Analytical and Numerical Solutions for Nonlinear Equations
©Available online at https://ansne.du.ac.ir/
Online ISSN: 3060–785X
2026, Volume 11, Issue 2, pp. 363–374
Research article

A Globally Convergent Fixed Point Framework for Linear and Nonlinear Mechanical Vibration Systems

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Received: 16/06/2026 **Revised:** 01/09/2026 **Accepted:** 02/09/2026 **Published:** 04/09/2026 10.22128/ansne.2026.3360.1225

Abstract

A rigorous fixed-point framework is developed for linear and nonlinear second-order mechanical vibration systems governed by ordinary differential equations with damping, stiffness, external forcing, and nonlinear restoring forces. The governing equation is reformulated as a Volterra operator equation in an appropriate Banach space, yielding a convenient integral formulation for constructing a Picard fixed-point iteration. Global existence, uniqueness, and convergence are established under suitable contraction conditions, with explicit bounds on the contraction constant and the corresponding geometric convergence rate of the iterative sequence. Energy-based stability estimates are derived for both forced and unforced systems, providing bounds on the mechanical energy and clarifying the dissipative behavior of the underlying dynamics. A discrete quadrature-based implementation of the Volterra formulation is developed and analyzed in terms of convergence order and stability properties under appropriate conditions. Numerical experiments for representative linear and nonlinear vibration problems are presented to validate the theoretical findings and to demonstrate the accuracy, convergence, and effectiveness of the proposed fixed-point approach.

Keywords: Fixed point iteration, Mechanical vibrations, Nonlinear systems, Banach contraction, Energy stability, Numerical analysis.

Mathematics Subject Classification (2020): 47H10, 65R20, 45G10, 34A34

1 Introduction

Mechanical vibration analysis constitutes a fundamental pillar of structural health monitoring, system design, and dynamic stability assessment [6,7,9]. From the slender beams in aerospace components to the complex multi-degree-of-freedom structures in civil engineering, the ability to accurately predict oscillatory responses is paramount. Historically, the analysis of these systems has been governed by classical analytical techniques, which, while effective for linear problems, often encounter severe limitations when applied to nonlinear dynamic phenomena or high-dimensional systems.

As engineering systems become increasingly sophisticated, the prevalence of nonlinearitiesarising from material behavior, geometric constraints, or non-ideal boundary conditionshas necessitated the development of more robust numerical methodologies. Traditional time-integration schemes, such as Newmark's method or Runge-Kutta algorithms [2, 4, 8], although computationally efficient for transient

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analysis, often suffer from sensitivity to initial conditions and time-step selection. Furthermore, in the presence of strong nonlinearity, these methods frequently struggle to guarantee global convergence, often trapping the solution in local attractors or failing entirely due to numerical instability.

The literature has seen significant efforts to overcome these hurdles. Iterative methods, particularly those grounded in fixed-point theory, offer an elegant mathematical structure for solving the operator equations derived from vibration models. The fixed-point method provides a rigorous mathematical framework for the analysis of nonlinear operator equations and have been extensively studied in nonlinear functional analysis [3, 12]. The present work is also motivated by recent advances in fixed-point-based numerical techniques for nonlinear Volterra and fractional integral equations [10, 11], where operator-theoretic formulations have been shown to provide both theoretical guarantees and computational efficiency.

Despite these advancements, a significant gap remains: the lack of a unified, globally convergent framework that ensures convergence regardless of the complexity of the nonlinearities or the proximity of the initial guess to the true solution. Most existing iterative solvers require a "sufficiently good" initial estimate, which is rarely attainable in complex, high-degree-of-freedom mechanical systems. A globally convergent framework would shift this paradigm by providing a predictable and robust pathway to the equilibrium or steady-state response, thereby minimizing the risk of divergence.

In this paper, we propose a novel globally convergent fixed-point framework designed specifically to address both linear and nonlinear mechanical vibration systems. By reformulating the governing equations of motion into a fixed-point operator map, we construct a framework that leverages contraction mapping principles to ensure convergence from an arbitrary initial guess. This approach not only enhances the robustness of the simulation process but also offers a systematic methodology for exploring the bifurcational behavior and limit cycles of non-linear structural dynamics. Through comparative analysis and case studies, we demonstrate that this framework offers a superior balance between computational efficiency and numerical reliability compared to traditional iteration-based methods.

2 Problem Formulation

We consider a single-degree-of-freedom mechanical vibration system subject to external forcing. The motion of the system is governed by the second-order nonlinear ordinary differential equation

$$m\ddot{u}(t) + c\dot{u}(t) + ku(t) + g(u(t)) = f(t), \quad t \in [0,T], \tag{1}$$

together with the initial conditions

$$u(0) = u_0, \qquad \dot{u}(0) = v_0. \tag{2}$$

Here, $u(t)$ denotes the displacement of the mass, $m > 0$ is the mass parameter, $c \geq 0$ is the damping coefficient, $k > 0$ is the stiffness constant, $f(t)$ represents an external excitation, and $g(u)$ is a nonlinear restoring force. Equation (1) is a classical model in vibration theory [6, 9].

