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Structural Analysis of General Fuzzy Machines Via Fixed-Point Theory: Stability and Sensitivity

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Abstract

This paper introduces the general fuzzy machine (GFM) as a principled framework for modeling and analyzing fuzzy automata through the lens of fixed-point theory. The system dynamics are formulated as a fixed-point equation $\mu = T_\theta(\mu)$ on the compact membership space $\mathscr{F}(Q) = [0,1]^n$, where T_θ is a parameterized fuzzy transition operator constructed from the aggregation functions F_1 and F_2, and the transition membership function δ. We investigate the existence, uniqueness, and parametric sensitivity of the fixed point $\mu^*(\theta)$ using the Banach contraction principle and the implicit function theorem. The contraction condition guarantees not only a unique steady-state fuzzy vector but also exponential convergence independent of the initial configuration. Analytical results include Lipschitz-type stability bounds that quantify the robustness of the equilibrium state under operator perturbations, and a closed-form expression for the sensitivity Jacobian $\frac{d\mu^*}{d\theta}$, which provides a precise measure of how the fixed point responds to small changes in the tunable parameters. Numerical examples confirm that actual sensitivities are often significantly smaller than the worst-case bounds, indicating that GFMs can exhibit robust behavior under parameter perturbations. We also identify structural limitations arising from non-contractive transition operators, where multiple fixed points may exist and Banach's theorem becomes inapplicable, underscoring the need for principled operator design in fuzzy systems when robustness and predictability are required.

Keywords: General fuzzy automata, Fuzzy transition operators, Fuzzy state space, Parametric sensitivity in automata, Automata-based analytical frameworks.

Mathematics Subject Classification (2020): 03D05, 68Q45, 68Q70

1 Introduction

Fuzzy automata theory, since its introduction by Wee and Fu [19], has provided a robust mathematical framework for modeling systems that operate under uncertainty, imprecision, and partial truth. By generalizing classical finite-state machines through fuzzy logic principles, fuzzy automata allow states and transitions to assume graded membership values, enabling applications in pattern recognition, natural language processing, and intelligent control systems [5, 10, 13, 14].

A significant advancement in this area is the concept of general fuzzy automata (GFA), introduced by Doostfatemeh and Kremer [8]. GFAs incorporate enhanced transition functions, multi-membership resolution mechanisms, and flexible state update rules, making them suitable for modeling adaptive systems with overlapping and continuous transitions. More recently, the notion of vector general fuzzy automata has been introduced by Shamsizadeh et al. [17, 18] as a refinement of the GFA model, enabling the representation of multi-dimensional fuzzy states and transitions. This vector-valued extension provides a richer algebraic structure for modeling complex systems where each state may be associated with multiple membership degrees. Abolpour et al. [1] further developed the theoretical foundations of general fuzzy automata by introducing a dynamic-logical perspective, which emphasizes the role of logical inference and dynamic behavior in fuzzy systems.

The functional-analytic treatment of fuzzy systems relies heavily on classical results from nonlinear analysis. Banach's contraction principle [3] provides a foundational tool for establishing existence and uniqueness of fixed points in complete metric spaces. The theory of functional analysis, as developed by Brezis [4] and Clarke [6], offers a comprehensive framework for studying differentiability and sensitivity in infinite-dimensional spaces. Rudin [16] provides the underlying principles of real and complex analysis that support these developments. Together, these classical texts form the mathematical backbone of our approach.

A central challenge in analyzing fuzzy automata is understanding their long-term behavior, particularly the existence and stability of fixed fuzzy states under iterative transitions. Fixed-point theory, a cornerstone of nonlinear analysis, provides the mathematical foundation for such investigations. Classical results such as Banach's contraction principle [3], Schauder's fixed-point theorem, and their fuzzy extensions [2, 11] have been instrumental in establishing conditions for convergence and uniqueness. Recent refinements using fuzzy metric spaces [9] have enabled more nuanced representations of uncertainty, allowing for convergence analysis in settings where traditional metrics are insufficient.

Sensitivity analysis has emerged as a key technique for evaluating the robustness of equilibrium fuzzy states with respect to parametric perturbations. In parametric fuzzy systems, the fixed point $\mu^*(\theta)$ of a transition operator T_θ depends implicitly on the parameter vector θ. Symbolic methods based on implicit differentiation [7] yield analytical expressions for the Jacobian $\frac{d\mu^*}{d\theta}$ under suitable regularity conditions. These are complemented by numerical techniques such as Jacobian estimation and perturbation bounds [15], which enable practical sensitivity evaluation even in complex settings. The interplay between theoretical bounds and numerical estimates provides a comprehensive understanding of system robustness.

To bridge theory and implementation, researchers have also explored the differentiability of the mapping $\theta \mapsto \mu^*(\theta)$ using tools from functional analysis, including Fréchet and Gateaux derivatives in Banach spaces [4, 6]. The classical text by Brezis [4] provides a rigorous treatment of Sobolev spaces and partial differential equations, while Clarke [6] offers a comprehensive approach to calculus of variations and optimal control. These techniques support gradient-based optimization and parameter tuning in fuzzy models, enabling systematic calibration of system parameters.

In parallel, data-driven approaches have emerged to complement analytical methods. Neuro-fuzzy systems and machine learning techniques have been applied to learn transition functions and optimize automata behavior based on observed data [12]. These hybrid models combine the interpretability of fuzzy logic with the adaptability of neural networks, offering new avenues for intelligent fuzzy system design. The integration of data-driven and analytical methods represents a promising direction for future research in fuzzy automata theory.

Despite these advances, a systematic framework for analyzing the structural properties of general fuzzy machines particularly the interplay between parametric variations, fixed-point uniqueness, and sensitivity remains largely absent in the literature. The existing results on fuzzy metric spaces [9] and common fixed points for fuzzy mappings [2] provide partial insights, but a unified treatment of parametric sensitivity in GFMs has not yet been developed. In this paper, we address this gap by presenting a comprehensive study of fixed-point structures in general fuzzy machines (GFMs) with parametric transition mappings. We adopt the GFM framework as an equivalent to the GFA model, retaining its structural features while enabling a functional-analytic treatment. Specifically, we establish conditions for the existence, uniqueness, and differentiability of fixed fuzzy states using Banach's contraction principle [3] and the implicit function theorem. We derive symbolic sensitivity bounds that quantify the robustness of the equilibrium state under parameter perturbations, drawing on the analytical techniques developed in functional analysis [4, 6, 16]. We validate our theoretical findings through numerical examples that demonstrate both contractive and non-contractive behaviors, with particular attention to the structural limitations identified in non-contractive mappings [15]. Our approach synthesizes concepts from fuzzy logic, automata theory, functional analysis, and computational intelligence, contributing to the growing body of literature on fuzzy dynamical systems and providing tools for principled operator design, parameter tuning, and robust system identification in fuzzy automata.

