalized quantity has the series representation

$$e^{2\gamma}\Theta_{\gamma,1/2,2} = \sum_{k=1}^{\infty} \frac{e^{-2(k-1)\gamma}}{\sqrt{2k}}.$$

Figure 1 visualizes its convergence to $2^{-1/2}$. The figure directly illustrates the asymptotic statement of Proposition 2.

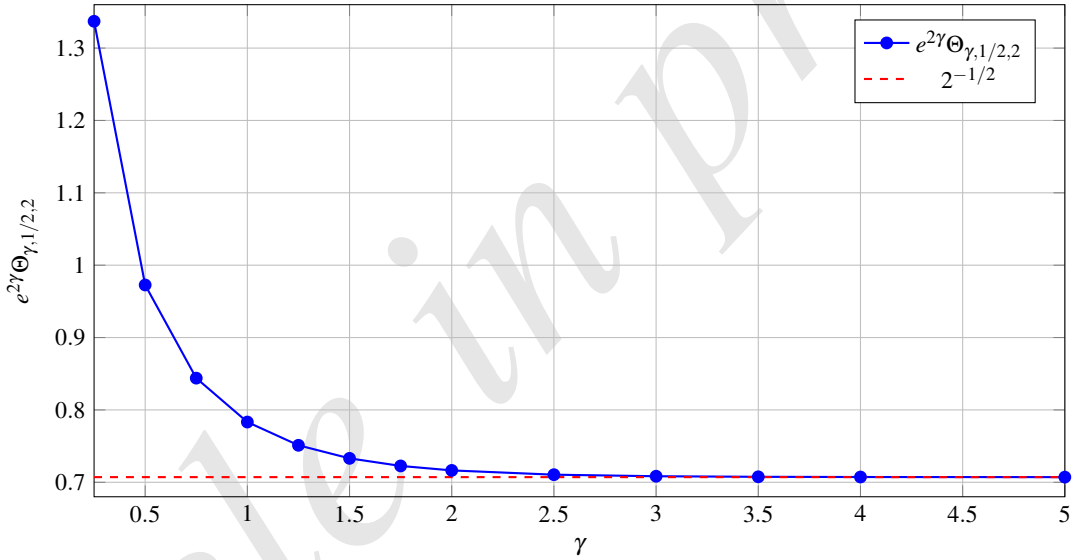


Figure 1. Convergence of the normalized upper-bound constant to $2^{-1/2}$ as γ increases.

Example 3 (A nonlinear system for the parameter λ)**.** Fix $p = q = 2$. Given a prescribed target value $\Theta_0 > 0$ for the constant of Theorem 1, we ask which parameters (γ, λ) produce $\Theta_{\gamma,\lambda,2} = \Theta_0$. This is relevant when the inequality is used as a quantitative estimate with a preassigned tolerance. A prescribed value of $\Theta_{\gamma,\lambda,2}$ alone leaves two parameters to determine. To give a concrete parameter-calibration example, we restrict attention to the illustrative family $\gamma = 2\lambda$. This restriction is a modelling choice, not a consequence of the inequality. We therefore consider the system

$$\begin{cases} \Theta_{\gamma,\lambda,2} = \Theta_0, \\ \gamma = 2\lambda. \end{cases}$$

Substituting the second equation into the first reduces the system to the scalar equation $h(\lambda) = \Theta_0$ below. We use $\Theta_0 = 0.05$ solely as a representative numerical target. By the series representation (9), this equation reduces to

$$h(\lambda) := \Theta_{2\lambda,\lambda,2} = \sum_{k=1}^{\infty} \frac{e^{-4k\lambda}}{(2k)^{\lambda}} = \Theta_0. \tag{38}$$

Each term of this series, namely $e^{-\lambda[4k+\log(2k)]}$, is strictly decreasing in λ; hence h, as the sum of strictly decreasing positive terms, is strictly decreasing on $(0,\infty)$. For continuity, note that on every compact interval $[a,b] \subset (0,\infty)$, the k-th term is bounded by e^{-4ka}, and the

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dominating series $\sum_{k\geq 1} e^{-4ka}$ converges; by the Weierstrass M-test, h is the uniform limit of continuous functions, hence continuous on $(0,\infty)$. Finally, $h(\lambda)\to\infty$ as $\lambda\downarrow 0$ because each term tends to 1, and $h(\lambda)\to 0$ as $\lambda\to\infty$ by dominated convergence, since for $\lambda\geq 1$ every term is bounded by $(2k)^{-1}e^{-4k}$. The intermediate value theorem therefore gives existence of a solution, and the strict monotonicity of h gives uniqueness, so (38) has exactly one solution for every $\Theta_0>0$.

