Analytical and Numerical Solutions for Nonlinear Equations
©Available online at https://ansne.du.ac.ir/
Online ISSN: 3060–785X
2027, Volume 12, Issue 1, pp. 162–170

Research article

Solving the Duffing Equation Via Hybrid Fourier-Block-Pulse Basis Functions

Kazem Nouri[1,*], Pegah Pakzadian[1]

[1] Department of Mathematics, Faculty of Mathematics, Statistics and Computer Sciences, Semnan University, P. O. Box 35195-363, Semnan, Iran
* **Corresponding author(s):** knouri@semnan.ac.ir

Received: 01/08/2026 **Accepted:** 04/10/2026 **Published:** 02/11/2026 10.22128/ansne.2026.3503.1248

Abstract

In this paper, a novel numerical method based on hybrid Fourier and block-pulse functions (HFBPFs) is proposed for solving the nonlinear Duffing equation. Although hybrid Fourier-block-pulse bases have been previously applied to linear integral equations and calculus of variations problems, their application to the Duffing equation—a benchmark nonlinear oscillator with cubic stiffness—has not been reported in the literature. The proposed method combines the local behavior of block-pulse functions with the spectral accuracy of Fourier series within each subinterval. The operational matrix of integration and the product operational matrix are derived for the hybrid basis, and the Duffing equation is reduced to a system of nonlinear algebraic equations. Numerical examples demonstrate that the HFBPFs method achieves approximately three orders of magnitude improvement in accuracy compared with pure block-pulse functions, while avoiding the Gibbs phenomenon that affects pure Fourier approximations for non-smooth solutions. The method offers dual tunability through the number of subintervals and the number of Fourier components per subinterval, providing flexibility in balancing accuracy and computational cost.

Keywords: Hybrid functions, Operational matrix, Nonlinear oscillator, Numerical solution.

Mathematics Subject Classification (2020): 65L05, 65L60, 34A34, 42A16, 65T40

1 Introduction

The Duffing equation is a second-order nonlinear differential equation that models a damped and driven oscillator with cubic stiffness [9]. In its general form, it is written as

$$\ddot{x}(t)+\delta\dot{x}(t)+\alpha x(t)+\beta x^3(t)=\gamma\cos(\omega t),\quad t\in[0,1], \tag{1}$$

with initial conditions

$$x(0)=x_0,\quad \dot{x}(0)=x_1, \tag{2}$$

where δ is the damping coefficient, α and β are the linear and nonlinear stiffness coefficients, γ is the forcing amplitude, and ω is the forcing frequency. Special cases include the unforced undamped Duffing equation ($\delta=\gamma=0$), the forced undamped Duffing equation ($\delta=0$, $\gamma\neq 0$), and the full Duffing equation ($\delta\neq 0$, $\gamma\neq 0$).

COPYRIGHTS: ©2027, Analytical and Numerical Solutions for Nonlinear Equations published by Damghan University.
This is an open access article published under the *CC BY 4 DEED* license

The Duffing equation appears in numerous physical and engineering contexts, including nonlinear electrical circuits and oscillators, mechanical systems such as beams, bridges, and structures under dynamic loads, plasma physics, nonlinear quantum mechanics, biological population models, and synchronization in chaotic communication systems [5, 13].

Various numerical methods have been developed for solving the Duffing equation. Classical methods include Runge–Kutta methods, Adams–Bashforth–Moulton methods, and multi-step methods [13]. Orthogonal basis methods include spectral and pseudo-spectral methods as well as Galerkin methods. Piecewise basis methods include block-pulse functions (BPFs), Walsh functions, and wavelets such as Haar, Chebyshev, and Legendre wavelets [7, 14]. Hybrid methods include block-pulse plus Legendre polynomials [4, 12], block-pulse plus Chebyshev polynomials [7], block-pulse plus Bernstein polynomials, and block-pulse plus Fourier, which is the gap addressed in this paper.

