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Simultaneous Profit Optimization and Market Price Agreement in Networks of LTI Duopoly Systems

Mehrasa Ayatollahi*

Department of Mathematics, Payame Noor University, Tehran, Iran

* **Corresponding author(s):** m_ayatollahi@pnu.ac.ir

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Abstract

This paper studies networks of dynamic duopoly systems with the objectives of maximization of firms' discounted profits and of convergence of market prices in all markets. We model each duopoly as a dynamical process of price adjustment, driven by production decisions of firms. The profit maximization problem is reformulated as a quadratic optimal control problem by representing the dynamics of the duopoly in a linear time-invariant (LTI) state-space form. In order to impose the price convergence between the markets, a consensus term is added to the cost function. We use the Hamilton–Jacobi–Bellman (HJB) approach to derive a control law that encourages profit maximization and market price convergence over a finite time horizon. The proposed framework reformulates the dynamic duopoly problem in networked markets within an optimal control framework, allowing simultaneous profit optimization and market price coordination.

Keywords: Duopoly system, Price agreement, LTI control system, Optimal control, State consensus.

Mathematics Subject Classification (2020): 49N10, 93C05, 93C15

1 Introduction

Optimal control theory offers a powerful framework for the analysis of dynamic systems in economics and finance. One important application of this framework is the study of duopoly markets, where two firms produce homogeneous products and interact strategically over time [3, 9, 14]. Traditional duopoly models have gradually evolved into dynamic differential game frameworks that incorporate time-dependent decision-making, adaptive consumer behavior, and network effects. In this context, differential game techniques have been widely employed to investigate competitive behavior, pricing dynamics and market evolution under network externalities [10].

Recently, the analysis of dynamic economic networks has attracted increasing attention in control theory [7, 11, 17]. Particularly, the duopoly competition has been extended to dynamic state-feedback strategies with consideration of partially observed consumer networks and robustness to external disturbances. These approaches emphasize stability, adaptability and convergence in market-oriented decision-making processes. [1]. Moreover, event-triggered control based distributed Nash equilibrium seeking approaches have been proposed for noncooperative duopoly games to guarantee system stability and improve the performance of multi-agent competitive systems [15].

In multi-agent control theory, consensus refers to the process through which interconnected agents asymptotically reach agreement on a common variable despite differences in initial conditions or locally available information. Integrating consensus mechanisms with optimal control theory provides a unified framework for simultaneously achieving performance optimization and coordination objectives [4,16]. This combination is not only theoretically significant but also practically relevant in modern economic infrastructures. For instance, networks of regional electricity markets can be modeled as interconnected duopoly systems, where each region is dominated by two major electricity producers. Such market structures are often seen in power systems that are not regulated whith a limited number of generators that compete within geographically separated but economically interconnected regions [6, 12]. In such environments, consensus-based coordination and optimal feedback control can contribute importantly to price stability, competition regulation and overall market efficiency.

From an economic perspective, price convergence is particularly desirable in interconnected or regulated markets, where excessive price disparities may reduce market efficiency and hinder market integration. In such environments, the consensus objective can be interpreted as a coordination mechanism that promotes balanced market behavior while preserving competitive incentives.

The existing literature has studied various aspects of the dynamic games and distributed coordination for economic systems. For instance, the work in [1] studied dynamic duopoly competition within a differential game framework using state-feedback strategies and online saddle-point dynamics under disturbances and partial observations. However, the considered framework focuses on an isolated duopoly market and does not address coordination or consensus among multiple interconnected markets. On the other hand, the consensus-based Nash equilibrium seeking methods for distributed multi-agent games have been developed in [18]. In these approaches the consensus mechanisms are essentially used for information exchange and equilibrium estimation, while the players are modeled by simplified integrator-type dynamics. So, the combination of dynamic duopoly models, optimal state-feedback control and consensus-based market coordination has not been much studied. Despite these advances, the incorporation of price consensus directly into the profit optimization objectives of dynamic duopoly systems remains insufficiently explored. In particular, the simultaneous achievement of individual discounted profit maximization and global market price convergence has not yet been adequately addressed within a unified state-space optimal control framework.

Classical differential game formulations for duopoly markets generally assign separate objective functions to individual firms and seek equilibrium strategies arising from strategic interactions. In contrast, the present work reformulates the dynamic duopoly problem within an optimal control framework, where discounted profit optimization and market price coordination are incorporated into a unified performance index. Furthermore, unlike conventional consensus-based approaches that primarily focus on coordination or information exchange, the proposed framework embeds the price agreement objective directly into the optimization criterion and derives an optimal state-feedback law through the HJB framework.