2 Preliminaries

We begin by recalling key notions from fuzzy automata theory and metric analysis that will be instrumental in formulating our main results.

Definition 1. *[4] Let X and Y be Banach spaces, and $U \subseteq X$ be an open set. A mapping $F : U \to Y$ is said to be continuously Fréchet differentiable on U if*

- *For every $x \in U$, the Fréchet derivative $DF(x) : X \to Y$ exists as a bounded linear operator.*
- *The mapping $x \mapsto DF(x) \in \mathscr{L}(X,Y)$ is continuous on U, where $\mathscr{L}(X,Y)$ denotes the space of bounded linear operators from X to Y.*

In other words, $F \in C^1(U)$, meaning that F is differentiable in the Fréchet sense and its derivative varies continuously with respect to the input point.

Definition 2. *[16] Let (X,d_X) and (Y,d_Y) be metric spaces, and let $f : X \to Y$ be a mapping. The function f is said to be uniformly continuous on X if*

$$\forall \varepsilon > 0,\ \exists \delta > 0 \text{ such that } \forall x_1, x_2 \in X, \quad d_X(x_1,x_2) < \delta \Rightarrow d_Y(f(x_1), f(x_2)) < \varepsilon.$$

Unlike pointwise continuity, where the choice of δ may depend on the location $x \in X$, uniform continuity requires a single δ that works uniformly for all pairs of points in the domain.

Definition 3. *[8] A general fuzzy machine (GFM) $\tilde{F}$ is an six-tuple machine denoted by $\tilde{F} = (Q, X, \tilde{R}, \tilde{\delta}, F_1, F_2)$, where*

- *$Q = \{q_1, q_2, ..., q_n\}$ is a finite set of states,*
- *$X = \{a_1, a_2, ..., a_m\}$ is a finite set of input symbols,*
- *$\tilde{R} \subseteq \tilde{P}(Q)$ is the set of fuzzy start states,*
- *$\tilde{\delta} : (Q \times [0,1]) \times X \times Q \to [0,1]$ is the augmented transition function,*
- *$F_1 : [0,1] \times [0,1] \to [0,1]$ is called the membership assignment function. The function $F_1(\mu, \delta)$, as is seen, is motivated by two parameters μ and δ, where μ is the membership value of a predecessor and δ is the value of a transition. In this definition, the process that takes place upon the transition from state q_i to q_j on an input a_k is represented as*

  $$\mu^{t+1}(q_j) = \tilde{\delta}((q_i, \mu^t(q_i)), a_k, q_j) = F_1(\mu^t(q_i), \delta(q_i, a_k, q_j)).$$

  Which means that membership value (MV) of the state q_j at time $t+1$ is computed by function F_1 using both the membership value of q_i at time t and the membership value of the transition. There are many options which can be used for the function $F_1(\mu, \delta)$. It can be for example $\max\{\mu, \delta\}, \min\{\mu, \delta\}, \frac{\mu+\delta}{2}$ or any other applicable mathematical function.

- *$F_2 : [0,1]^* \to [0,1]$ is called the multi-membership resolution function. The multi-membership resolution function resolves the multi-membership active states and assigns a single non-membership value to them.*

  $[0,1]^*$ is the set of elements in [0, 1]. The multi-membership resolution function F_2, is a function which specifies the strategy, that resolves the multi-membership active states and assigns a single mv to them.

Theorem 1. *[3] Let (X,d) be a nonempty complete metric space, and $T : X \to X$ be a mapping satisfying the contraction condition:*

$$d(T(x), T(y)) \leq \alpha\, d(x,y), \quad \forall x, y \in X$$

for some constant $\alpha \in [0,1)$. Then

- *T has a unique fixed point $x^* \in X$ such that $T(x^*) = x^*$.*
- *For any initial point $x_0 \in X$, the iterative sequence $x_{k+1} = T(x_k)$ converges to x^*.*
- *The convergence is geometric:*

  $$d(x_k, x^*) \leq \alpha^k\, d(x_0, x^*).$$

3 Membership Space and Transition Mapping

Definition 4. *Let $\mathscr{F}(Q)$ denote the space of fuzzy membership functions over a finite state set $Q = \{q_1, q_2, \ldots, q_n\}$, i.e.,*

$$\mathscr{F}(Q) = \{\mu : Q \to [0,1]\} = [0,1]^n.$$

The fuzzy transition operator $T : \mathscr{F}(Q) \to \mathscr{F}(Q)$ is defined componentwise as follows, for each $q_j \in Q$,

$$\mu^{t+1}(q_j) = T(\mu^t)(q_j) = \bigvee_{q_i \in Q} \bigvee_{x \in X} F_1\left(\mu^t(q_i),\ \delta(q_i,\ x,\ q_j)\right),$$

where $\delta : Q \times X \times Q \to [0,1]$ is the transition membership function of the GFM.

The transition operator T computes the new fuzzy state vector by aggregating all possible transitions into each state q_j, weighted by the current membership degrees and the transition possibilities.

This formulation embeds the dynamics of a general fuzzy machine into the compact space $[0,1]^n$. This embedding is the key step that enables us to apply tools from metric fixed-point theory — such as the Banach contraction principle — to the study of fuzzy automata, which is the central idea of this work.

Definition 5. *A fuzzy transition operator $T : \mathscr{F}(Q) \to \mathscr{F}(Q)$ is continuous if for every convergent sequence $\{\mu^k\} \subset \mathscr{F}(Q)$ with $\mu^k \to \mu$, we have*

$$\lim_{k \to \infty} T(\mu^k) = T(\mu).$$

Definition 6. *Let $\mathscr{F}(Q) = [0,1]^n$ be the fuzzy membership space over a finite state set $Q = \{q_1, q_2, \ldots, q_n\}$, equipped with a norm $\|\cdot\|$. A fuzzy transition operator $T : \mathscr{F}(Q) \to \mathscr{F}(Q)$ is said to be Lipschitz continuous on $\mathscr{F}(Q)$ if there exists a constant $L > 0$ such that*

$$\|T(\mu) - T(\nu)\| \le L\|\mu - \nu\|, \qquad \forall \mu, \nu \in \mathscr{F}(Q).$$

The minimal such constant L is called the Lipschitz constant of T. When $L < 1$, the operator exhibits contraction behavior, enabling unique fixed-point convergence via Banachs theorem. In general fuzzy machine, Lipschitz continuity serves as a key condition for assessing the robustness, stability, and differentiability of system trajectories with respect to initial states and parametric variations.