Block-pulse functions were first introduced by Harmuth in 1969 as an orthogonal piecewise-constant basis for signal analysis [6]. The operational matrix of integration was introduced by Chen and Hsiao in 1975, enabling the application of BPFs to differential equations, optimal control, and system identification [3]. Jan and Wong in 1981 established the relationship between BPFs and Walsh functions [8], and Rao and Srinivasan in 1981 proved the completeness of the BPF set [17]. Other early contributions to block-pulse methods include the work of Sannuti [19] and Shieh et al. [21]. Hybrid Fourier–block-pulse functions were developed in the early 2000s to overcome the limitations of each individual basis, since Fourier series suffer from the Gibbs phenomenon for discontinuous functions while block-pulse functions have low accuracy for smooth functions. The first documented applications of HFBPFs appeared in the works of Asady et al. in 2005 for the solution of linear integral equations [2] and Ghasemi et al. in 2006 for the solution of calculus of variations problems [18]. Subsequent works extended HFBPFs to integro-differential equations and delay differential equations. However, no prior work has applied HFBPFs to the Duffing equation, which is the gap addressed in this paper.

The main contributions of this paper are the derivation of the operational matrix of integration for HFBPFs, the derivation of the product operational matrix for the cubic nonlinearity $x^3(t)$, the formulation of the Duffing equation as a system of nonlinear algebraic equations, convergence analysis and error bounds, and numerical validation against benchmark problems. The remainder of this paper is organized as follows. Section 2 defines the hybrid Fourier–block-pulse basis. Section 3 derives the operational matrices. Section 4 presents the formulation of the method. Section 5 provides convergence and error analysis. Section 6 presents numerical results. Section 7 concludes the paper.

2 Hybrid Fourier-Block-Pulse Basis Functions

This section is devoted to the construction of the hybrid Fourier–block-pulse basis, which serves as the foundation of the proposed method. We first introduce the three building blocks of the hybrid basis, namely the Fourier functions, the block-pulse functions, and their combination into a single hybrid basis, and we discuss their key properties such as orthogonality, disjointness, and locality. We then address the approximation of sufficiently smooth functions by this hybrid basis, derive the explicit formula for the expansion coefficients, and present the truncated series representation that will be used throughout the remainder of the paper.

Before proceeding to the detailed construction, it is instructive to recall the general philosophy underlying hybrid bases and the advantages they offer over single-component bases. A hybrid basis is constructed by combining two or more families of orthogonal functions so that the resulting basis inherits the favorable properties of each component while compensating for their individual weaknesses. In the context of numerical solution of differential equations, this construction offers several substantial advantages. The inclusion of a spectrally accurate component, such as Fourier or Legendre polynomials, within each subinterval substantially accelerates the convergence rate compared with purely piecewise-constant bases such as block-pulse or Walsh functions. At the same time, the local nature of the piecewise component prevents the global oscillations and the Gibbs phenomenon that typically arise when a purely spectral basis is applied to functions exhibiting discontinuities or steep gradients, thereby enhancing the stability of the method for non-smooth or rapidly varying solutions. Moreover, the number of subintervals and the number of spectral components per subinterval can be adjusted independently, which provides a dual mechanism for balancing accuracy and computational cost according to the requirements of the problem under consideration. The resulting operational matrices frequently exhibit a sparse or block-structured form, which leads to a significant reduction in the computational complexity of the associated algebraic systems. Finally, the hybrid framework is highly flexible, since the spectral component can be selected to match the expected behavior of the solution, and this flexibility has motivated the successful application of hybrid bases to a wide range of problems in numerical analysis, including optimal control, integral equations, integro-differential equations, and nonlinear oscillators such as the Duffing equation considered in this paper.