In this paper, we propose a control-theoretic framework for coordinating a network of dynamic duopoly systems. The proposed approach is different from previous studies focusing mainly on isolated markets or equilibrium analysis, as it explicitly studies interacting duopolies in a networked setting and addresses both economic optimization and price consensus in a unified dynamic framework. Each duopoly is modeled in LTI state-space form, which allows the simultaneous use of optimal control and consensus theory. Moreover, the price coordination objective is directly included in the performance index, modeling centralized, regulatory or socially motivated coordination mechanisms.

The main contributions of this work are as follows: (i) The dynamic duopoly problem is reformulated as an optimal control problem for networks of duopoly systems; (ii) a consensus-based performance index is introduced to incorporate market price coordination directly into the optimization objective; and (iii) an HJB-based optimal feedback controller is developed to achieve simultaneous profit optimization and market price agreement.

The paper is organized as follows: Section 2 introduces the necessary preliminaries. Section 3 formulates the optimization problem and presents the main results. Section 4 provides a numerical example and simulation results. Finally, Section 5 concludes the paper with remarks and future research directions.

2 Preliminaries

In this section, we present necessary background required to understand the main topic of the article.

2.1 State-Space Representation

Consider a network of N continuous-time LTI systems. The j-th system dynamics are:

$$\dot{x}_j(t) = A_j x_j(t) + B_j u_j(t), \text{ for } j = 1, \ldots, N, \tag{1}$$

where $x_j \in \mathbb{R}^n$ is the state of the system (which can represent a set of economic variables), $u_j \in \mathbb{R}^m$ is the control input, and $A_j \in \mathbb{R}^{n\times n}$, $B_j \in \mathbb{R}^{n\times m}$ are system matrices.
The overall dynamic of the entire network can be expressed as:

$$\dot{X}(t) = A_D X(t) + B_D U(t), \tag{2}$$

where

$$X(t) = \Big[x_1(t), x_2(t), \ldots, x_N(t)\Big]^T, \qquad U(t) = \Big[u_1(t), u_2(t), \ldots, u_N(t)\Big]^T,$$

and

$$A_D = \operatorname{diag}(A_1, \ldots, A_N), \quad B_D = \operatorname{diag}(B_1, \ldots, B_N).$$

Definition 1 (see [13]). *The network (1) achieves state consensus if the following relation holds over a finite horizon:*

$$\forall j,k: \quad \|x_j(t) - x_k(t)\| \to 0. \tag{3}$$

In other words, state consensus refers to the agreement on a common value of a quantity of interest, reached despite differing initial states among network members.

Definition 2 (see [5]). *Let $S \in \mathbb{R}^{nN\times p}$ be an orthonormal matrix. The network (2) achieves consensus to the subspace $\mathscr{S} = span\{S\}$ if for any initial condition, $X(t)$ converges to a point in $\mathscr{S}$.*

The following lemma provides necessary and sufficient conditions under which state consensus can be achieved via state-feedback control.

Lemma 1 (see [5]). *For the network (2) there exists a control law $U = FX$ such that consensus to $\mathscr{S}$ is achieved if and only if there exists a positive definite matrix $\mathscr{X}$ and a matrix $\mathscr{Y}$ such that $F = \mathscr{Y}\mathscr{X}^{-1}$ and:*

$$\begin{aligned}
&i)\ \mathscr{X} = \mathscr{S}_\perp \mathscr{S}_\perp^T \mathscr{X} \mathscr{S}_\perp \mathscr{S}_\perp^T + \mathscr{S}\mathscr{S}^T \mathscr{X} \mathscr{S}\mathscr{S}^T,\\
&ii)\ A_D \mathscr{X}\mathscr{S} + B_D \mathscr{Y}\mathscr{S} = 0,\\
&iii)\ \mathscr{S}_\perp^T (A_D \mathscr{X} + \mathscr{X} A_D^T + B_D \mathscr{Y} + \mathscr{Y}^T B_D^T)\mathscr{S}_\perp < 0.
\end{aligned}$$

Here, the first condition ensures decomposition into consensus and disagreement subspaces, the second condition is about zero dynamics on agreement or consensus subspace and the third guarantees stability in the disagreement subspace.

