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Research article

A Hybrid Numerical Scheme for the Time-Fractional Telegraph Equation

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Abstract

This study introduces a hybrid approach based on a finite difference scheme and radial basis functions (RBF) for solving the Caputo time–fractional telegraph equation numerically. To handle the temporal fractional derivatives, a finite difference formulation of the L1/L2 type is adopted, while the spatial derivatives are approximated using an RBF collocation technique. This combination results in a discretized system characterized by a sparse matrix structure. Through the application of energy methods, the stability and convergence assessment of the temporal discretization is conducted, yielding a theoretical error estimate of $\mathscr{O}(\delta_t^{3-\alpha})$ in the time direction. The reliability and effectiveness of the proposed numerical scheme are verified through a representative test problem, confirming the capability of the proposed method to achieve high accuracy.

Keywords: Time–fractional telegraph equation, Caputo derivative, Collocation method, Radial basis functions, Stability and convergence.

Mathematics Subject Classification (2020): 35R11, 65M12, 65M70

1 Introduction

The mathematical description of wave–like phenomena subject to dissipative effects is a cornerstone of continuum modeling in science and engineering. Within this context, the classical telegraph equation, originally derived from the analysis of electrical signal propagation in transmission lines [1], has long served as a canonical model for signal transport in lossy media. In its standard hyperbolic form it synthesizes inertial, damping, restoring, and diffusive mechanisms into a single coherent framework, and has been successfully employed in applications ranging from persistent random walks and stochastic transport to reactive fluid dynamics and biological or neural signal transmission [2, 16]. Nevertheless, because the model is built upon integer–order temporal derivatives, it inherently reflects a local–in–time, Markovian dynamics, in which the future state depends only on the instantaneous present. This assumption fails to capture the hereditary behavior, long–range temporal dependence, and anomalous transport observed in many complex media.

The need to incorporate memory effects and non–Markovian behavior has elevated fractional calculus from a mathematical curiosity to an indispensable modeling tool. By extending differentiation and integration to arbitrary real (or complex) orders, fractional operators introduce nonlocal kernels that weight the entire past history of the state variable, thereby providing a natural mathematical framework for relaxation–oscillation and diffusion–wave phenomena with power–law memory [3,4,20]. Foundational monographs such as [6,7,18,19,24] have systematically developed the theory of fractional integrals and derivatives and documented their relevance in continuum mechanics, viscoelasticity, heat conduction, and statistical physics [5,21]. In particular, anomalous diffusion and wave propagation in heterogeneous or viscoelastic media, as well as non–Fourier heat waves and related generalized transport processes, are now routinely modeled by fractional integro–differential equations [17,20].

Embedding fractional derivatives into the classical telegraph framework leads to the time–fractional telegraph equation (TFTE)

$$ {}_0^C D_t^{\mu} u(x,t) + 2\alpha\, {}_0^C D_t^{\gamma} u(x,t) + \beta^2 u(x,t) = \kappa u_{xx}(x,t), \qquad 1<\mu\le 2,\ 0<\gamma\le 1, \tag{1} $$

where ${}_0^C D_t^{\nu}$ denotes the Caputo fractional derivative of order ν, which is particularly convenient because it allows the use of classical initial conditions [6, 7]. Equation (1) generalizes the classical telegraph equation [1, 15]: by tuning μ one can interpolate continuously between diffusion–dominated ($\mu \to 1$) and wave–dominated ($\mu \to 2$) regimes, while the parameter γ models damping mechanisms with memory. As a result, the TFTE provides a more flexible and physically realistic description of hybrid diffusion–wave processes in temporally nonlocal media, and is closely related to fractional diffusion–wave and fractional telegraph processes studied, for instance, in [3,4,8,22].

Despite its modeling advantages, the analytical treatment of (1) is highly nontrivial. Exact solutions obtained via Laplace–Fourier techniques are typically expressed in terms of Mittag–Leffler and Wright functions and are often limited to simple geometries and constant coefficients [9, 23, 24]. Their evaluation can be computationally demanding and ill–suited for practical problems with general initial–boundary data. This difficulty has spurred an extensive body of work on numerical algorithms for fractional evolution equations and, in particular, for fractional diffusion–wave and telegraph–type models; see, for example, [10,11,14,25,26] and the references therein.