Theorem 2. *Let $\tilde{F} = (Q, X, \tilde{R}, \tilde{\delta}, F_1, F_2)$ be a general fuzzy machine with state set $Q = \{q_1, \ldots, q_n\}$, and let $\mathscr{F}(Q) = [0,1]^n$ be its membership space equipped with the supremum norm $\|\cdot\|_\infty$. Suppose the transition operator $T : \mathscr{F}(Q) \to \mathscr{F}(Q)$ defined by*

$$(T\mu)(q_j) = \bigvee_{q_i \in Q} \bigvee_{x \in X} F_1(\mu(q_i), \delta(q_i, x, q_j))$$

satisfies the contraction inequality

$$\|T(\mu) - T(\nu)\|_\infty \le \alpha\|\mu - \nu\|_\infty, \qquad \forall \mu, \nu \in \mathscr{F}(Q), \tag{1}$$

for some constant $\alpha \in [0,1)$. Then:

- *The fuzzy machine $\tilde{F}$ possesses a unique steady-state membership vector $\mu^* \in \mathscr{F}(Q)$ satisfying $T(\mu^*) = \mu^*$, meaning that the machine eventually reaches an equilibrium fuzzy state.*
- *For any initial fuzzy state $\mu^0 \in \mathscr{F}(Q)$, the machine's evolution*

  $$\mu^{k+1} = T(\mu^k), \qquad k = 0, 1, 2, \ldots,$$

  converges to μ^*.
- *The convergence is geometric:*

  $$\|\mu^k - \mu^*\|_\infty \le \alpha^k\|\mu^0 - \mu^*\|_\infty, \qquad k \ge 0.$$

Thus, the machine asymptotically forgets its initial configuration and stabilizes to a unique fuzzy state, independent of where it started.

Proof. The space $\mathscr{F}(Q) = [0,1]^n$ is compact and complete since the state set Q is finite. Note that every $\mu \in \mathscr{F}(Q)$ represents a fuzzy state of the machine, and the operator T encodes all possible transitions of the GFM through the aggregation over all $q_i \in Q$ and $x \in X$. We first establish uniqueness. Suppose the fuzzy machine has two distinct steady states $\mu^*, \nu^* \in \mathscr{F}(Q)$, meaning that starting from either state, the machine remains unchanged under T, i.e., $T(\mu^*) = \mu^*$ and $T(\nu^*) = \nu^*$. Then, by the contraction inequality,

$$\|\mu^* - \nu^*\|_\infty = \|T(\mu^*) - T(\nu^*)\|_\infty \leq \alpha \|\mu^* - \nu^*\|_\infty.$$

Since $0 \leq \alpha < 1$, we must have $\|\mu^* - \nu^*\|_\infty = 0$, hence $\mu^* = \nu^*$. Therefore, the machine admits at most one equilibrium fuzzy state. To prove existence, take any initial fuzzy state $\mu^0 \in \mathscr{F}(Q)$. This represents the machine's starting configuration, which could be any arbitrary assignment of membership values to the states in Q. Let the machine evolve according to its transition rule:

$$\mu^{k+1} = T(\mu^k), \qquad k = 0, 1, 2, \dots$$

That is, at each step, for every target state $q_j \in Q$, the new membership value is computed by aggregating all possible predecessors $q_i \in Q$ and all input symbols $x \in X$, combining their current memberships via F_1 and the transition memberships $\delta(q_i, x, q_j)$.

Now, applying the contraction inequality iteratively to this sequence of machine states gives

$$\|\mu^{k+1} - \mu^k\|_\infty \leq \alpha \|\mu^k - \mu^{k-1}\|_\infty \leq \cdots \leq \alpha^k \|\mu^1 - \mu^0\|_\infty.$$

For $m > k$, the distance between two future states of the machine is bounded by the sum of consecutive changes:

$$\|\mu^m - \mu^k\|_\infty \leq \sum_{j=k}^{m-1} \|\mu^{j+1} - \mu^j\|_\infty \leq \|\mu^1 - \mu^0\|_\infty \sum_{j=k}^{m-1} \alpha^j = \|\mu^1 - \mu^0\|_\infty \cdot \alpha^k \frac{1 - \alpha^{m-k}}{1 - \alpha}.$$

As $k \to \infty$, the right-hand side tends to zero. Hence, the sequence of states generated by the machine is Cauchy. Since the membership space $\mathscr{F}(Q)$ is complete, this sequence converges to some $\mu^* \in \mathscr{F}(Q)$. Because T is contractive, it is continuous on $\mathscr{F}(Q)$. Passing to the limit in the machine's transition rule,

$$T(\mu^*) = T\left(\lim_{k\to\infty} \mu^k\right) = \lim_{k\to\infty} T(\mu^k) = \lim_{k\to\infty} \mu^{k+1} = \mu^*.$$

Thus, μ^* is a steady state of the fuzzy machine. Since we already proved uniqueness, this steady state is the only one.

Finally, for any step k, applying the contraction inequality to the current state μ^k and the steady state μ^* yields

$$\|\mu^{k+1} - \mu^*\|_\infty = \|T(\mu^k) - T(\mu^*)\|_\infty \leq \alpha \|\mu^k - \mu^*\|_\infty.$$

By induction,

$$\|\mu^k - \mu^*\|_\infty \leq \alpha^k \|\mu^0 - \mu^*\|_\infty, \qquad \forall k \geq 0.$$

This shows that the machine converges to its unique equilibrium state at an exponential rate, and the convergence is independent of the initial configuration. □

Theorem 3. *Let $(\mathscr{F}(Q), \|\cdot\|_\infty)$ be the fuzzy membership space of a general fuzzy machine. Let $T, T' : \mathscr{F}(Q) \to \mathscr{F}(Q)$ be two transition operators, where T' is contractive with constant $\alpha \in (0,1)$ and T has a fixed point $\mu^* \in \mathscr{F}(Q)$. Suppose that*

$$\sup_{\mu \in \mathscr{F}(Q)} \|T(\mu) - T'(\mu)\|_\infty \leq \varepsilon,$$

for some $\varepsilon > 0$. Let $\mu'^* \in \mathscr{F}(Q)$ denote the unique fixed point of T'. Then

$$\|\mu^* - \mu'^*\|_\infty \leq \frac{\varepsilon}{1 - \alpha}.$$

Proof. By the triangle inequality and the contraction property of T', we have,

$$\begin{aligned}\|\mu^* - \mu'^*\| &= \|T(\mu^*) - T'(\mu'^*)\| \\ &\leq \|T(\mu^*) - T'(\mu^*)\| + \|T'(\mu^*) - T'(\mu'^*)\| \\ &\leq \varepsilon + \alpha \|\mu^* - \mu'^*\|.\end{aligned}$$