In most optimization problems, the linear system (1) is associated with a quadratic cost criterion or performance index of the following general form that should be minimized:

$$J_j = \int_0^T (x_j^T(t) Q_j x_j(t) + u_j^T(t) R_j u_j(t))dt, \text{ for } j = 1, \ldots, N. \tag{4}$$

where $Q_j \in \mathbb{R}^{n\times n}$ is symmetric positive semi-definite and $R_j \in \mathbb{R}^{m\times m}$ is symmetric positive definite. An effective approach to solving the above optimization problem involves seeking a continuously differentiable value function of the following form:

$$V_j(t, x_j) = \min_{u_j} \Big(\int_0^T (x_j^T(t) Q_j x_j(t) + u_j^T(t) R_j u_j(t))dt \Big), \text{ for } j = 1, \ldots, N, \tag{5}$$

which satisfies the terminal condition $V_j(T, x_j) = 0$ and is the solution of a partial differential equation (PDE) known as HJB equation of the following form [2]:

$$-\frac{\partial V_j}{\partial t}(t, x_j) = \min_{u_j} \Big\{ (x_j^T Q_j x_j + u_j^T R_j u_j) + \frac{\partial V_j}{\partial x_j}(t, x_j)(A_j x_j + B_j u_j) \Big\}. \tag{6}$$

2.2 Dynamic Duopoly System

In this paper, we consider a network of N distinct duopoly systems that each of which has two firms. For the i-th $(i = 1,2)$ firm in the j-th $(j = 1,\ldots,N)$ duopoly system, let $v_{j_i}(t) \geqslant 0$ denote the output rate of each firm, $p_j(t)$ be the market price at time t and $a_j > 0$ be constant. Then, according to [7] the relationship between outputs of firms and the market price is stated by $p_j(t) = a_j - (v_{j_1}(t) + v_{j_2}(t))$. To reflect that market prices do not instantaneously adjust to the demand-implied price, Engwerda [8] introduces a price adjustment dynamic in which the current market price is a function of the difference between the current market price and the price indicated by the linear demand function that leads to the following mathematical model:

$$\dot{p}_j(t) = s_j\{a_j - (v_{j_1}(t) + v_{j_2}(t)) - p_j(t)\}, \quad p_j(0) = p_{j0}. \tag{7}$$

where $s_j > 0$ is the adjustment speed parameter.
In addition, the discount profit for the i-th firm $(i = 1,2)$ of the j-th duopoly system $(j = 1,\ldots,N)$ that should be maximized is:

$$J_{j_i} = \int_0^T e^{-rt}\{p_j(t)v_{j_i}(t) - c_{j_i}v_{j_i}(t) - \frac{1}{2}v_{j_i}^2(t)\}dt, \tag{8}$$

where $c_{j_i} \in (0, a_j)$ are fixed parameters and $r > 0$ is the discount rate.

3 Main Results

We assume that each member in the duopoly network (7) seeks to maximize its discounted profit (8). It should be emphasized that the proposed formulation differs from classical differential game approaches. In differential games, firms are usually treated as strategic players seeking Nash equilibrium solutions through individual objective functions. In the present work, the focus is not on equilibrium analysis but rather on constructing an optimal control framework in which the interactions among multiple duopoly systems are coordinated through a consensus-related performance criterion. This reformulation allows the simultaneous treatment of economic optimization and market price coordination within a unified HJB-based design framework. The reformulation of the dynamic duopoly problem into a unified HJB-based optimal control framework that simultaneously addresses profit optimization and market price coordination constitutes the main methodological distinction of the proposed approach from both classical differential game formulations and conventional consensus-based control frameworks. From an economic perspective, market price coordination is introduced as an additional design objective. Accordingly, the market prices of all network members are required to converge to a common value. In other words, the difference between the market prices of any two distinct members converges to zero over a finite horizon, i.e., $\forall j,k$: $\|p_j(t) - p_k(t)\| \to 0$. To reach both of these goals simultaneously, we are going to use the state space representation of duopoly systems (7). To this end, $for\ j = 1,\ldots,N\ and\ i = 1,2$ we introduce the state vector $x_{j_i}(t)$ and the control vector $u_{j_i}(t)$ as follows:

$$x_{j_i}(t) = e^{-rt/2}\begin{bmatrix} p_j(t) \\ 1 \end{bmatrix}, \quad u_{j_i}(t) = e^{-rt/2}v_{j_i}(t) + \begin{bmatrix} -1 & c_{j_i} \end{bmatrix} x_{j_i}(t). \tag{9}$$

Due to (9), the system matrices are given as:

$$A_{j_i} = \begin{bmatrix} -r/2 - 3s_j & (a_j + 2c_{j_i})s_j \\ 0 & -r/2 \end{bmatrix}, \quad B_{j_1} = B_{j_2} = \begin{bmatrix} -s_j \\ 0 \end{bmatrix}. \tag{10}$$