The numerical approximation of fractional partial differential equations poses distinctive challenges arising from the historicity of fractional derivatives. For Caputo operators, the value at a time level t_n involves an integral over the entire interval $[0,t_n]$, which implies that both the computational cost and memory requirements grow with the number of time steps [6, 11]. Effective numerical schemes must therefore strike a delicate balance: they must accurately approximate the nonlocal temporal dependence while keeping the algorithmic complexity and storage demand manageable. In the temporal domain, finite difference schemes tailored to fractional operators have become standard. Quadrature–based formulas of L1 type for derivatives of order $0<\nu\le 1$ and L2–type generalizations for $1<\nu\le 2$ provide stable and convergent discretizations of weakly singular kernels, yielding discrete convolution sums with algebraically decaying weights that faithfully represent fading memory [10,11,25,26].

For the spatial approximation, while low–order numerical methods are widely used, spectral techniques offer superior convergence properties for sufficiently smooth solutions. Classical references such as [12, 15, 27] document the potential of Fourier– and Chebyshev–based spectral methods to achieve exponential or spectral–order accuracy. More recently, wavelet–based spectral or collocation methods have received considerable attention in the context of fractional differential equations, as they combine high global accuracy with localization and multiresolution capabilities. In particular, Chebyshev wavelet systems and related constructions provide sparse operational matrices and are capable of efficiently resolving both smooth and localized structures; see, for instance, [13,28,29].

Motivated by these developments, the present work introduces a hybrid numerical framework for the one–dimensional TFTE (1) with prescribed initial–boundary data. Our approach couples

- a finite difference discretization in time of L1/L2 genre for the Caputo fractional derivatives, providing a stable and systematically refinable treatment of the nonlocal temporal operators [10,11,25,26].
- a high–order spatial approximation based on a collocation/spectral strategy employing Chebyshev–type wavelet basis functions (or closely related local bases), which yields sparse operational matrices and high spatial accuracy [12,13,27–29].

This strategy effectively exploits the complementary strengths of each component, the time discretization accurately captures the hereditary effects encoded by the fractional derivatives, while the spatial approximation delivers high resolution with relatively few degrees of freedom and a favorable algebraic structure of the resulting linear systems.

We carry out a rigorous numerical analysis of the discrete scheme. Using energy techniques in the spirit of [10, 25], we establish unconditional stability and derive error estimates that quantify the convergence rate in appropriate norms. The theoretical findings are supported by a series of numerical experiments for benchmark TFTE problems, which confirm the accuracy, computational efficiency, and algorithmic robustness of the present method and, in cases where comparative data exist, permit validation against previously developed approaches including those presented in [13,14,28].

This paper is structured in the following manner: The time–fractional telegraph equation and its mathematical description are presented in Section 2. Section 3 presents a rigorous stability and convergence analysis establishing an error estimate, while Section 4 details the RBF-based spatial discretization that yields a matrix structure. Section 5 then provides numerical validation confirming the theoretical findings. The paper concludes with Section 6, which offers a summary of the main findings and discusses several avenues for subsequent investigation.

2 Time Fractional Telegraph Equation

The section deals with the time–fractional telegraph equation written as

$$ {}_0^C D_t^{\alpha} u(x,t) + {}_0^C D_t^{\alpha-1} u(x,t) + u(x,t) = u_{xx}(x,t) + f(x,t), \quad (x,t) \in (0,1) \times (0,T], \tag{2} $$

subject to the initial conditions

$$ u(x,0) = g(x), \qquad \frac{\partial u}{\partial t}(x,0) = h(x), \quad x \in (0,1), \tag{3} $$

and the boundary condition

$$ u(0,t) = p(t), \qquad u(1,t) = q(t), \quad t \in (0,T). \tag{4} $$

Here $1 < \alpha < 2$, $T > 0$, and the functions $g(x)$, $h(x)$, $p(t)$, $q(t)$, and $f(x,t)$ are assumed to be sufficiently smooth functions. The operator ${}_0^C D_t^{\alpha} u(x,t)$ is denoted the Caputo fractional derivative of function u(x,t) with respect to t of fractional order α.