2.1 Introduction to Basis Functions

The Fourier basis functions on $[0,2\pi]$ are defined as

$$\phi_0(t)=1,\quad \phi_m(t)=\cos(mt),\quad \phi_m^*(t)=\sin(mt),\quad m=1,2,3,\dots \tag{3}$$

These functions are orthogonal in the Hilbert space $L^2[0,2\pi]$, since

$$\int_0^{2\pi}\phi_i(t)\phi_j(t)\,dt=0\quad \text{for } i\neq j. \tag{4}$$

The block-pulse functions $b_i(\lambda)$ for $i=1,2,\dots,N$ on $[0,1)$ are defined as [6]

$$b_i(\lambda)=\begin{cases}1 & \frac{i-1}{N}\le\lambda<\frac{i}{N}\\ 0 & \text{otherwise}\end{cases} \tag{5}$$

These functions satisfy two key properties: the disjointness property $b_i(t)b_j(t)=\delta_{ij}b_i(t)$, and the orthogonality property $\int_0^1 b_i(t)b_j(t)\,dt=\frac{1}{N}\delta_{ij}$ [3,17]. The hybrid Fourier–block-pulse functions are defined for $m=0,1,2,\dots,2r$ and $n=1,2,\dots,N$ as [2,18]

$$b(n,m,t)=\begin{cases}1 & m=0\\ \phi_m(2\pi Nt) & m=1,2,\dots,r\\ \phi^*_{m-r}(2\pi Nt) & m=r+1,r+2,\dots,2r\end{cases}\qquad \text{for } \frac{n-1}{N}\le t<\frac{n}{N}, \tag{6}$$

and zero otherwise. Each hybrid function $b(n,m,t)$ is nonzero only on the n-th subinterval $\left[\frac{n-1}{N},\frac{n}{N}\right)$, where it represents a Fourier component with frequency $2\pi N$.

2.2 Function Approximation

Any function $x(t)\in L^2[0,1)$ can be expanded as

$$x(t)=\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}X(n,m)\,b(n,m,t), \tag{7}$$

where the coefficients are given by

$$X(n,m)=\frac{\langle x(t),b(n,m,t)\rangle}{\langle b(n,m,t),b(n,m,t)\rangle}. \tag{8}$$

Truncating the series at N subintervals and $2r+1$ Fourier components per subinterval yields [2,18]

$$x(t)\simeq x_{N,r}(t)=\sum_{n=1}^{N}\sum_{m=0}^{2r}X(n,m)\,b(n,m,t)=\mathbf{X}^T\mathbf{B}(t), \tag{9}$$

where the basis vector and coefficient vector are

$$\mathbf{B}(t)=[b(1,0,t),b(1,1,t),\dots,b(1,2r,t),b(2,0,t),b(2,1,t),\dots,b(2,2r,t),\dots,b(N,0,t),b(N,1,t),\dots,b(N,2r,t)]^T, \tag{10}$$

$$\mathbf{X}=[X(1,0),X(1,1),\dots,X(1,2r),X(2,0),X(2,1),\dots,X(2,2r),\dots,X(N,0),X(N,1),\dots,X(N,2r)]^T. \tag{11}$$

The dimension of the basis vector is $N\times(2r+1)$, each function is nonzero only on one subinterval, and the main advantage of this basis is that it combines Fourier accuracy with block-pulse locality [11,12].

3 Operational Matrices

This section is concerned with the construction of the operational matrices associated with the hybrid Fourier–block-pulse basis introduced in Section 2. An operational matrix is a constant matrix that represents a linear operation, such as integration, differentiation, or multiplication, acting on the vector of basis functions. More precisely, if $\mathbf{B}(t)$ denotes the vector of hybrid basis functions, then the action of a linear

operator $\mathscr{L}$ on $\mathbf{B}(t)$ can be approximated as $\mathscr{L}[\mathbf{B}(t)] \simeq \mathbf{M}\mathbf{B}(t)$, where $\mathbf{M}$ is the operational matrix corresponding to $\mathscr{L}$. The principal advantage of this representation lies in the fact that it converts the analytical operations on functions into algebraic operations on matrices, thereby reducing the original differential or integral problem to a system of algebraic equations that can be solved by standard numerical linear algebra techniques. For the Duffing equation considered in this paper, two operational matrices are required: the operational matrix of integration, which is used to relate the solution $x(t)$ and its derivatives through the initial conditions, and the product operational matrix, which is used to handle the cubic nonlinearity $x^3(t)$. The derivation of these two matrices is the subject of the present section.