Furthermore, required matrices for performance (6) are as follows:

$$Q_{j_i} = -\frac{1}{2}\begin{bmatrix} 1 & -c_{j_i} \\ -c_{j_i} & c_{j_i}^2 \end{bmatrix}, \quad R_{j_1} = R_{j_2} = -\frac{1}{2}. \tag{11}$$

By incorporating relations (9)–(11) into (8), the discounted profit maximization becomes:

$$\max J_{j_i} = \int_0^T (x_{j_i}^T Q_{j_i} x_{j_i} + u_{j_i}^T R_{j_i} u_{j_i})dt. \tag{12}$$

Since $\max J_{j_i} = \min -J_{j_i}$ by selecting $\tilde{J}_{j_i} = -J_{j_i}$, $\tilde{Q}_{j_i} = -Q_{j_i}$ and $\tilde{R}_{j_i} = -R_{j_i}$ the corresponding minimization problem is:

$$\min \tilde{J}_{j_i} = \int_0^T (x_{j_i}^T \tilde{Q}_{j_i} x_{j_i} + u_{j_i}^T \tilde{R}_{j_i} u_{j_i}) dt. \tag{13}$$

It is noted that $\tilde{R}_{j_i} = \frac{1}{2}$ is positive definite and the eigenvalues of $\tilde{Q}_{j_i}$ are given by:

$$\lambda_1 = \tfrac{1}{2}(1 + c_{j_i}^2), \quad \lambda_2 = 0.$$

Hence, $\tilde{Q}_{j_i}$ is positive semidefinite. Now our objective is to ensure that market prices converge to a common value or equivalently, their difference converges to zero, i.e., $\forall j,k \quad \|p_j(t) - p_k(t)\| \to 0$. By using the state vector defined in (9), this requirement can be expressed in the form of a state-consensus condition as: $\forall j,k \quad \|x_{j_i}(t) - x_{k_i}(t)\| \to 0$. We incorporate this criterion into $\tilde{J}_{j_i}$. In this way, the final form for the optimization problem for the j-th ($j = 1, \ldots, N$) member of the network of duopoly systems converging to a common market price can be summarized as follows:

$$\begin{aligned} &\min \tilde{J}_{j_i} = \int_0^T \Big(\sum_{\substack{k=1 \\ k \neq j}}^{N} (x_{j_i} - x_{k_i})^T \tilde{Q}_{j_i} (x_{j_i} - x_{k_i}) + u_{j_i}^T \tilde{R}_{j_i} u_{j_i} \Big) dt \\ &s.t. \\ &\dot{x}_{j_i} = A_{j_i} x_{j_i} + B_{j_i} \sum_{i=1}^{2} u_{j_i}. \end{aligned} \tag{14}$$

It should be noted that the optimization problem in (14) is formulated using a single performance index that combines economic and coordination objectives. Therefore, the proposed framework should not be interpreted as a classical differential game, where individual players are typically associated with separate objective functions and equilibrium concepts. Instead, the interaction among duopoly systems is addressed through a unified optimal control formulation. The consensus term in the performance index penalizes excessive price differences among interconnected markets. From an economic perspective, this term represents a coordination objective that may arise in integrated or regulated market environments, where reducing large price disparities can improve market efficiency, enhance stability, and support market integration. Therefore, the proposed formulation seeks to balance individual profit optimization with coordinated market behavior through a unified optimal control framework.

The corresponding HJB quation to (14) is:

$$-\frac{\partial V_{j_i}}{\partial t}(t, x_{j_i}) = \min_u \left\{ \sum_{\substack{k=1 \\ k \neq j}}^{N} (x_{j_i} - x_{k_i})^T \tilde{Q}_{j_i} (x_{j_i} - x_{k_i}) + u_{j_i}^T \tilde{R}_{j_i} u_{j_i} + \frac{\partial V_{j_i}}{\partial x_{j_i}}(t, x_{j_i}) \left(A_{j_i} x_{j_i} + B_{j_i} \sum_{i=1}^{2} u_{j_i} \right) \right\}, \tag{15}$$

where V_{j_i} is a value function that should be determined subject to the boundary condition $V_{j_i}(T, x_{j_i}) = 0$.

Since the objective is formulated through a unified performance index, the resulting HJB equation characterizes an optimal control solution rather than a Nash equilibrium solution of a differential game.

Based on the above formulation, the following theorem presents the optimal control law that ensures both profit optimization and price consensus.