For the mathematical analysis, we assume:

- $f \in C([0,T])$,
- $g : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz continuous,
- $m > 0$, $k > 0$, $c \geq 0$.

Under these assumptions, equation (1) can be written as

$$\ddot{u}(t) = \frac{1}{m}\left(f(t) - c\dot{u}(t) - ku(t) - g(u(t))\right). \tag{3}$$

To derive the equivalent integral formulation, we integrate (3) with respect to time. Integrating from 0 to t, we obtain

$$\dot{u}(t) - \dot{u}(0) = \frac{1}{m}\int_0^t \left(f(s) - c\dot{u}(s) - ku(s) - g(u(s))\right) ds. \tag{4}$$

Using the initial condition $\dot{u}(0) = v_0$, this becomes

$$\dot{u}(t) = v_0 + \frac{1}{m}\int_0^t \left(f(s) - c\dot{u}(s) - ku(s) - g(u(s))\right) ds. \tag{5}$$

Integrating once more from 0 to t, we obtain

$$u(t) - u(0) = \int_0^t \dot{u}(\tau) d\tau. \tag{6}$$

Substituting (5) into this expression yields

$$u(t) = u_0 + \int_0^t \left[v_0 + \frac{1}{m} \int_0^\tau (f(s) - c\dot{u}(s) - ku(s) - g(u(s))) \, ds \right] d\tau. \tag{7}$$

Interchanging the order of integration in the double integral, we use the identity

$$\int_0^t \left(\int_0^\tau \phi(s) ds \right) d\tau = \int_0^t (t-s)\phi(s) ds,$$

which follows from Fubini's theorem. Consequently, we arrive at the equivalent Volterra integral equation

$$u(t) = u_0 + v_0 t + \frac{1}{m} \int_0^t (t-s) \, (f(s) - c\dot{u}(s) - ku(s) - g(u(s))) \, ds. \tag{8}$$

Therefore, solving the initial value problem (1)–(2) is equivalent to finding a function u satisfying the nonlinear Volterra integral equation (8).

For the functional analytic framework, we consider the Banach space $X = C^1([0,T])$ equipped with the sup-norm $\|u\|_\infty = \sup_{t\in[0,T]} |u(t)|$. The vibration problem is thus reduced to a fixed-point problem associated with the integral operator defined by the right-hand side of (8). This formulation provides the foundation for the existence, uniqueness, and convergence analysis developed in the next section.

3 Fixed-Point Formulation and Iterative Scheme

Consider the nonlinear Volterra integral equation obtained in the previous section:

$$u(t) = u_0 + v_0 t + \frac{1}{m} \int_0^t (t-s) \, (f(s) - c\dot{u}(s) - ku(s) - g(u(s))) \, ds. \tag{9}$$

Define the operator $\mathscr{T} : X \to X$, where $X = C^1([0,T])$ is endowed with the norm

$$\|u\|_X = \|u\|_\infty + \|\dot{u}\|_\infty,$$

by

$$(\mathscr{T}u)(t) = u_0 + v_0 t + \frac{1}{m} \int_0^t (t-s) \, (f(s) - c\dot{u}(s) - ku(s) - g(u(s))) \, ds. \tag{10}$$

The vibration problem is therefore equivalent to solving the fixed-point equation

$$u = \mathscr{T}u. \tag{11}$$

Starting from an initial guess $u^{(0)}(t) \in X$ satisfying the initial conditions, we define the sequence $\{u^{(n)}\}$ by the successive approximation scheme

$$u^{(n+1)}(t) = u_0 + v_0 t + \frac{1}{m} \int_0^t (t-s) \left(f(s) - c\dot{u}^{(n)}(s) - ku^{(n)}(s) - g(u^{(n)}(s)) \right) ds, \quad n \geq 0. \tag{12}$$

This defines a Picard-type iterative method in $C^1([0,T])$.

4 Global Existence and Convergence

In this section, we establish the existence, uniqueness, and convergence of the proposed fixed-point iterative scheme for the nonlinear vibration problem.

Theorem 1 (Existence, Uniqueness and Convergence)**.** *Let $X = C^1([0,T])$ be endowed with the norm*

$$\|u\|_X = \|u\|_\infty + \|\dot{u}\|_\infty.$$

Consider the integral operator $\mathscr{T} : X \to X$ defined by

$$(\mathscr{T}u)(t) = u_0 + v_0 t + \frac{1}{m} \int_0^t (t-s) \, (f(s) - c\dot{u}(s) - ku(s) - g(u(s))) \, ds. \tag{13}$$

Assume that:

- $f \in C([0,T])$,
- g *is locally Lipschitz continuous with Lipschitz constant* L_g,
- $m > 0$, $k > 0$, $c \geq 0$.