Rearranging terms yields,

$$(1-\alpha)\|\mu^* - \mu'^*\| \leq \varepsilon \quad \Rightarrow \quad \|\mu^* - \mu'^*\| \leq \frac{\varepsilon}{1-\alpha}.$$

□

Theorem 4. *Let $T : \mathscr{F}(Q) \to \mathscr{F}(Q)$ be the transition operator of a general fuzzy machine $\tilde{F}$, as defined in Definition 2.1, and suppose T is a contraction with constant $\alpha_T \in (0,1)$ on the complete membership space $\mathscr{F}(Q) = [0,1]^n$ equipped with the supremum norm. Consider the damped iteration operator $S : \mathscr{F}(Q) \to \mathscr{F}(Q)$ defined by*

$$S(\mu) := \alpha T(\mu) + (1-\alpha)\mu, \qquad 0 < \alpha \leq 1.$$

The operator S models a machine whose state update at each step is a convex combination of the standard fuzzy transition and the current state, thereby incorporating a memory effect into the dynamics. Then S is a contraction on $\mathscr{F}(Q)$ with constant $\lambda = \alpha\alpha_T + (1-\alpha) < 1$. Consequently, the damped iteration $\mu^{k+1} = S(\mu^k)$ converges to the unique fixed point μ^* of T. Moreover, for every $0 < \alpha < 1$, the convergence rate satisfies $\lambda < \alpha_T$, meaning the damped machine reaches its steady state asymptotically faster than the original machine evolving under T.

Proof. Let $\mu, \nu \in \mathscr{F}(Q)$ be arbitrary fuzzy states. Since $\mathscr{F}(Q)$ is a convex subset of the normed space $\mathbf{R}^n$, the expression $\alpha T(\mu) + (1-\alpha)\mu$ is well-defined and belongs to $\mathscr{F}(Q)$. Using the properties of the supremum norm and the contraction inequality for T, we obtain

$$\begin{aligned}
\|S(\mu) - S(\nu)\|_\infty &= \|\alpha(T(\mu) - T(\nu)) + (1-\alpha)(\mu - \nu)\|_\infty \\
&\leq \alpha\|T(\mu) - T(\nu)\|_\infty + (1-\alpha)\|\mu - \nu\|_\infty \\
&\leq \alpha\alpha_T\|\mu - \nu\|_\infty + (1-\alpha)\|\mu - \nu\|_\infty \\
&= \big(\alpha\alpha_T + 1 - \alpha\big)\|\mu - \nu\|_\infty.
\end{aligned}$$

Set $\lambda := \alpha\alpha_T + (1-\alpha)$. Since $\alpha_T < 1$ and $0 < \alpha \leq 1$, it follows that

$$\lambda = 1 - \alpha(1-\alpha_T) < 1,$$

so S is a contraction on the complete space $\mathscr{F}(Q)$. By the Banach fixed-point theorem, S has a unique fixed point $\mu^* \in \mathscr{F}(Q)$. From the definition of S, this point satisfies

$$\mu^* = \alpha T(\mu^*) + (1-\alpha)\mu^*,$$

which implies $T(\mu^*) = \mu^*$. Hence the fixed point of the damped operator coincides with the unique steady state of the original machine.

Finally, for any $0 < \alpha < 1$, we have

$$\lambda - \alpha_T = (1-\alpha)(1-\alpha_T) > 0,$$

so $\lambda < \alpha_T$. Thus the damped iteration contracts at a strictly smaller rate than the direct iteration, resulting in asymptotically faster convergence to the machine's equilibrium state. □

Each fuzzy transition function δ_x is represented by a transition matrix $D_x \in [0,1]^{n \times n}$, defined as

$$D_x = [\delta(q_i, x, q_j)]_{i,j},$$

where each entry $\delta(q_i, x, q_j)$ denotes the fuzzy degree of transition from state q_i to state q_j under input $x \in X$. For a fixed $F_1 = \min$ and $\bigvee = \max$, the fuzzy transition operator $T : \mathscr{F}(Q) \to \mathscr{F}(Q)$ introduced in Definition 2.1 reduces to the max-min composition form:

$$(T(\mu))_j = \max_{x \in X} \max_{i \in \{1,\ldots,n\}} \min(\mu_i, (D_x)_{ij}).$$

Equivalently, in matrix-vector notation,

$$T(\mu) = \max_{x \in X}(\mu \circ D_x),$$

where $\circ$ denotes the max-min composition, i.e.,

$$(\mu \circ D_x)_j = \max_i \min(\mu_i, (D_x)_{ij}).$$

A common stopping criterion for the iterative process $\mu^{k+1} = T(\mu^k)$ is

$$\|\mu^{k+1} - \mu^k\|_\infty < \varepsilon,$$

for a prescribed tolerance $\varepsilon > 0$.

Example 1. Let $Q = \{q_1, q_2, q_3\}$ and $X = \{x\}$. Denote $\delta_{ij} = \delta(q_i, x, q_j)$ and consider the transition matrix

$$D = \begin{pmatrix} 0.8 & 0.1 & 0.0 \\ 0.3 & 0.7 & 0.4 \\ 0.2 & 0.5 & 0.9 \end{pmatrix}.$$

Equip $\mathscr{F}(Q) = [0,1]^3$ with the supremum norm and define the fuzzy transition operator $T : \mathscr{F}(Q) \to \mathscr{F}(Q)$ by the standard maxmin composition:

$$T(\mu)(q_j) = \max_{1 \le i \le 3} \min(\mu(q_i), \delta_{ij}), \qquad j = 1,2,3.$$

Take the initial membership vector $\mu^0 = [0.6, 0.4, 0.2]$. A direct computation gives:

$$\begin{aligned} T(\mu^0)(q_1) &= \max\{\min(0.6,0.8), \min(0.4,0.3), \min(0.2,0.2)\} = 0.6, \\ T(\mu^0)(q_2) &= \max\{\min(0.6,0.1), \min(0.4,0.7), \min(0.2,0.5)\} = 0.4, \\ T(\mu^0)(q_3) &= \max\{\min(0.6,0.0), \min(0.4,0.4), \min(0.2,0.9)\} = 0.4. \end{aligned}$$

Thus $\mu^1 = [0.6, 0.4, 0.4]$. Applying T once more yields

$$\mu^2 = T(\mu^1) = [0.6, 0.4, 0.4] = \mu^1,$$

so the iteration stabilizes at $\mu^* = [0.6, 0.4, 0.4]$, which is a fixed point of T.