2.1 Time Discretization of the Caputo Fractional Derivative Derivative

The Caputo fractional derivative of $u(x,,t)$ with respect to t of fractional order α is defined by

$$ {}_0^C D_t^{\alpha} u(x,t) = \begin{cases} \dfrac{1}{\Gamma(1-\alpha)} \displaystyle\int_0^t \frac{\partial u(x,s)}{\partial s} \frac{ds}{(t-s)^{\alpha}}, & 0 < \alpha < 1, \\ \dfrac{1}{\Gamma(2-\alpha)} \displaystyle\int_0^t \frac{\partial^2 u(x,s)}{\partial s^2} \frac{ds}{(t-s)^{\alpha-1}}, & 1 < \alpha < 2. \end{cases} \tag{5} $$

If $1 < \alpha < 2$, then $0 < \alpha - 1 < 1$, and so,

$$ {}_0^C D_t^{\alpha-1} u(x,t) = \frac{1}{\Gamma(1-(\alpha-1))} \int_0^t \frac{\partial u(x,s)}{\partial s} \frac{ds}{(t-s)^{\alpha-1}} = \frac{1}{\Gamma(2-\alpha)} \int_0^t \frac{\partial u(x,s)}{\partial s} \frac{ds}{(t-s)^{\alpha-1}}. \tag{6} $$

We introduce a uniform partition of the time interval using N subdivisions, where N is a positive integer, such that the step size is given by $\delta t = \frac{T}{N}$ and define

$$ t_n = n\delta t, \qquad n = 0, 1, 2, \ldots, N. $$

The approximation scheme in time is conveniently described by introducing the following notation:

$$ u^{n-\frac{1}{2}} = \frac{1}{2}\left(u^n + u^{n-1}\right), \tag{7} $$

$$ \delta_t u^{n-\frac{1}{2}} = \frac{1}{\delta t}\left(u^n - u^{n-1}\right), \tag{8} $$

where

$$ u^n = u(x, t_n). $$

For simplicity, we set

$$ v(x,t) = u_t(x,t) = \frac{\partial u(x,t)}{\partial t}, \tag{9} $$

$$ w(x,t) = \frac{1}{\Gamma(2-\alpha)} \int_0^t \frac{\partial v(x,s)}{\partial s} \frac{ds}{(t-s)^{\alpha-1}}, \tag{10} $$

$$ z(x,t) = \frac{1}{\Gamma(2-\alpha)} \int_0^t \frac{\partial u(x,s)}{\partial s} \frac{ds}{(t-s)^{\alpha-1}}. \tag{11} $$

Applying the Taylor expansion to (9), we obtain

$$v^{n-\frac{1}{2}} = \delta t u^{n-\frac{1}{2}} + T_1^{n-\frac{1}{2}}. \tag{12}$$

The numerical scheme for Eq. (2) is then given by

$$w^{n-\frac{1}{2}} + z^{n-\frac{1}{2}} + u^{n-\frac{1}{2}} = \Delta u^{n-\frac{1}{2}} + f^{n-\frac{1}{2}} + T_2^{n-\frac{1}{2}}, \qquad n \geq 1, \tag{13}$$

where $T_1^{n-\frac{1}{2}}$ and $T_2^{n-\frac{1}{2}}$ are local truncation errors satisfying

$$\left| T_1^{n-\frac{1}{2}} \right| \leq C_1(\delta t)^2, \qquad \left| T_2^{n-\frac{1}{2}} \right| \leq C_2(\delta t)^2. \tag{14}$$

The temporal discretization of (10) and (11) evaluated at the nodal point t_n produces

$$w(x,t_n) = \frac{1}{\Gamma(2-\alpha)} \int_0^{t_n} \frac{\partial v(x,t)}{\partial t} \frac{dt}{(t_n - t)^{\alpha-1}},$$

$$z(x,t_n) = \frac{1}{\Gamma(2-\alpha)} \int_0^{t_n} \frac{\partial u(x,t)}{\partial t} \frac{dt}{(t_n - t)^{\alpha-1}}.$$

From [30], based on finite differences approximations of $w(x,t_n)$ and $z(x,t_n)$ we obtain

$$w^n = a_0 \left[b_0 v^n - \sum_{k=1}^{n-1} (b_{n-k-1} - b_{n-k}) v^k - b_{n-1} v^0 \right] + \mathscr{O}\left((\delta t)^{3-\alpha} \right), \tag{15}$$

$$z^n = a_0 \left[b_0 u^n - \sum_{k=1}^{n-1} (b_{n-k-1} - b_{n-k}) u^k - b_{n-1} u^0 \right] + \mathscr{O}\left((\delta t)^{3-\alpha} \right). \tag{16}$$