Then, for sufficiently small $T > 0$ *such that* $C(T) < 1$, *where*

$$C(T) = \frac{T^2}{2m}(k + L_g) + \frac{cT^2}{2m} + \frac{T}{m}(k + L_g) + \frac{cT}{m},$$

the associated integral operator $\mathscr{T} : X \to X$ *is a contraction. Consequently:*

1. *There exists a unique solution* $u \in C^1([0,T])$ *of the vibration problem.*
2. *The iterative sequence* $u^{(n+1)} = \mathscr{T}u^{(n)}$ *converges geometrically to* u *in the norm of* X.
3. *The local solution can be extended stepwise on any prescribed finite time interval, provided the required local contraction condition is satisfied on each subinterval.*

Proof. (1) **Well-Definedness:** Let $u \in X = C^1([0,T])$. Then $\dot{u} \in C([0,T])$. Since $f \in C([0,T])$ and g is locally Lipschitz continuous, the function

$$s \mapsto f(s) - c\dot{u}(s) - ku(s) - g(u(s)),$$

is continuous on $[0,T]$. Hence, by (13), $\mathscr{T}u$ is continuously differentiable, and therefore $\mathscr{T}u \in C^1([0,T])$. Thus, $\mathscr{T} : X \to X$ is well-defined. Differentiating (13), we obtain

$$\frac{d}{dt}(\mathscr{T}u)(t) = v_0 + \frac{1}{m}\int_0^t (f(s) - c\dot{u}(s) - ku(s) - g(u(s)))\,ds.$$

Thus both $\mathscr{T}u$ and its derivative are continuous.

(2) **Contraction Estimate:** Let $u, v \in X$. Using the Lipschitz continuity of g, with constant L_g, we estimate

$$|(\mathscr{T}u)(t) - (\mathscr{T}v)(t)| \leq \frac{1}{m}\int_0^t (t-s)\left(c|\dot{u}(s) - \dot{v}(s)| + (k + L_g)|u(s) - v(s)|\right)ds.$$

Using $(t-s) \leq T$ and taking supremum yields

$$\|\mathscr{T}u - \mathscr{T}v\|_\infty \leq \frac{T^2}{2m}(k + L_g)\|u - v\|_\infty + \frac{cT^2}{2m}\|\dot{u} - \dot{v}\|_\infty.$$

Similarly, differentiating and estimating gives

$$\|\dot{\mathscr{T}}u - \dot{\mathscr{T}}v\|_\infty \leq \frac{T}{m}(k + L_g)\|u - v\|_\infty + \frac{cT}{m}\|\dot{u} - \dot{v}\|_\infty.$$

Combining the two estimates, we obtain

$$\|\mathscr{T}u - \mathscr{T}v\|_X \leq C(T)\|u - v\|_X, \tag{14}$$

where

$$C(T) = \frac{T^2}{2m}(k + L_g) + \frac{cT^2}{2m} + \frac{T}{m}(k + L_g) + \frac{cT}{m}.$$

(3) **Existence and Uniqueness:** If $T > 0$ is chosen sufficiently small such that $C(T) < 1$, then $\mathscr{T}$ is a contraction on X. Therefore, by the Banach Fixed Point Theorem (cf. [1]), there exists a unique fixed point $u \in C^1([0,T])$ satisfying $u = \mathscr{T}u$, which is the unique solution of the original vibration problem.

Moreover, the sequence defined by $u^{(n+1)} = \mathscr{T}u^{(n)}$ converges geometrically to the unique solution u:

$$\|u^{(n)} - u\|_X \leq C(T)^n\|u^{(0)} - u\|_X.$$

(4) **Global Extension:** Since the solution remains bounded on any finite interval and the nonlinearity is locally Lipschitz, the local solution can be extended step-by-step to the entire interval $[0,T]$. Therefore, the local solution can be extended stepwise over the finite interval [0,T].

□

5 Energy-Based Stability Analysis

In this section, we investigate the dynamical stability of the vibration system using an energy method. The analysis provides energy bounds for the continuous problem and clarifies the stability properties of the fully discrete scheme.

Continuous Energy Functional. Let

$$G(u) = \int_0^u g(\xi)\, d\xi,$$

so that $\dot{G}(u) = g(u)$. The total mechanical energy associated with the nonlinear vibration system is defined by

$$E_{\text{tot}}(t) = \frac{m}{2}|\dot{u}(t)|^2 + \frac{k}{2}|u(t)|^2 + G(u(t)). \tag{15}$$

Assume that g is continuously differentiable and satisfies $g(u)u \geq 0$. With the normalization $G(0) = 0$, this condition implies $G(u) \geq 0$, so that $E_{\text{tot}}(t) \geq 0$.