However, the zero vector $\mathbf{0} = [0,0,0]$ is also a fixed point, since $\min(0, \delta_{ij}) = 0$ for all i, j. Hence T admits multiple fixed points, and uniqueness fails. Moreover, the maxmin composition operator is well known to be non-expansive with respect to the supremum norm:

$$\|T(\mu) - T(\nu)\|_\infty \le \|\mu - \nu\|_\infty, \qquad \forall \mu, \nu \in \mathscr{F}(Q),$$

with Lipschitz constant equal to 1. Therefore, T is not a contraction, and Banach's fixed-point theorem does not apply. Nevertheless, for the specific initial vector μ^0, the iteration converges in one step to μ^*.

Example 2. Let $Q = \{q_1, q_2, q_3\}$ and $X = \{x\}$. We consider a fuzzy transition operator constructed from the product assignment function $F_1(\mu(q_i), \delta_{ij}) = \mu(q_i) \cdot \delta_{ij}$ and aggregation $\bigvee = \max$. Membership vectors $\mu \in \mathscr{F}(Q) = [0,1]^3$ are equipped with the supremum norm.

Let the unscaled transition matrix be

$$D = \begin{pmatrix} 0.6 & 0.2 & 0.1 \\ 0.4 & 0.9 & 0.5 \\ 0.3 & 0.2 & 0.8 \end{pmatrix}, \qquad \delta_{ij} = \delta(q_i, x, q_j).$$

To enforce contractivity, we scale the matrix by $\beta = 0.5$, yielding

$$D' = 0.5 \cdot D = \begin{pmatrix} 0.30 & 0.10 & 0.05 \\ 0.20 & 0.45 & 0.25 \\ 0.15 & 0.10 & 0.40 \end{pmatrix}, \quad \text{with } \alpha := \max_{i,j} \delta'_{ij} = 0.45 < 1.$$

The transition operator is defined componentwise as

$$T(\mu)(q_j) = \max_{1 \le i \le 3} \mu(q_i) \cdot \delta'_{ij}, \qquad j = 1,2,3.$$

For arbitrary $\mu, \nu \in \mathscr{F}(Q)$, we have

$$\begin{aligned}|T(\mu)(q_j) - T(\nu)(q_j)| &= \left|\max_i \mu(q_i)\delta'_{ij} - \max_i \nu(q_i)\delta'_{ij}\right| \\ &\le \max_i \left|\mu(q_i)\delta'_{ij} - \nu(q_i)\delta'_{ij}\right| \\ &\le \max_i \delta'_{ij} \cdot |\mu(q_i) - \nu(q_i)| \\ &\le \alpha\|\mu - \nu\|_\infty.\end{aligned}$$

Taking the supremum over j gives

$$\|T(\mu) - T(\nu)\|_\infty \le \alpha\|\mu - \nu\|_\infty,$$

so T is a contraction with constant $\alpha = 0.45$. By Banach's theorem, T has a unique fixed point μ^*, and the iteration $\mu^{k+1} = T(\mu^k)$ converges geometrically.

Now take two initial vectors:

$$\mu^0 = [0.9,\ 0.4,\ 0.6], \qquad \nu^0 = [0.5,\ 0.6,\ 0.7].$$

A direct computation gives

$$\begin{aligned}T(\mu^0) &= [0.27,\ 0.18,\ 0.24], \\ T(\nu^0) &= [0.15,\ 0.27,\ 0.28].\end{aligned}$$

Then

$$\|\mu^0 - \nu^0\|_\infty = 0.4, \qquad \|T(\mu^0) - T(\nu^0)\|_\infty = 0.12,$$

and indeed $0.12 \le 0.45 \cdot 0.4 = 0.18$.

Since $T(\mathbf{0}) = \mathbf{0}$, the zero vector is a fixed point. By uniqueness, $\mu^* = \mathbf{0}$. Starting from μ^0, the Picard iterates are

$$\begin{aligned}\mu^1 &= T(\mu^0) = [0.27,\ 0.18,\ 0.24], \\ \mu^2 &= T(\mu^1) \approx [0.081,\ 0.081,\ 0.096], \\ \mu^3 &= T(\mu^2) \approx [0.0243,\ 0.03645,\ 0.0384].\end{aligned}$$

These converge rapidly to zero, consistent with the geometric bound

$$\|\mu^k - \mathbf{0}\|_\infty \le (0.45)^k \|\mu^0 - \mathbf{0}\|_\infty.$$

Example 3. (Fixed-Point Behavior of a Nonlinear Transition Mapping) Let $T : [0,1]^3 \to [0,1]^3$ be a nonlinear fuzzy transition operator defined componentwise by

$$T(\mu) = \begin{pmatrix}\nu_1(\mu_2, \mu_3) \\ \nu_2(\mu_1, \mu_3) \\ \nu_3(\mu_1, \mu_2)\end{pmatrix},$$

where each function $\nu_i : [0,1]^2 \to [0,1]$ is given by

$$\nu_i(u, v) := \min\{1,\ \alpha_i u + \beta_i v\}, \qquad i = 1,2,3.$$

Fix the parameters,

$$\alpha_1 = 1.2, \quad \beta_1 = 0.5; \qquad \alpha_2 = 0.6, \quad \beta_2 = 0.6; \qquad \alpha_3 = 0.3, \quad \beta_3 = 0.3.$$

A fixed point $\mu^* = (\mu_1^*, \mu_2^*, \mu_3^*) \in [0,1]^3$ must satisfy,

$$\begin{cases}\mu_1^* = \min\{1,\ 1.2\mu_2^* + 0.5\mu_3^*\}, \\ \mu_2^* = \min\{1,\ 0.6\mu_1^* + 0.6\mu_3^*\}, \\ \mu_3^* = \min\{1,\ 0.3\mu_1^* + 0.3\mu_2^*\}.\end{cases}$$

There are $2^3 = 8$ possible clipping patterns (each component either clipped to 1 or equal to its linear argument). We analyze the most plausible pattern,

$$(\mathrm{P}): \quad \mu_1^* = 1, \quad \mu_2^* < 1, \quad \mu_3^* < 1.$$

Under pattern (P), the fixed-point equations reduce to,

$$\begin{cases} \mu_2^* = 0.6 \cdot 1 + 0.6\mu_3^*, \\ \mu_3^* = 0.3 \cdot 1 + 0.3\mu_2^*. \end{cases}$$

Solving this linear system yields,

$$\mu_2^* = \frac{39}{41} \approx 0.95122, \qquad \mu_3^* = \frac{24}{41} \approx 0.58537.$$

To verify consistency of pattern (P), we compute the unclipped expression for μ_1^*,

$$1.2 \cdot \frac{39}{41} + 0.5 \cdot \frac{24}{41} = \frac{58.8}{41} \approx 1.434 > 1,$$

so the min operator returns 1 as assumed. A brief check of other patterns shows that none yields a self-consistent solution within $[0,1]^3$. Hence, the admissible pattern is unique and the fixed point is,

$$\mu^* = \left(1,\ \tfrac{39}{41},\ \tfrac{24}{41}\right) \approx (1,\ 0.95122,\ 0.58537)\ .$$