Let the operator $AP(v^n;q)$ be defined through the relation:

$$AP(v^n;q) = b_0 v^n - \sum_{k=1}^{n-1} (b_{n-k-1} - b_{n-k}) v^k - b_{n-1} q. \tag{17}$$

Using the initial conditions (3), namely

$$u^0 = u(x,0) = u(x,t_0) = g, \qquad v^0 = v(x,0) = v(x,t_0) = h,$$

we obtain the fully discrete formulation:

$$w^{n-\frac{1}{2}} = a_0 AP(v^{n-\frac{1}{2}}, h) + T_3^{n-\frac{1}{2}}, \tag{18}$$

$$z^{n-\frac{1}{2}} = a_0 AP(u^{n-\frac{1}{2}}, g) + T_4^{n-\frac{1}{2}}, \tag{19}$$

where

$$|T_3^{n-\frac{1}{2}}| \leq C_3 \delta t^{3-\alpha} \quad \text{and} \quad |T_4^{n-\frac{1}{2}}| < C_4 \delta t^{3-\alpha}.$$

By substitution (12) in (18) we have

$$w^{n-k} = a_0 AP(\delta t u^{n-\frac{1}{2}}, h) + a_0 AP(T_1^{n-\frac{1}{2}}, 0) + T_3^{n-\frac{1}{2}} \tag{20}$$

Now by using (19) and (20) in (12), we have

$$a_0 AP(\delta t u^{n-\frac{1}{2}}, h) + a_0 AP(T_1^{n-\frac{1}{2}}, 0) + T_3^{n-\frac{1}{2}} + a_0 AP(u^{n-\frac{1}{2}}, g) + T_4^{n-\frac{1}{2}} + u^{n-\frac{1}{2}} = \Delta u^{n-\frac{1}{2}} + f^{n-\frac{1}{2}} + T_2^{n-\frac{1}{2}}.$$

Equivalently

$$a_0 AP(\delta t u^{n-\frac{1}{2}}, h) + a_0 AP(u^{n-\frac{1}{2}}, g) + u^{n-\frac{1}{2}} = \Delta u^{n-\frac{1}{2}} + f^{n-\frac{1}{2}} + T^{n-\frac{1}{2}}, \tag{21}$$

where

$$|T^{n-\frac{1}{2}}| \leq C \delta t^{3-\alpha},$$

and

$$C = \left\{ \left[\frac{2C_1}{(2-\alpha)\Gamma(2-\alpha)} + C_3 + C_4 \right] + C_2 \right\}.$$

Upon neglecting the truncation error term $T^{n-\frac{1}{2}}$ from the Eq. (21), and replacing u^n with its numerical approximation U^n, we arrive at the following representation for the approximate solution:

$$a_0 AP(\delta t U^{n-\frac{1}{2}} + U^{n-\frac{1}{2}}, h+g) + U^{n-\frac{1}{2}} = \Delta U^{n-\frac{1}{2}} + f^{n-\frac{1}{2}}, \qquad 1 \le n \le N. \tag{22}$$

The foregoing expression can be expressed equivalently as

$$\mathscr{L} U^n = \mathscr{F}^n,$$

where

$$\begin{cases} \mathscr{L} U^n = \dfrac{1}{\delta t\,\Gamma(2-\alpha)} \dfrac{b_0}{\delta t} U^n + \dfrac{1}{\delta t\,\Gamma(2-\alpha)} \dfrac{b_0}{2} U^n + \dfrac{1}{2} U^n - \dfrac{1}{2} \Delta U^n, \\ \mathscr{F}^n = \dfrac{1}{\delta t\,\Gamma(2-\alpha)} \dfrac{b_0}{\delta t} U^{n-1} - \dfrac{1}{\delta t\,\Gamma(2-\alpha)} \dfrac{b_0}{2} U^{n-1} - \dfrac{1}{2} U^{n-1} + \dfrac{1}{2} \Delta U^{n-1} \\ \qquad + \dfrac{1}{\delta t\,\Gamma(2-\alpha)} \displaystyle\sum_{k=1}^{n-1} \big(b_{n-k-1} - b_{n-k}\big)\big(\delta t U^{k-\frac{1}{2}} + U^{k-\frac{1}{2}}\big) + \dfrac{1}{\delta t\,\Gamma(2-\alpha)} b_{n-1}(h+g) + f^{n-\frac{1}{2}}. \end{cases}$$

3 Error Analysis

This section is dedicated to assessing both the stability and the convergence behavior of the numerical scheme, with a theoretical error bound of $\mathscr{O}(\delta t^{3-\alpha})$ established in the L_2-norm.