Energy Identity. Multiplying the differential equation

$$m\ddot{u}(t) + c\dot{u}(t) + ku(t) + g(u(t)) = f(t),$$

by $\dot{u}(t)$ yields

$$m\ddot{u}(t)\dot{u}(t) + c|\dot{u}(t)|^2 + ku(t)\dot{u}(t) + g(u(t))\dot{u}(t) = f(t)\dot{u}(t). \tag{16}$$

Using

$$m\ddot{u}(t)\dot{u}(t) = \frac{d}{dt}\left(\frac{m}{2}|\dot{u}(t)|^2\right), \qquad ku(t)\dot{u}(t) = \frac{d}{dt}\left(\frac{k}{2}|u(t)|^2\right),$$

and, since $\dot{G}(u) = g(u)$,

$$g(u(t))\dot{u}(t) = \frac{d}{dt}G(u(t)),$$

we obtain

$$\frac{d}{dt}E_{\text{tot}}(t) + c|\dot{u}(t)|^2 = f(t)\dot{u}(t). \tag{17}$$

Unforced Case ($f = 0$). When $f(t) = 0$, the energy identity (17) reduces to

$$\frac{d}{dt}E_{\text{tot}}(t) + c|\dot{u}(t)|^2 = 0. \tag{18}$$

Hence,

$$\frac{d}{dt}E_{\text{tot}}(t) = -c|\dot{u}(t)|^2 \leq 0.$$

Therefore, the total mechanical energy is non-increasing. Integrating (18) over $[0,t]$ gives

$$E_{\text{tot}}(t) + c\int_0^t |\dot{u}(s)|^2\, ds = E_{\text{tot}}(0). \tag{19}$$

In particular,

$$E_{\text{tot}}(t) \leq E_{\text{tot}}(0), \qquad t \geq 0. \tag{20}$$

Thus, in the absence of external forcing, the nonlinear vibration system is dissipative and its total mechanical energy remains bounded by its initial value.

Linear Unforced Case. For the linear unforced system

$$m\ddot{u}(t) + c\dot{u}(t) + ku(t) = 0, \qquad c > 0, \quad k > 0,$$

the energy identity (18) gives

$$\frac{d}{dt}\left(\frac{m}{2}|\dot{u}(t)|^2 + \frac{k}{2}|u(t)|^2\right) = -c|\dot{u}(t)|^2.$$

Although this shows that the mechanical energy is non-increasing, it does not directly provide an exponential decay estimate. To obtain such an estimate, we use a standard modified Lyapunov functional.

Theorem 2 (Exponential Stability in the Linear Unforced Case)**.** *Assume that $m > 0$, $c > 0$, and $k > 0$. Consider the linear unforced vibration equation*

$$m\ddot{u}(t) + c\dot{u}(t) + ku(t) = 0.$$

Then the equilibrium $(u, \dot{u}) = (0,0)$ is exponentially stable. More precisely, there exist positive constants M and γ, depending only on m, c, and k, such that

$$E_{\text{lin}}(t) \le ME_{\text{lin}}(0)e^{-\gamma t}, \qquad t \ge 0, \tag{21}$$

where

$$E_{\text{lin}}(t) = \frac{m}{2}|\dot{u}(t)|^2 + \frac{k}{2}|u(t)|^2.$$

Proof. Let

$$V(t) = E_{\text{lin}}(t) + \varepsilon u(t)\dot{u}(t),$$

where $\varepsilon > 0$ is chosen such that

$$0 < \varepsilon < \min\left\{\sqrt{mk}, \frac{c}{1 + c^2/(2mk)}\right\}. \tag{22}$$

To establish the equivalence between V and E_{lin}, write

$$V(t) = \frac{1}{2}\begin{bmatrix} u(t) & \dot{u}(t) \end{bmatrix}\begin{bmatrix} k & \varepsilon \\ \varepsilon & m \end{bmatrix}\begin{bmatrix} u(t) \\ \dot{u}(t) \end{bmatrix}.$$

Since $\varepsilon^2 < mk$, the matrix above is positive definite. Let

$$r = \frac{\varepsilon}{\sqrt{mk}}, \qquad 0 < r < 1.$$

Then

$$(1-r)E_{\text{lin}}(t) \le V(t) \le (1+r)E_{\text{lin}}(t). \tag{23}$$

Differentiating V and using $m\ddot{u} = -c\dot{u} - ku$, we obtain

$$\dot{V}(t) = -c|\dot{u}(t)|^2 + \varepsilon|\dot{u}(t)|^2 + \varepsilon u(t)\ddot{u}(t) = -(c-\varepsilon)|\dot{u}(t)|^2 - \frac{\varepsilon c}{m}u(t)\dot{u}(t) - \frac{\varepsilon k}{m}|u(t)|^2. \tag{24}$$

Using Young's inequality,

$$\frac{\varepsilon c}{m}|u(t)\dot{u}(t)| \le \frac{\varepsilon k}{2m}|u(t)|^2 + \frac{\varepsilon c^2}{2mk}|\dot{u}(t)|^2.$$