At μ^*, the first component is clipped, so the derivative of v_1 vanishes. For the other two components, the min is inactive, and the partial derivatives equal the coefficients. Therefore, the Jacobian matrix $J := DT(\mu^*)$ is,

$$J = \begin{pmatrix} 0 & 0 & 0 \\ 0.6 & 0 & 0.6 \\ 0.3 & 0.3 & 0 \end{pmatrix}.$$

The eigenvalues of J are,

$$\lambda_1 = 0, \qquad \lambda_{2,3} = \pm\sqrt{0.18} \approx \pm 0.4243,$$

so the spectral radius is $\rho(J) = \sqrt{0.18} \approx 0.4243 < 1$. By linearized stability theory, this implies that μ^* is locally asymptotically stable. To illustrate convergence, consider the initial condition, $\mu^{(0)} = (0.2,\ 0.5,\ 0.7)$. Iterating $\mu^{(k+1)} = T(\mu^{(k)})$, we obtain the following rounded values, The sequence rapidly converges to $\mu^* \approx (1,\ 0.9513,\ 0.5851)$, with stability to four decimal places by iteration 11. This numerical

Table 1. Picard iterates for the nonlinear transition mapping

Iteration k	$\mu_1^{(k)}$	$\mu_2^{(k)}$	$\mu_3^{(k)}$
0	0.2000	0.5000	0.7000
1	0.9500	0.5400	0.2100
2	0.7530	0.6960	0.4470
3	1.0000	0.7200	0.4347
4	1.0000	0.8608	0.5160
5	1.0000	0.9096	0.5582
6	1.0000	0.9349	0.5729
7	1.0000	0.9447	0.5805
8	1.0000	0.9483	0.5834
9	1.0000	0.9500	0.5855
10	1.0000	0.9513	0.5850
11	1.0000	0.9513	0.5851

behavior confirms the local attraction predicted by the Jacobian analysis. Note that intermediate non-monotonicity is expected due to the nonlinear clipping.

4 Sensitivity Analysis

In practical implementations of general fuzzy machines (GFMs), the transition operator T depends on a set of tunable parameters θ, such as the weighting coefficients (α_i, β_i) that determine the influence of different fuzzy-state components. These parameters are typically obtained through system identification, empirical calibration, or optimization routines.

However, in realistic scenarios, the parameter set θ is rarely known with perfect accuracy. It may be subject to estimation errors, measurement noise, or environmental fluctuations. Consequently, it is essential to quantify how sensitive the equilibrium configuration of the GFM is to small perturbations in these parameters.

Formally, let

$$T(\mu;\theta) : \mathscr{F}(Q) \times \Theta \to \mathscr{F}(Q)$$

denote the parametric transition mapping, where $\mu \in \mathscr{F}(Q) = [0,1]^n$ is the fuzzy-state vector and

$$\theta = (\alpha_1, \beta_1, \ldots, \alpha_n, \beta_n) \in \Theta \subseteq \mathbf{R}^{2n}$$

is the vector of tunable coefficients, with Θ denoting the feasible parameter space (e.g., $\alpha_i, \beta_i \geq 0$ and $\alpha_i + \beta_i \leq 1$ for stability). The long-term behavior of the system is characterized by the fixed point μ^* satisfying

$$T(\mu^*;\theta) = \mu^*. \tag{2}$$

Assuming that T is continuously differentiable with respect to θ and that the fixed point μ^* depends smoothly on θ a condition that holds when μ^* lies in the interior of the domain and the aggregation functions are smooth we define the parametric sensitivity of μ^* with respect to θ_i as

$$S^{\mu^*}_{\theta_i} := \frac{\partial \mu^*}{\partial \theta_i}.$$

This derivative quantifies the local responsiveness of the equilibrium state to infinitesimal changes in the parameter θ_i. A large value of $\|S^{\mu^*}_{\theta_i}\|$ indicates that small perturbations in θ_i may induce significant deviations in the fixed point, potentially leading to instability or degraded performance. Conversely, a small value suggests robust behavior under parameter uncertainty.

For comparative purposes, we also define a normalized relative sensitivity index

$$S^{\text{rel}}_{\theta_i} := \frac{\|\Delta\mu^*\|/\|\mu^*\|}{\Delta\theta_i/\theta_i}, \qquad \theta_i \neq 0,$$

which captures the proportional deviation of the fixed point in response to relative changes in θ_i. This index is particularly useful when the parameters and state variables operate on different numerical scales.

The following proposition establishes an upper bound on the sensitivity $\|S^{\mu^*}_{\theta_i}\|$ in terms of the Lipschitz constants of the transition mapping, thereby linking parametric robustness directly to the contractive properties of the GFM.

Proposition 1. *Let $T : \mathscr{F}(Q) \times \Theta \to \mathscr{F}(Q)$ be continuously differentiable, where $\mathscr{F}(Q) = [0,1]^n$ is a complete normed space and $\Theta \subseteq \mathbf{R}^{2n}$ is open. Suppose that for each $\theta \in \Theta$, the map $T(\cdot;\theta)$ is a contraction with constant $L_\mu < 1$, and that the derivative with respect to θ satisfies $\|D_\theta T(\mu;\theta)\| \leq L_\theta$. Then the unique fixed point $\mu^*(\theta)$ satisfies the sensitivity bound*

$$\left\| \frac{\partial \mu^*}{\partial \theta} \right\| \leq \frac{L_\theta}{1 - L_\mu}.$$

Proof. For each fixed $\theta \in \Theta$, since $T(\cdot;\theta)$ is a contraction with constant $L_\mu < 1$, the Banach fixed-point theorem guarantees the existence and uniqueness of a fixed point $\mu^*(\theta) \in \mathscr{F}(Q)$. Since T is continuously differentiable with respect to both arguments and $\|D_\mu T(\mu^*(\theta);\theta)\| \leq L_\mu < 1$, the implicit function theorem applies to the equation

$$G(\mu,\theta) := T(\mu;\theta) - \mu = 0.$$

The derivative of G with respect to μ is $D_\mu G(\mu;\theta) = D_\mu T(\mu;\theta) - I$, which is invertible because $\|D_\mu T(\mu;\theta)\| \leq L_\mu < 1$. Hence, $\mu^*(\theta)$ is continuously differentiable with respect to θ. Differentiating the fixed-point equation $T(\mu^*(\theta);\theta) = \mu^*(\theta)$ with respect to θ gives

$$\frac{\partial \mu^*}{\partial \theta} = D_\mu T(\mu^*;\theta) \cdot \frac{\partial \mu^*}{\partial \theta} + D_\theta T(\mu^*;\theta),$$

hence

$$(I - D_\mu T(\mu^*;\theta))\frac{\partial \mu^*}{\partial \theta} = D_\theta T(\mu^*;\theta).$$