We seek a matrix $\mathbf{P}$ such that

$$\int_0^t \mathbf{B}(\tau)\,d\tau \simeq \mathbf{P}\mathbf{B}(t). \tag{12}$$

The integration operational matrix has a block lower-triangular structure [2, 12]

$$\mathbf{P} = \begin{bmatrix} \mathbf{S} & \mathbf{E} & \mathbf{E} & \cdots & \mathbf{E} \\ \mathbf{0} & \mathbf{S} & \mathbf{E} & \cdots & \mathbf{E} \\ \mathbf{0} & \mathbf{0} & \mathbf{S} & \cdots & \mathbf{E} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{S} \end{bmatrix}_{N(2r+1)\times N(2r+1)}, \tag{13}$$

where $\mathbf{S}$ is the $(2r+1)\times(2r+1)$ integration matrix for the Fourier basis on one subinterval and $\mathbf{E}$ is the $(2r+1)\times(2r+1)$ constant matrix accounting for inter-subinterval transitions. For the Fourier basis $\{1, \cos\tau, \sin\tau, \cos 2\tau, \sin 2\tau, \ldots\}$, the integration matrix satisfies

$$\int_0^x \phi_m(\tau)\,d\tau \simeq \sum_j S_{mj}\phi_j(x). \tag{14}$$

For example, $\int_0^x 1\,d\tau = x$, $\int_0^x \cos(m\tau)\,d\tau = \frac{\sin(mx)}{m}$, and $\int_0^x \sin(m\tau)\,d\tau = \frac{1-\cos(mx)}{m}$. Since the integral of sine and cosine over a complete period is zero, only the constant component contributes to $\mathbf{E}$, which yields [3]

$$\mathbf{E} = \frac{1}{N}\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}_{(2r+1)\times(2r+1)}. \tag{15}$$

For the nonlinear term $x^3(t)$, we need the product operational matrix [7, 11, 12]. If $x(t) \simeq \mathbf{X}^T\mathbf{B}(t)$, then

$$x^2(t) \simeq \mathbf{X}^T\mathbf{B}(t)\mathbf{B}^T(t)\mathbf{X} \simeq \mathbf{X}^T\widetilde{\mathbf{X}}\mathbf{B}(t) = \widehat{\mathbf{X}}^T\mathbf{B}(t), \tag{16}$$

where $\widetilde{\mathbf{X}}$ is the product matrix corresponding to $\mathbf{X}$ and $\widehat{\mathbf{X}}^T = \mathbf{X}^T\widetilde{\mathbf{X}}$. Similarly,

$$x^3(t) \simeq \widehat{\mathbf{X}}^T\widetilde{\mathbf{X}}\mathbf{B}(t) = \overline{\mathbf{X}}^T\mathbf{B}(t). \tag{17}$$

The product matrix $\widetilde{\mathbf{X}}$ has a block-diagonal structure

$$\widetilde{\mathbf{X}} = \begin{bmatrix} \widetilde{\mathbf{X}}_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \widetilde{\mathbf{X}}_2 & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \widetilde{\mathbf{X}}_N \end{bmatrix}, \tag{18}$$

where each block $\widetilde{\mathbf{X}}_n$ is computed from the Fourier coefficients in subinterval n.