Theorem 1. *Assume a network of N duopoly systems with dynamics (7) and discounted profits (8). To achieve individual profit maximization together with market price convergence, the optimization problem can be reformulated in the following state-space form:*

$$\begin{aligned} &\min \tilde{J}_{j_i} = \int_0^T \Big(\sum_{\substack{k=1 \\ k \neq j}}^{N} (x_{j_i} - x_{k_i})^T \tilde{Q}_{j_i} (x_{j_i} - x_{k_i}) + u_{j_i}^T \tilde{R}_{j_i} u_{j_i} \Big) dt \\ &s.t. \\ &\dot{x}_{j_i} = A_{j_i} x_{j_i} + B_{j_i} \sum_{i=1}^{2} u_{j_i}, \end{aligned} \tag{16}$$

with the optimal state-feedback control law as:

$$u_{j_i} = -\tilde{R}_{j_i}^{-1} B_{j_i}^T \Big(S_{j_i} x_{j_i} + \sum_{k \neq j} G_{j_i}^T x_{k_i} \Big), \tag{17}$$

where $S_{j_i} \in \mathbb{R}^{n\times n}$ and $G_{j_i} \in \mathbb{R}^{n\times n}$ are positive definite and symmetric matrices satisfying:

$$\begin{aligned} &\dot{S}_{j_i} + S_{j_i}A_{j_i} + A_{j_i}^T S_{j_i} - S_{j_i}B_{j_i}\tilde{R}_{j_i}^{-1}B_{j_i}^T S_{j_i} + \tilde{Q}_{j_i} = 0, \quad S_{j_i}(T) = 0, \\ &\dot{G}_{j_i} + (A_{j_i} - B_{j_i}\tilde{R}_{j_i}^{-1}B_{j_i}^T S_{j_i})^T G_{j_i} - \tilde{Q}_{j_i} = 0, \quad G_{j_i}(T) = 0. \end{aligned} \tag{18}$$

Proof. The state-space representation of the optimization problem (16) has already been established via relations (9)–(14). For the remainder of the proof, consider the following candidate value function:

$$V_{j_i}(t, x_{j_i}) = \tfrac{1}{2} x_{j_i}^T S_{j_i} x_{j_i} + \sum_{\substack{k=1 \\ k\neq j}}^{N} x_{k_i}^T G_{j_i} x_{j_i}, \tag{19}$$

where $S_{j_i} \in \mathbb{R}^{n\times n}$ and $G_{j_i} \in \mathbb{R}^{n\times n}$ are positive definite matrices with continuously differentiable entries. Differentiating (19) and substituting into the HJB equation (15) yields:

$$-\tfrac{1}{2} x_{j_i}^T \dot{S}_{j_i} x_{j_i} - x_{j_i}^T \sum_{\substack{k=1 \\ k\neq j}}^{N} \dot{G}_{j_i} x_{k_i} = \min_{u_{j_i}} \left\{ \sum_{\substack{k=1 \\ k\neq j}}^{N} (x_{j_i} - x_{k_i})^T \tilde{Q}_{j_i} (x_{j_i} - x_{k_i}) + u_{j_i}^T \tilde{R}_{j_i} u_{j_i} + \left(x_{j_i}^T S_{j_i} + \sum_{\substack{k=1 \\ k\neq j}}^{N} x_{k_i}^T G_{j_i} \right) \left(A_{j_i} x_{j_i} + B_{j_i} \sum_{i=1}^{2} u_{j_i} \right) \right\}. \tag{20}$$

To handle the recent equation, take the derivative of the right-hand side with respect to u_{j_i} and set it equal to zero:

$$\frac{d}{du_{j_i}} \left\{ \sum_{\substack{k=1 \\ k\neq j}}^{N} (x_{j_i} - x_{k_i})^T \tilde{Q}_{j_i} (x_{j_i} - x_{k_i}) + u_{j_i}^T \tilde{R}_{j_i} u_{j_i} + \left(x_{j_i}^T S_{j_i} + \sum_{\substack{k=1 \\ k\neq j}}^{N} x_{k_i}^T G_{j_i} \right) \left(A_{j_i} x_{j_i} + B_{j_i} \sum_{i=1}^{2} u_{j_i} \right) \right\} = 0, \tag{21}$$

which leads to the optimal control law:

$$u_{j_i}^* = -\tilde{R}_{j_i}^{-1} B_{j_i}^T \left(S_{j_i} x_{j_i} + \sum_{\substack{k=1 \\ k\neq j}}^{N} G_{j_i}^T x_{k_i} \right). \tag{22}$$

To verify optimality, consider the second derivative:

$$\frac{d^2}{du_{j_i}^2} \left\{ \sum_{\substack{k=1 \\ k\neq j}}^{N} (x_{j_i} - x_{k_i})^T \tilde{Q}_{j_i} (x_{j_i} - x_{k_i}) + u_{j_i}^T \tilde{R}_{j_i} u_{j_i} + \left(x_{j_i}^T S_{j_i} + \sum_{\substack{k=1 \\ k\neq j}}^{N} x_{k_i}^T G_{j_i} \right) \left(A_{j_i} x_{j_i} + B_{j_i} \sum_{i=1}^{2} u_{j_i} \right) \right\} = \tilde{R}_{j_i}. \tag{23}$$

Since $\tilde{R}_{j_i}$ is positive definite, the control law (22) is optimal.