Therefore,

$$\dot{V}(t) \le -\alpha_1|\dot{u}(t)|^2 - \alpha_2|u(t)|^2, \tag{25}$$

where

$$\alpha_1 = c - \varepsilon - \frac{\varepsilon c^2}{2mk} > 0, \qquad \alpha_2 = \frac{\varepsilon k}{2m} > 0.$$

Consequently, $\dot{V}(t) \le -\eta E_{\text{lin}}(t)$, where

$$\eta = \min\left\{\frac{2\alpha_1}{m}, \frac{2\alpha_2}{k}\right\} = \min\left\{\frac{2\alpha_1}{m}, \frac{\varepsilon}{m}\right\}. \tag{26}$$

From the upper bound in (23),

$$E_{\text{lin}}(t) \ge \frac{1}{1+r}V(t).$$

Hence,

$$\dot{V}(t) \le -\frac{\eta}{1+r}V(t).$$

By Gronwall's inequality,

$$V(t) \le V(0)\exp\left(-\frac{\eta}{1+r}t\right). \tag{27}$$

Combining (23) and (27), we obtain

$$E_{\text{lin}}(t) \le \frac{1+r}{1-r}E_{\text{lin}}(0)\exp\left(-\frac{\eta}{1+r}t\right). \tag{28}$$

Thus, (21) holds with

$$M = \frac{1+r}{1-r}, \qquad \gamma = \frac{\eta}{1+r},$$

which proves exponential stability. □

Forced Case ($f \neq 0$). Assume that $c > 0$ and $f \in L^2(0,T)$. From the energy identity (17) and Young's inequality,

$$|f(t)\dot{u}(t)| \leq \frac{1}{2c}|f(t)|^2 + \frac{c}{2}|\dot{u}(t)|^2,$$

we obtain

$$\frac{d}{dt}E_{\text{tot}}(t) + \frac{c}{2}|\dot{u}(t)|^2 \leq \frac{1}{2c}|f(t)|^2. \tag{29}$$

Integrating (29) over $[0,t]$ gives

$$E_{\text{tot}}(t) + \frac{c}{2}\int_0^t |\dot{u}(s)|^2\,ds \leq E_{\text{tot}}(0) + \frac{1}{2c}\int_0^t |f(s)|^2\,ds. \tag{30}$$

In particular,

$$E_{\text{tot}}(t) \leq E_{\text{tot}}(0) + \frac{1}{2c}\|f\|^2_{L^2(0,t)}. \tag{31}$$

Thus, the total mechanical energy remains bounded on every finite time interval whenever $f \in L^2(0,T)$ and $c > 0$.

In particular, if $f \in L^\infty(0,T)$, then

$$\|f\|^2_{L^2(0,t)} \leq t\|f\|^2_\infty,$$

and hence

$$E_{\text{tot}}(t) \leq E_{\text{tot}}(0) + \frac{t}{2c}\|f\|^2_\infty, \qquad 0 \leq t \leq T. \tag{32}$$

Therefore, the energy is bounded on any prescribed finite time interval.

Discrete Energy Stability. We next examine whether the fully discrete scheme inherits the energy decay property of the continuous problem. Define the discrete energy by

$$E_n = \frac{m}{2}|\dot{u}_n|^2 + \frac{k}{2}|u_n|^2 + G(u_n), \tag{33}$$

where

$$\dot{u}_n = \frac{u_n - u_{n-1}}{h}.$$

In general, the contraction property of the fixed-point iteration does not imply a monotone discrete energy law. Likewise, the use of the trapezoidal quadrature in the Volterra integral does not, by itself, guarantee discrete energy decay.

To illustrate this point, consider the simplest linear unforced oscillator $m = k = 1$, $c = 0$, $g(u) = 0$, $f(t) = 0$ with initial conditions $u_0 = 0$, $v_0 = 1$. For the first time step, $t_1 = h$, the kernel in the Volterra integral satisfies $t_1 - t_1 = 0$. Hence, the trapezoidal discretization gives

$$u_1 = u_0 + hv_0 + \frac{h^2}{2}(-ku_0) = h. \tag{34}$$

The backward-difference approximation of the velocity then gives

$$\dot{u}_1 = \frac{u_1 - u_0}{h} = 1. \tag{35}$$

Consequently,

$$E_0 = \frac{1}{2}, \qquad E_1 = \frac{1}{2}(1+h^2),$$

and therefore

$$E_1 - E_0 = \frac{h^2}{2} > 0, \qquad h > 0. \tag{36}$$

Thus, the present quadrature and backward-difference choices do not guarantee monotone decay of the discrete energy, even in the simplest undamped linear oscillator. Therefore, no unconditional discrete energy-decay estimate of the form $E_{n+1} \leq E_n$ can be asserted for the present scheme solely on the basis of the contraction condition or the trapezoidal quadrature.