Theorem 1. *Under the assumption that $U^n \in H_0^1(\Omega)$, the discrete formulation (22) is stable, and the following estimate holds:*

$$\|U^n\|^2 \le C\Big(\|g\|^2 + \|\nabla g\|^2 + \|h+g\|^2 + \max_{1\le j\le n} \|f^{j-\frac{1}{2}}\|^2\Big),$$

where $C > 0$ is independent of δt and n.

Proof. Multiplying (22) by $\big(\delta_t U^{n-\frac{1}{2}} + U^{n-\frac{1}{2}}\big)$ and integrating over $\Omega = [0,1]$ gives the following, where $(\cdot,\cdot)$ is used for inner product:

$$a_0 \Bigg\{ b_0\big(\delta_t U^{n-\frac{1}{2}} + U^{n-\frac{1}{2}}, \delta_t U^{n-\frac{1}{2}} + U^{n-\frac{1}{2}}\big) + \sum_{k=1}^{n-1} \big(b_{n-k-1} - b_{n-k}\big)\big(\delta_t U^{k-\frac{1}{2}} + U^{k-\frac{1}{2}},\, \delta_t U^{n-\frac{1}{2}} + U^{n-\frac{1}{2}}\big)$$
$$- b_{n-1}\big(h,\, \delta_t U^{n-\frac{1}{2}} + U^{n-\frac{1}{2}}\big) \Bigg\} + \big(U^{n-\frac{1}{2}}, \delta_t U^{n-\frac{1}{2}} + U^{n-\frac{1}{2}}\big) = \big(\Delta U^{n-\frac{1}{2}}, \delta_t U^{n-\frac{1}{2}} + U^{n-\frac{1}{2}}\big) + \big(f^{n-\frac{1}{2}}, \delta_t U^{n-\frac{1}{2}} + U^{n-\frac{1}{2}}\big).$$

Now we are using the following relations:

$$\big(U^{n-\frac{1}{2}},\, \delta_t U^{n-\frac{1}{2}} + U^{n-\frac{1}{2}}\big) = \big(U^{n-\frac{1}{2}},\, \delta_t U^{n-\frac{1}{2}}\big) + \big(U^{n-\frac{1}{2}},\, U^{n-\frac{1}{2}}\big).$$

Indeed,

$$\begin{aligned} \big(U^{n-\frac{1}{2}},\, \delta_t U^{n-\frac{1}{2}}\big) &= \int_0^1 \frac{U^n + U^{n-1}}{2} \frac{U^n - U^{n-1}}{\delta_t}\, dx + \int_0^1 (U^{n-\frac{1}{2}})^2\, dx \\ &= \frac{1}{2\delta_t} \int_0^1 \big((U^n)^2 - (U^{n-1})^2\big)\, dx + \int_0^1 (U^{n-\frac{1}{2}})^2\, dx = \frac{1}{2\delta_t}\big(\|U^n\|^2 - \|U^{n-1}\|^2\big) + \|U^{n-\frac{1}{2}}\|^2, \end{aligned}$$

Moreover,

$$\big(\Delta U^{n-\frac{1}{2}},\, \delta_t U^{n-\frac{1}{2}} + U^{n-\frac{1}{2}}\big) = -\big(\nabla U^{n-\frac{1}{2}}, \nabla(\delta_t U^{n-\frac{1}{2}} + U^{n-\frac{1}{2}})\big).$$

Using

$$\nabla U^{n-\frac{1}{2}} = \frac{\nabla U^n + \nabla U^{n-1}}{2}, \qquad \nabla \delta_t U^{n-\frac{1}{2}} = \frac{\nabla U^n - \nabla U^{n-1}}{\delta_t},$$

we obtain

$$\left(\Delta U^{n-\frac{1}{2}}, \delta_t U^{n-\frac{1}{2}} + U^{n-\frac{1}{2}}\right) = -\int_0^1 \frac{\nabla U^n + \nabla U^{n-1}}{2} \cdot \frac{\nabla U^n - \nabla U^{n-1}}{\delta_t}\, dx - \int_0^1 |\nabla U^{n-\frac{1}{2}}|^2\, dx$$
$$= -\frac{1}{2\delta_t}\left(\|\nabla U^n\|^2 - \|\nabla U^{n-1}\|^2\right) - \|\nabla U^{n-\frac{1}{2}}\|^2.$$