To establish state consensus, rewrite (22) in overal form by using (2):

$$U = -R_D^{-1} B_D^T (S_D + G_D) X_D, \tag{24}$$

where $R_D = \mathrm{diag}\{\tilde{R}_{j_i}\}$, $B_D = \mathrm{diag}\{B_{j_i}\}$, $S_D = \mathrm{diag}\{S_{j_i}\}$, and $G_D = \mathrm{diag}\{G_{j_i}^T\}$. Since both S_D and G_D are positive definite, their sum is also positive definite and invertible. So, define:

$$\mathscr{X} = (S_D + G_D)^{-1}, \quad \mathscr{Y} = -R_D^{-1} B_D^T. \tag{25}$$

Then, with $F = \mathscr{Y}\mathscr{X}^{-1}$, we obtain $U = FX_D$, and the closed-loop system becomes:

$$\dot{X}_D = \left(A_D - B_D R_D^{-1} B_D^T (S_D + G_D) \right) X_D. \tag{26}$$

To verify the conditions of Lemma 1, note that $\mathscr{X} = (S_D + G_D)^{-1}$ is positive definite. The matrix S_D captures individual (consensus) dynamics, while G_D captures coupling (disagreement) dynamics. Hence, the state space is decomposed into orthogonal consensus and disagreement subspaces, i.e., condition (i) holds.

From (26) and the definition of $\mathscr{S}$ in lemma 1, we obtain:

$$\left(A_D - B_D R_D^{-1} B_D^T (S_D + G_D) \right) \mathscr{S} = 0, \tag{27}$$

which, using (25), yields:

$$(A_D + B_D \mathscr{Y} \mathscr{X}^{-1}) \mathscr{S} = 0. \tag{28}$$

Right multiplying (28) by $\mathscr{S}^T \mathscr{X} \mathscr{S}$ gives:

$$(A_D \mathscr{X} + B_D \mathscr{Y}) \mathscr{S} = 0. \tag{29}$$

Finally, consider the Lyapunov function:

$$V(t) = X_D^T (S_D + G_D) X_D. \tag{30}$$

Since $S_D + G_D$ is positive definite, it follows that $\dot{V}(t) < 0$, implying asymptotic stability of the closed-loop system. Restricting to the disagreement subspace yields:

$$\mathscr{S}_\perp^T (A_D \mathscr{X} + \mathscr{X} A_D^T + B_D \mathscr{Y} + \mathscr{Y}^T B_D^T) \mathscr{S}_\perp < 0, \tag{31}$$

which guarantees condition (iii).

Therefore, all conditions of Lemma 1 are satisfied and this completes the proof. □

4 Numerical Example

Example 1. Consider a network of $N = 3$ duopoly systems. For each system $j = 1, 2, 3$, the demand and price adjustment parameters are defined as:

$$a_j = 100, \quad s_j = 0.4, \quad r = 0.05.$$

The marginal production costs of the two firms in each system are given by:

$$c_{j_1} = 20, \quad c_{j_2} = 25.$$

Under these parameters, the market price dynamics for the jth system are described by:

$$\dot{p}_j(t) = 0.4\left(100 - \left(v_{j_1}(t) + v_{j_2}(t)\right) - p_j(t)\right).$$

Each firm seeks to maximize its discounted profit over a finite time horizon T, defined as:

$$J_{j_1} = \int_0^T e^{-0.05t} \left(p_j(t) v_{j_1}(t) - 20 v_{j_1}(t) - \tfrac{1}{2} v_{j_1}^2(t) \right) dt,$$

$$J_{j_2} = \int_0^T e^{-0.05t} \left(p_j(t) v_{j_2}(t) - 25 v_{j_2}(t) - \tfrac{1}{2} v_{j_2}^2(t) \right) dt.$$

By the above assumptions, the control objective is:

(i) To maximize the discounted profit of each firm;