Since $\|D_\mu T(\mu^*;\theta)\| \le L_\mu < 1$, the Neumann series gives

$$(I - D_\mu T(\mu^*;\theta))^{-1} = \sum_{k=0}^{\infty} (D_\mu T(\mu^*;\theta))^k,$$

and therefore

$$\left\|(I - D_\mu T(\mu^*;\theta))^{-1}\right\| \le \frac{1}{1-L_\mu}.$$

Taking norms on both sides of the differentiated equation yields

$$\left\|\frac{\partial \mu^*}{\partial \theta}\right\| \le \left\|(I - D_\mu T(\mu^*;\theta))^{-1}\right\| \cdot \|D_\theta T(\mu^*;\theta)\| \le \frac{L_\theta}{1-L_\mu}.$$

This completes the proof. □

The bound provides a worst-case estimate for the responsiveness of the fixed point to parametric variations. A smaller value of L_θ indicates that the transition operator is less sensitive to perturbations in the tunable parameters, while a smaller L_μ corresponds to a stronger contraction in the fuzzy state space. Both conditions contribute to greater robustness of the fixed point under parameter uncertainty or environmental fluctuations.

Example 4. To illustrate the sensitivity estimate derived in Proposition 1, we consider a simple linear parametric transition operator on $\mathscr{F}(Q) = [0,1]^3$, endowed with the sup-norm. For a fuzzy-state vector $\mu \in \mathbf{R}^3$ and parameter vector

$$\theta = (\alpha_1, \beta_1, \alpha_2, \beta_2, \alpha_3, \beta_3),$$

define the transition mapping

$$T(\mu;\theta) = M(\theta)\mu + b,$$

where

$$M(\theta) = \begin{pmatrix} 0 & \alpha_1 & \beta_1 \\ \alpha_2 & 0 & \beta_2 \\ \alpha_3 & \beta_3 & 0 \end{pmatrix}, \qquad b = \begin{pmatrix} 0.10 \\ 0.05 \\ 0 \end{pmatrix}.$$

The fixed point μ^* satisfies the linear system $(I - M)\mu^* = b$. We fix the nominal parameter values:

$$\alpha_1 = 0.50, \quad \beta_1 = 0.20; \qquad \alpha_2 = 0.40, \quad \beta_2 = 0.30; \qquad \alpha_3 = 0.20, \quad \beta_3 = 0.30,$$

yielding

$$M = \begin{pmatrix} 0 & 0.50 & 0.20 \\ 0.40 & 0 & 0.30 \\ 0.20 & 0.30 & 0 \end{pmatrix}.$$

Solving $(I - M)\mu^* = b$ numerically gives

$$\mu^* \approx \begin{pmatrix} 0.193181 \\ 0.152597 \\ 0.084416 \end{pmatrix},$$

which satisfies $T(\mu^*;\theta) \approx \mu^*$ up to numerical precision.

To compute the sensitivity of μ^* with respect to α_1, we differentiate the fixed-point equation implicitly. Let $E^{(1,2)} = \partial M/\partial \alpha_1$ denote the matrix with a single unit entry at row 1, column 2:

$$E^{(1,2)} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Differentiating $(I-M)\mu^* = b$ with respect to α_1 yields

$$-(\partial M/\partial \alpha_1)\mu^* + (I-M)\frac{\partial \mu^*}{\partial \alpha_1} = 0,$$

and thus the closed-form sensitivity expression:

$$\frac{\partial \mu^*}{\partial \alpha_1} = (I-M)^{-1}(E^{(1,2)}\mu^*). \tag{3}$$

Evaluating $E^{(1,2)}\mu^* = (\mu_2^*, 0, 0)^\top \approx (0.152597, 0, 0)^\top$, and solving the system $(I-M)x = E^{(1,2)}\mu^*$ yields

$$\frac{\partial \mu^*}{\partial \alpha_1} \approx \begin{pmatrix} 0.225397 \\ 0.113939 \\ 0.079281 \end{pmatrix}.$$

Hence, the actual sup-norm sensitivity is

$$\|S_{\alpha_1}^{\mu^*}\|_\infty = \left\| \frac{\partial \mu^*}{\partial \alpha_1} \right\|_\infty \approx 0.225397.$$

To compare with the theoretical bound, we compute the Lipschitz constants. The Lipschitz constant of T with respect to μ in the sup-norm is the induced matrix norm,

$$L_\mu = \|M\|_\infty = \max_i \sum_j |M_{ij}| = \max\{0.70,\ 0.70,\ 0.50\} = 0.70.$$

Since $L_\mu < 1$, the mapping is contractive and the fixed point is unique. For the parameter α_1, we estimate the Lipschitz constant as

$$L_\theta = \left\| \frac{\partial M}{\partial \alpha_1} \right\|_\infty = \|E^{(1,2)}\|_\infty = 1.$$

Proposition 1 then yields the conservative bound

$$\|S_{\alpha_1}^{\mu^*}\|_\infty \le \frac{L_\theta}{1-L_\mu} = \frac{1}{1-0.70} = \frac{1}{0.30} \approx 3.3333,$$

which safely overestimates the true sensitivity 0.225397 and confirms the robustness of the fixed point under small perturbations in α_1.

Theorem 5. *Let $T : \mathscr{F}(Q) \times \Theta \to \mathscr{F}(Q)$ be continuously differentiable in a neighborhood of (μ^*, θ^*), where $\mathscr{F}(Q) = [0,1]^n$ and $\Theta \subseteq \mathbf{R}^p$ is open. Suppose $T(\mu^*; \theta^*) = \mu^*$ and $I - D_\mu T(\mu^*, \theta^*)$ is invertible. Then there exists a neighborhood $\mathscr{U}$ of θ^* such that the unique fixed point $\mu^*(\theta)$ is continuously differentiable on $\mathscr{U}$ and*

$$\frac{d\mu^*}{d\theta} = \big(I - D_\mu T(\mu^*, \theta^*)\big)^{-1} D_\theta T(\mu^*, \theta^*).$$

Proof. Let $\mu^*(\theta) \in \mathscr{F}(Q)$ denote the fixed point of the machine's transition operator, which depends on the parameter vector θ. By definition, this fixed point satisfies

$$T(\mu^*(\theta); \theta) = \mu^*(\theta),$$

for all θ in a neighborhood of θ^*. This equation expresses the fact that when the machine is in its equilibrium state, one additional application of the transition operator leaves the fuzzy membership vector unchanged.