So we have,

$$\begin{aligned} a_0 \Bigg\{ b_0 \left\| \delta_t U^{n-\frac{1}{2}} + U^{n-\frac{1}{2}} \right\|^2 - \sum_{k=1}^{n-1} (b_{n-k-1} - b_{n-k}) \left\| \delta_t U^{k-\frac{1}{2}} + U^{k-\frac{1}{2}} \right\| \left\| \delta_t U^{n-\frac{1}{2}} + U^{n-\frac{1}{2}} \right\| \\ - b_{n-1} \left\| h + g \right\| \left\| \delta_t U^{n-\frac{1}{2}} + U^{n-\frac{1}{2}} \right\| \Bigg\} + \frac{1}{2\delta_t}\left(\|U^n\|^2 - \|U^{n-1}\|^2\right) + \|U^{n-\frac{1}{2}}\|^2 \\ \leq -\|\nabla U^{n-\frac{1}{2}}\|^2 - \frac{1}{2\delta t}\left(\|\nabla U^n\|^2 - \|\nabla U^{n-1}\|^2\right) + \|f^{n-\frac{1}{2}}\| \left\| \delta_t U^{n-\frac{1}{2}} + U^{n-\frac{1}{2}} \right\|. \end{aligned} \tag{23}$$

Summing equation (23) for each n from 1 to m leads to the expression

$$\begin{aligned} a_0 \sum_{n=1}^{m} \Bigg\{ b_0 \left\| \delta_t U^{n-\frac{1}{2}} + U^{n-\frac{1}{2}} \right\|^2 - \sum_{k=1}^{n-1} (b_{n-k-1} - b_{n-k}) \| \delta_t U^{k-\frac{1}{2}} + U^{k-\frac{1}{2}} \| \, \| \delta_t U^{n-\frac{1}{2}} + U^{n-\frac{1}{2}} \| \\ - b_{n-1} \|h+g\| \, \| \delta_t U^{n-\frac{1}{2}} + U^{n-\frac{1}{2}} \| \Bigg\} + \frac{1}{2\delta_t}\left(\|U^m\|^2 - \|U^0\|^2\right) + \sum_{n=1}^{m} \|U^{n-\frac{1}{2}}\|^2 \\ \leq -\frac{1}{2\delta_t}\left(\|\nabla U^m\|^2 - \|\nabla U^0\|^2\right) - \sum_{n=1}^{m} \|\nabla U^{n-\frac{1}{2}}\|^2 + \sum_{n=1}^{m} \|f^{n-\frac{1}{2}}\| \, \| \delta_t U^{n-\frac{1}{2}} + U^{n-\frac{1}{2}} \|. \end{aligned}$$

By using the inequality $|xy| \leq \frac{x^2}{2\theta} + \frac{\theta}{2} y^2$ with a suitable choice $\theta = \dfrac{t_m^{1-\alpha}}{\Gamma(2-\alpha)}$, we get

$$\sum_{n=1}^{m} \left\| f^{n-\frac{1}{2}} \right\| \left\| \delta_t U^{n-\frac{1}{2}} + U^{n-\frac{1}{2}} \right\|^2 \leq \frac{\Gamma(2-\alpha)}{2 t_m^{1-\alpha}} \sum_{n=1}^{\infty} \|f^{n-\frac{1}{2}}\|^2 + \frac{t_m^{1-\alpha}}{2\Gamma(2-\alpha)} \sum_{n=1}^{m} \left\| \delta_t U^{n-\frac{1}{2}} + U^{n-\frac{1}{2}} \right\|^2.$$