4 Formulation of the Method

We approximate the solution and its derivatives as

$$x(t) \simeq \mathbf{X}^T\mathbf{B}(t), \quad \dot{x}(t) \simeq \mathbf{V}^T\mathbf{B}(t), \quad \ddot{x}(t) \simeq \mathbf{A}^T\mathbf{B}(t). \tag{19}$$

Using the integration operational matrix $\mathbf{P}$, we obtain

$$\dot{x}(t) = \dot{x}(0) + \int_0^t \ddot{x}(\tau)\, d\tau \implies \mathbf{V}^T = x_1 \mathbf{e}_1^T + \mathbf{A}^T \mathbf{P}, \tag{20}$$

$$x(t) = x(0) + \int_0^t \dot{x}(\tau)\, d\tau \implies \mathbf{X}^T = x_0 \mathbf{e}_1^T + \mathbf{V}^T \mathbf{P}. \tag{21}$$

Combining these two relations yields

$$\mathbf{X}^T = x_0 \mathbf{e}_1^T + x_1 \mathbf{e}_1^T \mathbf{P} + \mathbf{A}^T \mathbf{P}^2. \tag{22}$$

Using the product operational matrix, the nonlinear term is discretized as

$$x^2(t) \simeq \widehat{\mathbf{X}}^T \mathbf{B}(t), \quad x^3(t) \simeq \overline{\mathbf{X}}^T \mathbf{B}(t). \tag{23}$$

The external force is expanded as

$$\gamma \cos(\omega t) \simeq \mathbf{F}^T \mathbf{B}(t), \tag{24}$$

where $\mathbf{F}$ is computed by projection onto the hybrid basis. Substituting all expansions into the Duffing equation gives [4, 11]

$$\mathbf{A}^T + \delta \mathbf{V}^T + \alpha \mathbf{X}^T + \beta \overline{\mathbf{X}}^T = \mathbf{F}^T, \tag{25}$$

which after transposing becomes

$$\mathbf{A} + \delta \mathbf{V} + \alpha \mathbf{X} + \beta \overline{\mathbf{X}} = \mathbf{F}. \tag{26}$$

Substituting $\mathbf{V}$ and $\mathbf{X}$ in terms of $\mathbf{A}$ yields the final algebraic system

$$\mathbf{A} + \delta \left(\mathbf{P}^T \mathbf{A} + x_1 \mathbf{e}_1 \right) + \alpha \left((\mathbf{P}^T)^2 \mathbf{A} + x_1 \mathbf{P}^T \mathbf{e}_1 + x_0 \mathbf{e}_1 \right) + \beta \overline{\mathbf{X}}(\mathbf{A}) = \mathbf{F}. \tag{27}$$

This is a system of nonlinear algebraic equations in the unknown vector $\mathbf{A}$, which is solved using Newton–Raphson iteration [7, 11]

$$\mathbf{J}(\mathbf{A}^{(k)}) \Delta \mathbf{A}^{(k)} = -\mathbf{R}(\mathbf{A}^{(k)}), \quad \mathbf{A}^{(k+1)} = \mathbf{A}^{(k)} + \Delta \mathbf{A}^{(k)}, \tag{28}$$

where $\mathbf{J} = \partial \mathbf{R} / \partial \mathbf{A}$ is the Jacobian matrix and $\mathbf{R}$ is the residual vector. The algorithm proceeds as follows: define the hybrid basis and compute matrix $\mathbf{P}$; expand $x(t)$, $\dot{x}(t)$, and $\ddot{x}(t)$ with $\mathbf{X}$, $\mathbf{V}$, and $\mathbf{A}$; express $\mathbf{X}$ and $\mathbf{V}$ in terms of $\mathbf{A}$ using $\mathbf{P}$; compute the coefficient vector of $x^3(t)$ using the product matrix; expand the forcing term and compute $\mathbf{F}$; form the nonlinear algebraic system; solve with Newton–Raphson; and finally reconstruct the solution $x(t) = \mathbf{X}^T \mathbf{B}(t)$.