Since h is defined by the series (38), no closed form is available; the equation must be solved numerically. We use Newton's method [14, 15], with the derivative h' evaluated by term-by-term differentiation of the series (38); starting from $\lambda^{(0)}=0.5$ and taking $\Theta_0=0.05$, the iterations converge quadratically to

$$\lambda^*\approx 0.648819, \qquad \gamma^*=2\lambda^*\approx 1.297639, \tag{39}$$

as reported in Table 2.

Table 2. Newton iterations for (38) with $\Theta_0=0.05$.

k	$\gamma^{(k)}$	$\lambda^{(k)}$	$\lvert F\rvert$
1	1.203068	0.601534	1.32×10^{-2}
2	1.286802	0.643401	1.36×10^{-3}
3	1.297488	0.648744	1.86×10^{-5}
4	1.297639	0.648819	3.62×10^{-9}
5	1.297639	0.648819	1.38×10^{-16}

Direct evaluation of (38) at λ^* confirms that $\Theta_{2\lambda^*,\lambda^*,2}\approx 0.05$ to the precision shown, so λ^* solves (38) within numerical tolerance.

4 Conclusion

We established a discrete Hardy–Hilbert-type inequality for the exponential-power kernel, together with a discrete Hardy-type inequality (Theorem 2) and an integrated exponential-kernel inequality (Theorem 3). We also determined the dependence of the explicit upper-bound constant $\Theta_{\gamma,\lambda,p}$ in (2) on the damping parameter. We close with the following remarks.

(a) **Sharpness of the explicit constant.** As shown in Remark 8, the explicit bound is not sharp at any fixed admissible parameter choice: if $C^*_{\gamma,\lambda,p}$ denotes the smallest admissible constant, then

$$C^*_{\gamma,\lambda,p}<\Theta_{\gamma,\lambda,p} \qquad (p>1,\ \gamma,\lambda>0). \tag{40}$$

Nevertheless, Proposition 3 shows that

$$\frac{C^*_{\gamma,\lambda,p}}{\Theta_{\gamma,\lambda,p}}\longrightarrow 1 \qquad \text{as } \gamma\to\infty,$$

with a relative error of order $O(e^{-\gamma\min(p,q)})$. In the symmetric case $p=q=2$, the best constant also has a spectral characterization as the largest eigenvalue of a positive compact operator.

(b) **Weighted sequence spaces.** A natural extension is to weighted sequence spaces, where the kernel structure suggests that a gamma-integral representation of the constant should carry over, provided the weights satisfy suitable growth conditions. Identifying the class of weights for which the constant remains finite is an interesting problem for future work.

(c) **Discrete versus continuous.** The present results do not establish that the discrete inequality implies its continuous counterpart, or conversely; a precise comparison requires a separate discretization or limiting argument. For $0<\lambda<1$, the elementary inequality $e^{-x}/(1-e^{-x})\leq 1/x$ gives the upper bound

$$\Theta_{\gamma,\lambda,p}\leq p^{-1/p}q^{-1/q}\,\Gamma(1-\lambda)\,\gamma^{\lambda-1},$$

which coincides with the leading asymptotic term in Proposition 1 (since $\pi/(\Gamma(\lambda)\sin(\pi\lambda))=\Gamma(1-\lambda)$) and matches the continuous constant of Chesneau [12]. This shows that, in the regime $0<\lambda<1$, the discrete bound is consistent with the continuous one up to the constant factor. Whether the discrete inequality implies the continuous one (or conversely) remains open.

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(d) **Further directions.** Further questions concern multidimensional sums, more general exponentially damped kernels, and quantitative bounds for the best constant in the small-γ regime.

Authors' Contributions

A. Azizi developed the main results, carried out the proofs, and prepared the manuscript. M. T. Heydari contributed to the formulation of the problem, the mathematical analysis, and the revision of the manuscript. Both authors read and approved the final version of the manuscript.

Data Availability

Not applicable. This study does not involve the generation or analysis of datasets.

Code Availability

No standalone computer code is provided for this study. The numerical results were obtained using the series representations and Newton's method described in the manuscript.

Conflicts of Interest

The authors declare that there is no conflict of interest.

Ethical Considerations

Not applicable. This research is entirely theoretical and does not involve human participants, animals, or personal data.

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