Differentiating both sides of this identity with respect to the parameter vector θ captures how the fixed point responds to infinitesimal changes in the machine's tunable parameters. Applying the chain rule yields

$$\frac{\partial T}{\partial \mu}(\mu^*(\theta), \theta) \cdot \frac{d\mu^*}{d\theta} + \frac{\partial T}{\partial \theta}(\mu^*(\theta), \theta) = \frac{d\mu^*}{d\theta}.$$

The term $\frac{\partial T}{\partial \mu}$ represents the sensitivity of the transition rule to variations in the current fuzzy state, while $\frac{\partial T}{\partial \theta}$ quantifies how the transition rule itself changes when the machine's parameters are perturbed.

Rearranging the differentiated equation gives

$$\left(I - \frac{\partial T}{\partial \mu}(\mu^*(\theta), \theta)\right) \cdot \frac{d\mu^*}{d\theta} = \frac{\partial T}{\partial \theta}(\mu^*(\theta), \theta).$$

The matrix $I - \frac{\partial T}{\partial \mu}$ describes how a small deviation from the fixed point is amplified or damped by the machine's dynamics. Its invertibility at (μ^*, θ^*) ensures that the fixed point is locally isolated and that the sensitivity is well-defined. Evaluating the expression at the nominal parameter value $\theta = \theta^*$ and using the invertibility assumption, we obtain the sensitivity formula

$$\frac{d\mu^*}{d\theta} = \left(I - \frac{\partial T}{\partial \mu}(\mu^*, \theta^*)\right)^{-1} \cdot \frac{\partial T}{\partial \theta}(\mu^*, \theta^*).$$

This formula provides a quantitative measure of how the machine's fixed point shifts in response to small changes in its internal parameters, which is essential for robustness analysis of fuzzy systems in uncertain environments. □

Theorem 5 establishes that the fixed point $\mu^*(\theta)$ of the parametric transition operator $T(\cdot;\theta)$ is continuously differentiable with respect to the transition parameters θ. The Jacobian $\frac{d\mu^*}{d\theta}$ quantifies the sensitivity of each component of the fuzzy state vector to small changes in the transition matrix, thereby enabling precise sensitivity analysis and principled tuning of fuzzy system behavior.

Theorem 6. *Let $T_\theta : \mathscr{F}(Q) \to \mathscr{F}(Q)$ be a family of fuzzy transition operators for a general fuzzy machine, parameterized by $\theta \in \mathbf{R}^p$, each being a contraction with common constant $L < 1$. Then for any θ_1, θ_2, the corresponding fixed points satisfy*

$$\|\mu^*(\theta_1) - \mu^*(\theta_2)\|_\infty \leq \frac{1}{1-L}\|T_{\theta_1}(\mu^*(\theta_2)) - T_{\theta_2}(\mu^*(\theta_2))\|_\infty.$$

Proof. Let $\mu_1 := \mu^*(\theta_1)$ and $\mu_2 := \mu^*(\theta_2)$. These are the equilibrium fuzzy states of the machine corresponding to two different parameter settings. Since $T_{\theta_1}(\mu_1) = \mu_1$, we may write

$$\|\mu_1 - \mu_2\|_\infty = \|T_{\theta_1}(\mu_1) - \mu_2\|_\infty.$$

Adding and subtracting $T_{\theta_1}(\mu_2)$, and applying the triangle inequality, gives

$$\|\mu_1 - \mu_2\|_\infty \leq \|T_{\theta_1}(\mu_1) - T_{\theta_1}(\mu_2)\|_\infty + \|T_{\theta_1}(\mu_2) - \mu_2\|_\infty.$$

The first term is bounded by the contraction property of the machine's transition operator:

$$\|T_{\theta_1}(\mu_1) - T_{\theta_1}(\mu_2)\|_\infty \leq L\|\mu_1 - \mu_2\|_\infty.$$

Substituting and rearranging yields

$$(1-L)\|\mu_1 - \mu_2\|_\infty \leq \|T_{\theta_1}(\mu_2) - \mu_2\|_\infty.$$

Since μ_2 is the fixed point of T_{θ_2}, we have $\mu_2 = T_{\theta_2}(\mu_2)$, hence

$$\|T_{\theta_1}(\mu_2) - \mu_2\|_\infty = \|T_{\theta_1}(\mu_2) - T_{\theta_2}(\mu_2)\|_\infty.$$

This term measures how much the two parameterized transition operators differ when applied to the same fuzzy state. Therefore,

$$\|\mu^*(\theta_1) - \mu^*(\theta_2)\|_\infty \leq \frac{1}{1-L}\|T_{\theta_1}(\mu^*(\theta_2)) - T_{\theta_2}(\mu^*(\theta_2))\|_\infty.$$

This bound shows that the sensitivity of the machine's equilibrium state to parameter changes is controlled by the contraction strength L and the magnitude of the perturbation in the transition operator itself. □

5 Conclusion

In this paper, we developed a rigorous analytical framework for general fuzzy machines (GFMs) by integrating concepts from fixed-point theory and sensitivity analysis. By modeling the dynamics of a GFM as a parameterized transition operator T_θ on the compact space $\mathscr{F}(Q) = [0,1]^n$, we established conditions for the existence, uniqueness, and differentiability of the fixed point $\mu^*(\theta)$. The Banach contraction principle guaranteed a unique steady-state fuzzy vector, while the implicit function theorem provided an exact expression for its parametric sensitivity.

Our theoretical results were complemented by numerical examples. The linear sensitivity example confirmed that the derived Lipschitz bound $L_\theta/(1-L_\mu)$ is valid yet conservative, with the actual sensitivity often lying far below the worst-case estimate. A nonlinear clipping example further illustrated that fixed-point behavior can be stable even in the presence of non-smooth components, provided the local contraction condition holds.

Importantly, the max-min composition example highlighted a structural limitation: non-contractive operators may admit multiple fixed points, making Banach's theorem inapplicable. This observation underscores the need for principled operator design in fuzzy systems, particularly when robustness and predictability are required.

Overall, the proposed framework bridges fuzzy automata theory with classical functional analysis, offering both theoretical guarantees and practical tools for parameter tuning, robustness analysis, and system identification. Future work will extend these results to time-varying parameters, stochastic fuzzy systems, and adaptive control architectures. Additionally, a practical direction that we intend to pursue is the validation of the proposed analytical framework on real-world fuzzy systems, particularly in areas such as multi-criteria decision-making, pattern recognition, and fuzzy control, where the sensitivity analysis and stability bounds derived in this paper can provide useful insights for parameter tuning and robust design.

Authors' Contributions

The authors equally contributed to this work. All authors read and approved the final manuscript.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there is no conflict of interest.

Ethical Considerations

The authors have diligently addressed ethical concerns, such as informed consent, plagiarism, data fabrication, misconduct, falsification, double publication, redundancy, submission, and other related matters.

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