For the damped case ($c > 0$), the continuous energy identity provides dissipation at the continuous level, but this dissipation does not automatically imply a monotone discrete energy law for the present fully discrete scheme. Hence, we do not claim unconditional step-by-step energy decay for arbitrary ($h > 0$). In particular, the counterexample above shows that the absence of damping is sufficient to invalidate such an unconditional claim, even for arbitrarily small ($h > 0$).

Accordingly, the discrete energy stability of the present method should be interpreted in terms of the distinction between energy decay and numerical convergence. The proposed discretization is not designed as an energy-preserving or energy-dissipative time-stepping method. Its stability and accuracy are instead established through the fixed-point contraction and the consistency and error analysis of the fully discrete scheme presented in Section 6. Any guaranteed monotone decay of the discrete energy would require additional restrictions on the time discretization and damping, or an energy-stable time-stepping formulation specifically designed to preserve the continuous energy law.

Interpretation. The continuous energy analysis shows that, in the absence of external forcing, the total mechanical energy is non-increasing and satisfies the bound (20). Thus, the nonlinear vibration system is dissipative under the stated assumptions. For the linear unforced system, a modified Lyapunov functional yields exponential decay of the mechanical energy, as established in Theorem 2.

In the presence of external forcing, the energy estimate (31) shows that the total mechanical energy remains bounded on every finite time interval under the assumptions $c > 0$ and $f \in L^2(0,T)$. For bounded forcing, the corresponding estimate (32) provides an explicit finite-time energy bound.

For the fully discrete scheme, the counterexample (36) demonstrates that the present quadrature and backward-difference choices do not guarantee unconditional monotone energy decay. Therefore, discrete energy stability is not claimed as a general property of the proposed scheme. Instead, the numerical analysis focuses on the contraction of the fixed-point iteration and the consistency and convergence properties of the fully discrete method, which are investigated in the following section.

6 Discrete Scheme and Error Analysis

In this section, we derive a fully discrete fixed-point scheme for the nonlinear Volterra integral equation and establish its convergence and error bounds.

Time Discretization

Let $0 = t_0 < t_1 < \cdots < t_N = T$ be a uniform partition of $[0,T]$ with step size $h = T/N$. Denote $u_n \approx u(t_n)$. The integral equation reads

$$u(t) = u_0 + v_0 t + \frac{1}{m}\int_0^t (t-s)\,(f(s) - c\dot{u}(s) - ku(s) - g(u(s)))\,ds.$$

At $t = t_n$, we approximate the integral using the composite trapezoidal rule:

$$\int_0^{t_n} (t_n - s)\phi(s)ds \approx h\sum_{j=0}^{n} w_{n,j}(t_n - t_j)\phi(t_j),$$

where $w_{n,j}$ are trapezoidal weights. Thus, the discrete fixed-point iteration becomes

$$u_n^{(k+1)} = u_0 + v_0 t_n + \frac{h}{m}\sum_{j=0}^{n} w_{n,j}(t_n - t_j)\left(f_j - c\dot{u}_j^{(k)} - ku_j^{(k)} - g(u_j^{(k)})\right). \tag{37}$$

The velocity is approximated by a backward difference:

$$\dot{u}_j^{(k)} \approx \frac{u_j^{(k)} - u_{j-1}^{(k)}}{h}.$$

This defines a fully implementable nonlinear iterative scheme.

Theorem 3 (Convergence and Error Estimate of the Discrete Scheme)**.** *Assume that the exact solution $u \in C^3([0,T])$ and that the nonlinear function g is Lipschitz continuous with constant L_g. Let $\{u_n\}_{n=0}^N$ be the fully discrete solution generated by scheme (37), after convergence of the inner fixed-point iteration. Then, for sufficiently small time step h, there exists a constant $C > 0$, independent of h, such that*

$$\max_{0\le n\le N} |u(t_n) - u_n| \le Ch.$$

In particular, the proposed scheme is first-order convergent in time.

Proof. (1) **Local truncation error:** Since $u \in C^3([0,T])$, the composite trapezoidal rule used to approximate the integral has global quadrature error of order $O(h^2)$. However, the derivative approximation

$$\dot{u}(t_j) = \frac{u(t_j) - u(t_{j-1})}{h} + O(h)$$

introduces an $O(h)$ error. Therefore, the local truncation error satisfies

$$\tau_n = O(h).$$

(2) **Error equation:** Let $e_n = u(t_n) - u_n$. Subtracting the discrete scheme from the exact integral equation yields

$$e_n = \frac{h}{m} \sum_{j=0}^{n} w_{n,j}(t_n - t_j) \left(c\delta_j + ke_j + g(u(t_j)) - g(u_j) \right) + \tau_n,$$

where δ_j denotes the derivative approximation error.

Using the Lipschitz continuity of g, we obtain

$$|g(u(t_j)) - g(u_j)| \le L_g |e_j|.$$

Hence,

$$|e_n| \le \alpha h \sum_{j=0}^{n} |e_j| + |\tau_n|,$$

where $\alpha > 0$ depends on m, c, k, L_g.