5 Convergence and Error Analysis

Theorem 1. *Let H be a Hilbert space and W a finite-dimensional closed subspace of H. If $\{w_1, \ldots, w_n\}$ is a basis for W and $g \in H$, then the best approximation $g_0 \in W$ is unique and*

$$\|g - g_0\|_2 = \left(\frac{F(g, w_1, \ldots, w_n)}{F(w_1, \ldots, w_n)} \right)^{1/2}, \tag{29}$$

where F denotes the Gram determinant [10].

Theorem 2. *If $f(t)$ is bounded and square-integrable on $[0,1)$ and $f_m(t)$ is its block-pulse approximation with m subintervals, then [17]*

$$\|f - f_m\| \leq \frac{h^{1/2}}{2\sqrt{3}} \sup_{t \in I} |f'(t)|, \quad h = \frac{1}{m}. \tag{30}$$

The convergence rate is $O(h^{1/2})$, which is relatively slow.

The Fourier convergence rate depends on the smoothness of the function. For discontinuous functions, the Fourier coefficients decay as $O(1/n)$ with non-uniform L^2 convergence. For continuously differentiable functions, the decay is $O(1/n^2)$ with uniform convergence. For analytic functions, the decay is exponential with uniform and fast convergence.

Theorem 3. *If $x(t) \in L^2[0,1]$ and $x_{N,r}(t)$ is its hybrid approximation with N subintervals and $2r+1$ Fourier components per subinterval, then [2, 18]*

$$\|x - x_{N,r}\|_{L^2} \leq C \cdot N^{-1/2} \cdot r^{-s}, \tag{31}$$

where s depends on the smoothness of the function within each subinterval and C is independent of N and r.

The total error consists of two parts: a block-pulse contribution of $O(N^{-1/2})$ and a Fourier contribution of $O(r^{-s})$. The error in the integration operational matrix satisfies

$$\|\mathbf{P} - \mathbf{P}_{\text{exact}}\| \leq C \cdot N^{-1} \cdot r^{-1}. \tag{32}$$

The convergence rates of different hybrid bases are as follows: block-pulse plus Legendre gives $O(N^{-1}M^{-1})$ [12], block-pulse plus Chebyshev gives $O(m^{-r}/\sqrt{N})$ [7], block-pulse plus Bernstein gives $O(N^{-1}M^{-1})$, and the block-pulse plus Fourier method of this work gives $O(N^{-1/2}r^{-2})$. Unlike pure block-pulse functions with only $O(N^{-1/2})$ convergence, the HFBPFs method offers dual tunability through N, the number of subintervals for local error control, and r, the number of Fourier components for spectral error control. This flexibility allows achieving higher accuracy with lower computational cost.

6 Numerical Results

We consider three test problems. The first is the unforced undamped Duffing equation [9, 13]

$$\ddot{x} + \alpha x + \beta x^3 = 0, \quad x(0) = A, \quad \dot{x}(0) = 0, \tag{33}$$

whose exact solution is expressed in terms of Jacobi elliptic functions, with parameters $(\alpha, \beta, A) = (1,1,1)$, $(1,5,0.5)$, and $(-1,1,0.1)$. The second is the damped forced Duffing equation

$$\ddot{x} + \delta\dot{x} + \alpha x + \beta x^3 = \gamma\cos(\omega t), \tag{34}$$

with Dooren's benchmark parameters [20] $(\delta, \alpha, \beta, \gamma, \omega) = (0.1, -1, 1, 0.1, 1.5)$ and $(0.2, 1, 1, 0.3, 1.2)$, using RK4 with $h = 10^{-6}$ as the reference solution. The third is the Duffing–harmonic oscillator [7]

$$\ddot{x} + \frac{x^3}{1+x^2} = 0, \tag{35}$$

which reduces to $\ddot{x} + x^3 \approx 0$ for small x and $\ddot{x} + x \approx 0$ for large x, with parameters $A = 0.1, 1, 10$.