(ii) To ensure convergence of market prices to a common value, i.e.,

$$\forall j \neq k, \quad \|p_j(t) - p_k(t)\| \to 0.$$

By reformulating the network using relations (9)–(12), for $j = 1, 2, 3$ the system matrices are obtained as:

$$A_{j_1} = \begin{bmatrix} -1.225 & 56 \\ 0 & -0.025 \end{bmatrix}, \quad A_{j_2} = \begin{bmatrix} -1.225 & 60 \\ 0 & -0.025 \end{bmatrix}, \quad B_{j_1} = B_{j_2} = \begin{bmatrix} -0.4 \\ 0 \end{bmatrix}.$$

The corresponding cost matrices for $j = 1, 2, 3$ are given by:

$$\tilde{Q}_{j_1} = -Q_{j_1} = \tfrac{1}{2} \begin{bmatrix} 1 & -20 \\ -20 & 400 \end{bmatrix}, \quad \tilde{Q}_{j_2} = -Q_{j_2} = \tfrac{1}{2} \begin{bmatrix} 1 & -25 \\ -25 & 625 \end{bmatrix},$$

$$\tilde{R}_{j_1} = \tilde{R}_{j_2} = -R_{j_i} = \tfrac{1}{2}.$$

By solving the ODEs in (18), the optimal control laws for $j = 1,2,3$ are obtained as:

$$u^*_{j_1} = \begin{bmatrix} 0.1591 & 0.6937 \end{bmatrix} x_{j_1} - \begin{bmatrix} 0.2208 & 1391.7091 \end{bmatrix} \sum_{\substack{k=1 \\ k\neq j}}^{3} x_{k_1},$$

$$u^*_{j_2} = \begin{bmatrix} 0.1591 & 0.3442 \end{bmatrix} x_{j_2} - \begin{bmatrix} 0.2208 & 1503.3473 \end{bmatrix} \sum_{\substack{k=1 \\ k\neq j}}^{3} x_{k_2}.$$

To evaluate the effectiveness of the proposed coordination strategy, different initial price conditions are considered:

$$p_1(0) = 100, \quad p_2(0) = 50, \quad p_3(0) = 10.$$

The corresponding initial state vectors are:

$$x_1(0) = \begin{bmatrix} 100 \\ 1 \end{bmatrix}, \quad x_2(0) = \begin{bmatrix} 50 \\ 1 \end{bmatrix}, \quad x_3(0) = \begin{bmatrix} 10 \\ 1 \end{bmatrix}.$$

The numerical simulations show that, for all firms, the proposed optimal feedback control law drives the market prices of all duopolies to a common value, despite different initial conditions. This confirms that profit optimization and price consensus are simultaneously achieved for each firm.

Figure 1 shows evolution of market prices $p_1(t), p_2(t)$ and $p_3(t)$ under the proposed optimal consensus-based feedback controller. The trajectories converge to a common value despite different initial conditions.

Figure 2 shows Consensus errors between market prices of the duopoly systems. All pairwise price differences asymptotically converge to zero.

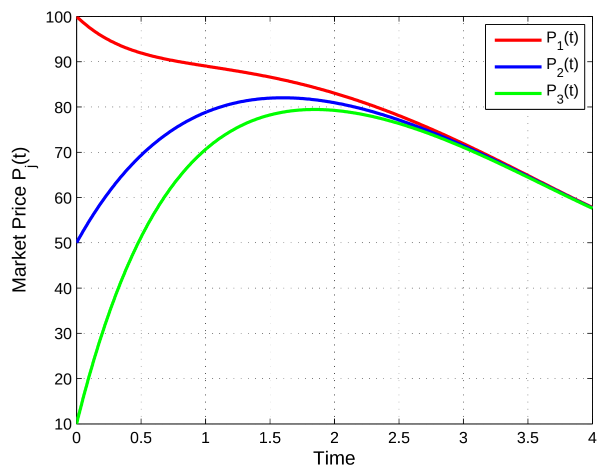

Figure 1. Evolution of market prices.

Table 1 compares the accumulated discounted profits obtained using the proposed consensus-based controller and the baseline controller without the consensus coupling term. Although the accumulated profit of the third market is reduced, the first and second markets exhibit profit improvements. Consequently, the total accumulated discounted profit of the network increases from 7425.2 to 10098.4, corresponding to an improvement of approximately 36%. These results indicate that the proposed controller enhances the overall economic performance of the interconnected markets while simultaneously achieving market price coordination.

The above results also provide evidence that, for the considered numerical example, the consensus objective does not conflict with profit maximization. Instead, the proposed controller simultaneously promotes market price agreement and improves the overall accumulated discounted profit of the interconnected network compared with the baseline controller.