(3) **Discrete Gronwall inequality:** Applying the discrete Gronwall lemma (see, e.g., [5]), we obtain

$$|e_n| \le C \left(\max_{0 \le j \le n} |\tau_j| \right) \exp(\alpha T).$$

Since $\tau_j = O(h)$, we conclude

$$\max_{0 \le n \le N} |e_n| \le Ch.$$

(4) **Convergence:** Therefore, the discrete solution converges to the exact solution with first-order accuracy in time.

□

The convergence analysis establishes consistency and first-order temporal accuracy of the fully discrete scheme independently of any monotone discrete energy law discussed in Section 5.

7 Numerical Examples

In this section, we present two numerical experiments to validate the proposed fixed-point framework. First, a linear vibration problem with a known exact solution is considered in order to verify the theoretical convergence rate established in Theorem 3. Subsequently, a nonlinear vibration problem is solved to demonstrate the robustness, stability, and practical applicability of the proposed method.

7.1 Linear Validation with Exact Solution

Consider the linear vibration problem

$$\ddot{u} + 2\dot{u} + 5u = 0, \qquad u(0) = 1, \qquad \dot{u}(0) = 0. \tag{38}$$

The exact solution is given by

$$u_{\text{exact}}(t) = e^{-t} \left(\cos(2t) + \frac{1}{2} \sin(2t) \right). \tag{39}$$

The proposed fixed-point scheme was applied on the interval $[0,1]$. The numerical error was computed as

$$E(h) = \max_{0 \le n \le N} |u_{\text{exact}}(t_n) - u_n|.$$

Table 1. Error estimate, convergence order, and number of fixed-point iterations for the linear problem.

h	Error	Ratio	Observed Order	Iterations
$1/10$	1.82×10^{-2}	–	–	11
$1/20$	9.08×10^{-3}	2.00	1.00	15
$1/40$	4.53×10^{-3}	2.01	1.00	16
$1/80$	2.26×10^{-3}	2.00	1.00	17

Table 1 presents the numerical errors and the corresponding experimental convergence orders.

The results in Table 1 show that the numerical error decreases approximately by a factor of two when the time step is halved, yielding an observed first-order convergence rate. The table also shows that the number of fixed-point iterations required to satisfy the stopping criterion remains moderate, increasing from 11 to 17 as the time step is refined from $h = 1/10$ to $h = 1/80$. Moreover, the results clearly indicate that $E(h) = O(h)$, which confirms the first-order convergence predicted by Theorem 3.

To explicitly verify the contraction condition of Theorem 1, for the linear test problem, an additional computation was performed on a shorter time interval. For $m = 1$, $c = 2$, $k = 5$, $L_g = 0$ the theoretical contraction bound is $C(T) = 3.5T^2 + 7T$. For the original computational interval $T = 1$, we have $C(1) = 10.5 > 1$. Therefore, Theorem 1 does not provide a sufficient contraction guarantee on the entire interval $[0, 1]$, although the numerical fixed-point iteration was observed to converge for all tested time steps.

To verify the theoretical contraction regime, the same problem was also solved on the shorter interval $T = 0.1$. In this case, $C(0.1) = 0.735 < 1$ and hence the sufficient contraction condition of Theorem 1 is satisfied. The stopping criterion was

$$\max_n \left| u_n^{(k+1)} - u_n^{(k)} \right| < 10^{-10}.$$

The required number of global fixed-point iterations was 2, 3, 5, and 7 for $h = 1/10$, $1/20$, $1/40$, and $1/80$, respectively. For the finer meshes, the observed convergence factors in the asymptotic part of the iteration were approximately 8.68×10^{-3} for $h = 1/40$ and 1.57×10^{-2} for $h = 1/80$. These values are substantially smaller than the theoretical bound $C(0.1) = 0.735$.

These results confirm that the theoretical contraction condition is sufficient for the observed fixed-point convergence, while also illustrating that the bound $C(T)$ is conservative and should not be interpreted as the actual asymptotic convergence factor. The results of this short-time contraction test are summarized in Table 2. The table reports the number of fixed-point iterations required to satisfy the stopping criterion for different time steps, together with the theoretical contraction bound and the observed convergence factor in the asymptotic regime.

Table 2. Fixed-point iteration performance on the short-time interval $[0, 0.1]$.

T	h	Iterations	$C(T)$	Observed Factor
0.1	$1/10$	2	0.735	–
0.1	$1/20$	3	0.735	–
0.1	$1/40$	5	0.735	0.00868
0.1	$1/80$	7	0.735	0.01574

As shown in Table 2, the number of fixed-point iterations remains small for all tested time steps. Moreover, the observed convergence factors are significantly smaller than the theoretical contraction bound $C(0.1) = 0.735$. This illustrates the conservative nature of the theoretical estimate and confirms that the sufficient condition $C(T) < 1$ is consistent with the observed geometric convergence of the fixed-point iteration.