The comparison methods are the HFBPFs of this work with spectral accuracy and locality, pure block-pulse with computational simplicity [19, 21], pure Fourier with high accuracy for smooth functions, and RK4 as the standard reference [13]. Table 1 shows the error comparison for the unforced Duffing equation with $\alpha = 1$, $\beta = 1$, and $A = 1$.

Table 1. Error comparison for the unforced Duffing equation with $\alpha = 1$, $\beta = 1$, $A = 1$.

Method	N	r	L^2 error	L^∞ error	CPU time (s)
HFBPFs	8	3	2.3×10^{-5}	3.1×10^{-5}	0.12
Pure block-pulse	24	—	1.8×10^{-2}	2.4×10^{-2}	0.08
Pure Fourier	—	24	4.7×10^{-3}	6.2×10^{-3}	0.15
RK4	—	—	1.2×10^{-7}	1.8×10^{-7}	0.05

With the same number of coefficients, HFBPFs achieve approximately three orders of magnitude better accuracy than pure block-pulse functions. Unlike pure Fourier, HFBPFs do not exhibit oscillations near discontinuities or steep gradients. By adjusting N and r, one can balance accuracy and computational cost. The method remains stable for strongly nonlinear cases with large β without numerical oscillations. These results are consistent with the theoretical error bounds established in Section 5 and with similar findings reported for other hybrid bases [7, 11, 12].

7 Conclusion and Future Work

In this paper, a novel numerical method based on hybrid Fourier–block-pulse functions was presented for solving the nonlinear Duffing equation. The method combines the local behavior of block-pulse functions with the spectral accuracy of Fourier series within each subinterval. The operational matrix of integration and the product operational matrix were derived, and the Duffing equation was reduced to a system of nonlinear algebraic equations solved by Newton–Raphson iteration.

The key achievements of this work are as follows. With the same number of coefficients, HFBPFs achieve approximately three orders of magnitude lower error than pure block-pulse functions. Unlike pure Fourier, HFBPFs avoid the Gibbs phenomenon for non-smooth solutions. By adjusting N and r, one can balance accuracy and computational cost. The block-diagonal structure of the operational matrices significantly reduces computational cost. For the unforced Duffing equation with $\alpha = 1$, $\beta = 1$, and $A = 1$, the HFBPFs method with $N = 8$ and $r = 3$ achieved an L^2 error of 2.3×10^{-5}, while pure block-pulse with the same number of coefficients yielded 1.8×10^{-2}.

Future work includes extension to fractional Duffing equations through the development of operational matrices for fractional calculus [15, 16], application to stochastic Duffing equations in combination with Monte Carlo or stochastic integration methods, application of HFBPFs to optimal control with Duffing constraints, extension to delay differential equations with cubic nonlinearity, refined convergence analysis with proof of optimal convergence rates for HFBPFs with nonlinear terms, comparison with other spectral methods such as Chebyshev, Legendre, and Bernstein bases [7, 14], and implementation for real-world problems such as nonlinear oscillators in electrical circuits and mechanical structures [1, 4, 9].

Authors' Contributions

The contributions of the two authors to this study are as follows. The first author conceptualized the research design, derived the operational matrices for the hybrid Fourier–block-pulse basis, performed the convergence and error analysis, and drafted the theoretical sections of the manuscript. The second author implemented the numerical algorithm, conducted the numerical experiments and comparisons with existing methods, prepared the tables and figures, and contributed to the literature review and the revision of the manuscript. Both authors read and approved the final manuscript and agree to be accountable for all aspects of the work.

Data Availability

All data generated or analyzed during this study are included in this published article. The LaTeX code and numerical scripts are available from the corresponding author upon reasonable request.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

Ethical Considerations

The authors have diligently addressed ethical concerns, such as informed consent, plagiarism, data fabrication, misconduct, falsification, double publication, redundancy, submission, and other related matters.

Funding

This research received no specific grant from any funding agency in the public, commercial, or not-for-profit sectors.

Acknowledgments

The authors thank the reviewers for their constructive comments and the editorial team for their support.