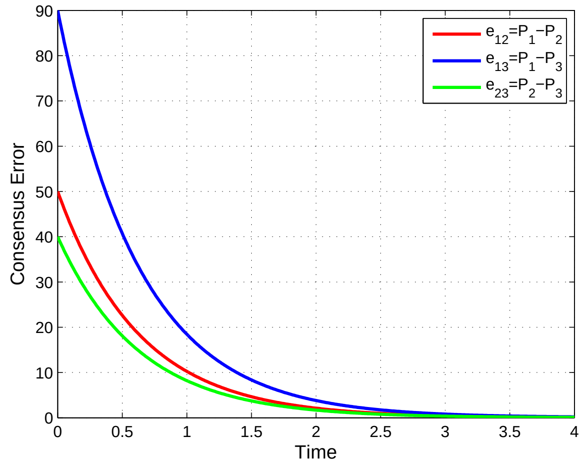

Figure 2. Consensus errors.

Table 1. Comparison of accumulated discounted profits for the proposed consensus-based controller and the baseline controller without the consensus coupling term.

Controller	Market 1	Market 2	Market 3	Total
Baseline	4061.1	1539.6	1824.5	7425.2
Proposed	6788.9	1552.4	1757.1	10098.4

To investigate the sensitivity of the proposed controller to the discount rate, additional simulations were carried out for different values of r. The corresponding accumulated discounted profits and final consensus errors are reported in Table 2. As shown in Table 2, the proposed

Table 2. Sensitivity analysis of the proposed controller with respect to the discount rate.

r	Total Profit	Final Consensus Error
0.02	9486.4	0.1879
0.05	10098.4	0.1995
0.10	11083.4	0.2205

controller maintains satisfactory performance over the considered range of discount rates. Although both the accumulated discounted profit and the final consensus error exhibit moderate variations as the discount rate changes, the controller consistently achieves simultaneous profit optimization and market price coordination. These results indicate that the proposed framework is not overly sensitive to moderate changes in the discount rate and preserves its effectiveness under different economic conditions.

To further examine the scalability of the proposed framework, the original network was expanded by adding one additional interconnected duopoly market, resulting in a four-market network. The same controller design procedure was applied without any modification.

Figure 3 illustrates the market price trajectories for the expanded four-market network. It can be observed that all market prices converge to a common value despite the increased network size. Since the same controller design and optimization framework are directly applicable after expanding the network, this example demonstrates the scalability of the proposed approach to larger interconnected market networks.

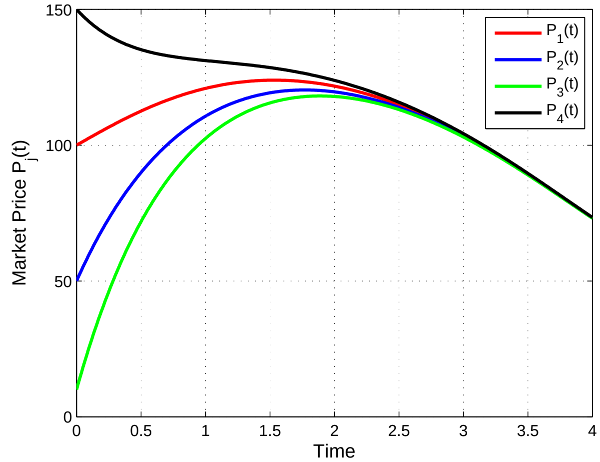

Figure 3. Evolution of market prices for the expanded four-market network.

5 Conclusion

In this paper, a network of dynamic duopoly systems has been considered with the aim of jointly optimizing discounted profits of firms and requiring the convergence of market prices in all markets. The original dynamic duopoly problem was reformulated as a quadratic optimal control problem by modeling the duopoly dynamics of each market through a price adjustment mechanism and representing the overall system in an LTI state-space form. For the price consensus, a term based on the deviation between the states of different duopoly systems was embedded directly into the performance index. An optimal state-feedback control law was derived using the HJB approach. The proposed control strategy ensures convergence of the market prices while maintaining the individual profit optimization. Unlike classical differential game formulations that assign separate objective functions to individual players and seek equilibrium strategies, the proposed framework employs a unified performance index with an embedded consensus objective. This enables discounted profit optimization and market price coordination to be addressed simultaneously through a common state-feedback design. Results show that consensus based optimal control is a useful tool for the coordination of dynamic competitive markets. Future work may generalize this framework to time-varying network topologies, stochastic disturbances or more general demand functions.

Data Availability

All data in the paper are available from the corresponding authors upon reasonable request.

Conflicts of Interest

The author declares that there is no conflict of interest.

Ethical Considerations

This study does not involve human or animal subjects, and therefore no ethics approval is required.

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