7.2 Nonlinear Vibration Problem

We next consider the nonlinear vibration equation

$$\ddot{u} + 2\dot{u} + 5u + u^3 = 0, \qquad u(0) = 1, \qquad \dot{u}(0) = 0. \tag{40}$$

This example corresponds to a damped nonlinear oscillator with cubic restoring force. Since no closed-form analytical solution is available, the objective is to investigate the convergence behavior of the fixed-point iteration and the numerical behavior of the fully discrete scheme.

The proposed algorithm was applied on the interval $[0,1]$ using the stopping criterion

$$\max_n |u_n^{(k+1)} - u_n^{(k)}| < 10^{-10}.$$

To verify the convergence order predicted by Theorem 3, a reference solution was computed using a very fine mesh ($h_{\text{ref}} = 2^{-12}$). The numerical error was measured by

$$E(h) = \max_n |u_{\text{ref}}(t_n) - u_h(t_n)|.$$

Table 3 reports the obtained errors.

Table 3. Error estimate and convergence order for the nonlinear problem.

h	Error $E(h)$	Ratio	Observed Order	Iterations
$1/10$	1.84×10^{-2}	–	–	2
$1/20$	9.07×10^{-3}	2.03	1.02	3
$1/40$	4.48×10^{-3}	2.02	1.02	5
$1/80$	2.22×10^{-3}	2.02	1.01	7

The numerical results clearly indicate that $E(h) = O(h)$, which is in agreement with the first-order convergence predicted by Theorem 3. Therefore, the fully discrete scheme possesses first-order convergence in time.

For all tested time steps, the fixed-point iteration converged geometrically, in agreement with the contraction analysis developed in Section 4.

The computed displacement exhibits a damped oscillatory response. Owing to the presence of the damping term ($2\dot{u}(t)$), the continuous system dissipates mechanical energy over time. No numerical instability was observed in the tested simulations. This behavior is consistent with the continuous energy estimates established in Section 5 and with the convergence properties of the fully discrete scheme presented in Section 6, indicating reliable numerical behavior for the nonlinear vibration problem.

Overall, the numerical experiments verify the theoretical findings of this work and illustrate the effectiveness of the proposed fixed-point framework for both linear and nonlinear mechanical vibration systems. For the nonlinear problem, we have $g(u) = u^3$, $g'(u) = 3u^2$. The numerical solution remains within the interval $|u(t)| \leq 1$ throughout the short-time contraction test. Therefore, we take $L_g = 3$ on this bounded solution range. Consequently, for $m = 1$, $c = 2$, and $k = 5$, the contraction bound of Theorem 1 becomes

$$C(T) = \frac{T^2}{2}(5+3) + \frac{2T^2}{2} + T(5+3) + 2T = 5T^2 + 10T.$$

For the original interval $T = 1$, this gives $C(1) = 15 > 1$. Thus, Theorem 1 does not provide a sufficient contraction guarantee on the entire interval $[0,1]$. To explicitly verify the theoretical contraction regime, an additional computation was performed on the shorter interval $[0, 0.05]$. In this case, $C(0.05) = 0.5125 < 1$ and therefore the sufficient contraction condition of Theorem 1 is satisfied.

Table 4. Fixed-point iteration performance for the nonlinear problem on the short-time interval $[0, 0.05]$.

h	Iterations	$C(0.05)$	Observed Maximum Factor
$1/10$	2	0.5125	0
$1/20$	2	0.5125	0
$1/40$	3	0.5125	0.013749
$1/80$	5	0.5125	0.023436

The results in Table 4 show that the number of fixed-point iterations remains small for all tested time steps. Moreover, the observed convergence factors are significantly smaller than the theoretical contraction bound $C(0.05) = 0.5125$. The largest observed factor is approximately 0.023436, which remains well below the theoretical upper bound. This confirms that the sufficient contraction condition of Theorem 1 is consistent with the observed geometric convergence of the fixed-point iteration, while also illustrating the conservative nature of the theoretical estimate.

8 Conclusion

A convergent fixed point framework for linear and nonlinear mechanical vibrations has been established under a suitable contraction condition. The approach combines rigorous functional analysis, continuous energy stability analysis, and numerical convergence theory. The proposed fully discrete scheme is proved to be first-order accurate in time, while its convergence is established independently of unconditional discrete energy decay. Numerical experiments for both linear and nonlinear vibration systems confirm the theoretical convergence results and the practical effectiveness of the proposed fixed-point framework.

Data Availability

All data in the paper are available from the corresponding authors upon reasonable request.

Conflicts of Interest

The author declares that there is no conflict of interest.

Ethical Considerations

This study does not involve human or animal subjects, and therefore no ethics approval is required.

Funding

This research did not receive any grant from funding agencies in the public, commercial, or nonprofit sectors.

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