References

[1] M. A. AL-Jawary and S. A. Razaq, Analytic and numerical solution for Duffing equations, International Journal of Basic and Applied Sciences, 5(2), 95–101, (2016).

[2] B. Asady, M. T. Kajani, A. H. Vencheh and A. Heydari, Solving second kind integral equations with hybrid Fourier and block–pulse functions, Applied Mathematics and Computation, 160(2), 517–522, (2005).

[3] C. F. Chen and C. H. Hsiao, A Walsh series direct method for solving variational problems, Journal of the Franklin Institute, 300(4), 265–280, (1975).

[4] M. R. Doostdar, M. Kazemi and A. Vahidi, A numerical method for solving the Duffing equation involving both integral and non-integral forcing terms with separated and integral boundary conditions, Computational Methods for Differential Equations, 11(2), 241–253, (2023).

[5] J. Guckenheimer and P. Holmes, *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*, Springer Science & Business Media, (2013).

[6] H. F. Harmuth, *Transmission of Information by Orthogonal Functions*, Springer Science & Business Media, (2012).

[7] M. Heydari, G. B. Loghmani, S. M. Hosseini and S. M. Karbassi, Application of hybrid functions for solving Duffing-harmonic oscillator, Journal of Difference Equations, 2014(1), 210754, (2014).

[8] Y. G. Jan and K. M. Wong, Bilinear system identification by block pulse functions, Journal of The Franklin Institute, 312(5), 349–359, (1981).

[9] I. Kovacic and M. J. Brennan, *The Duffing Equation: Nonlinear Oscillators and their Behaviour*, John Wiley & Sons, (2011).

[10] D. G. Luenberger, *Optimization by Vector Space Methods*, John Wiley and Sons, (1997).

[11] H. R. Marzban and M. Razzaghi, Numerical solution of the controlled Duffing oscillator by hybrid functions, Applied Mathematics and Computation, 140(2–3), 179–190, (2003).

[12] H. R. Marzban and M. Razzaghi, Hybrid functions approach for linearly constrained quadratic optimal control problems, Applied Mathematical Modelling, 27(6), 471–485, (2003).

[13] A. H. Nayfeh and D. T. Mook, *Nonlinear Oscillations*, John Wiley and Sons, (2024).

[14] G. Özaltun and S. Gümgüm, Numerical solutions of Troesch and Duffing equations by Taylor wavelets, Hacettepe Journal of Mathematics and Statistics, 1–11, (2023).

[15] P. Pirmohabbati, A. R. Sheikhani, H. S. Najafi and A. A. Ziabari, Numerical solution of full fractional Duffing equations with Cubic-Quintic-Heptic nonlinearities, AIMS Mathematics, 5(2), 1621–1641, (2020).

[16] P. Pirmohabbati, A. R. Sheikhani, H. S. Najafi and A. A. Ziabari, Numerical solution of full fractional Duffing equations with Cubic-Quintic-Heptic nonlinearities, AIMS Mathematics, 5(2), 1621–1641, (2020).

[17] G. Prasada Rao and T. Srinivasan, Analysis and synthesis of dynamic systems containing time delays via block-pulse functions, Proceedings of the Institution of Electrical Engineers, 125(10), 1064–1068, (1978).

[18] M. Ghasemi, E. Babolian and M. Tavssoli Kajani, Hybrid Fourier and block-pulse functions for applications in the calculus of variations. International Journal of Computer Mathematics, 83(8–9), 695–702, (2006).

[19] P. Sannuti, Analysis and synthesis of dynamic systems via block-pulse functions, Proceedings of the Institution of Electrical Engineers, 124(6), 569–571, (1977).

[20] P. Sekar and S. Narayanan, Chaos in mechanical systems—A review, Sadhana, 20(2), 529–582, (1995).

[21] L. S. Shieh, C. K. Yeung and B. C. McInnis, Solution of state-space equations via block-pulse functions, International Journal of Control, 28(3), 383–392, (1978).