Now, using the above relation together from [30] for operator AP, we have

$$\begin{aligned} \frac{t_m^{1-\alpha}}{2\,\delta_t\,\Gamma(2-\alpha)} \delta_t \sum_{n=1}^{m} \left\| \delta_t U^{n-\frac{1}{2}} + U^{n-\frac{1}{2}} \right\|^2 - \frac{t_m^{2-\alpha}}{2\,\delta_t\,\Gamma(3-\alpha)} \|h+g\|^2 + \frac{1}{2\delta_t}\left(\|U^m\|^2 - \|U^0\|^2\right) + \sum_{n=1}^{m} \|U^{n-\frac{1}{2}}\|^2 \leq -\frac{1}{2\delta_t}\left(\|\nabla U^m\|^2 - \|\nabla U^0\|^2\right) \\ - \sum_{n=1}^{m} \|\nabla U^{n-\frac{1}{2}}\|^2 + \frac{\Gamma(2-\alpha)}{2 t_m^{1-\alpha}} \sum_{n=1}^{m} \|f^{n-\frac{1}{2}}\|^2 + \frac{t_m^{1-\alpha}}{2\Gamma(2-\alpha)} \sum_{n=1}^{m} \|\delta_t U^{n-\frac{1}{2}} + U^{n-\frac{1}{2}}\|^2. \end{aligned}$$

By simplifying the above equation and switching indices from n to m, one finally arrives at

$$\|U^n\|^2 + \|\nabla U^n\|^2 + 2\delta_t \sum_{k=1}^{n} \|U^{k-\frac{1}{2}} + \nabla U^{k-\frac{1}{2}}\|^2 \leq \left(\|U^0\|^2 + \|\nabla U^0\|^2 + \frac{t_n^{2-\alpha}}{\Gamma(3-\alpha)} \|h+g\|^2 + \Gamma(2-\alpha) t_n^{\alpha-1} \delta_t \sum_{j=1}^{n} \|f^{j-\frac{1}{2}}\|^2 \right),$$

$$\begin{aligned} \|U^n\|^2 &\leq \left(\|U^0\|^2 + \|\nabla U^0\|^2\right) + \frac{t_n^{2-\alpha}}{\Gamma(3-\alpha)} \|h+g\|^2 + t_n^{\alpha-1} \Gamma(2-\alpha) n \delta_t \max_{1\leq j\leq n} \|f^{j-\frac{1}{2}}\|^2, \\ &\leq \left(\|U^0\|^2 + \|\nabla U^0\|^2\right) + \frac{T^{2-\alpha}}{\Gamma(3-\alpha)} \|h+g\|^2 + T^{\alpha} \Gamma(2-\alpha) \max_{1\leq j\leq n} \|f^{j-\frac{1}{2}}\|^2, \\ &\left(\|\nabla g\|^2 + \|g\|^2\right) + \frac{T^{2-\alpha}}{\Gamma(3-\alpha)} \|h+g\|^2 + T^{\alpha} \Gamma(2-\alpha) \max_{1\leq j\leq n} \|f^{j-\frac{1}{2}}\|^2, \end{aligned}$$

where

$$h = U_t^0, \qquad g = U^0.$$

So,

$$\|U^n\|^2 \leq C\left(\|\nabla g\|^2 + \|g\|^2 + \|h+g\|^2 + \max_{1\leq j\leq n} \|f^{j-\frac{1}{2}}\|^2 \right),$$

where

$$C = 1 + \frac{T^{2-\alpha}}{\Gamma(3-\alpha)} + T^{\alpha}\Gamma(2-\alpha)$$

The desired result is now established, completing the proof of the theorem. □

Theorem 2. *Let u^n and U^n both belong to $H_0^1(\Omega)$ and are the exact solution and the corresponding numerical approximation solution of (19) and (22), respectively. Then the numerical scheme (22) has convergence order $\mathscr{O}(\delta_t^{3-\alpha})$ in the L_2-norm.*

Proof. Let

$$E^n = u^n - U^n, \qquad n \geq 1,$$

denote the error at time level n and set $E^0 = 0$. Subtracting the numerical scheme (22) from the exact formulation (21) yields the following relation

$$a_0\left\{b_0\left(\delta_t E^{n-\frac{1}{2}} + E^{n-\frac{1}{2}}\right) - \sum_{k=1}^{n-1}(b_{n-k-1} - b_{n-k})\left(\delta_t E^{k-\frac{1}{2}} + E^{k-\frac{1}{2}}\right)\right\} + E^{n-\frac{1}{2}} = \Delta E^{n-\frac{1}{2}} + T^{n-\frac{1}{2}}, \tag{24}$$

where $T^{n-\frac{1}{2}}$ denotes the local truncation error satisfying $\|T^{n-\frac{1}{2}}\| \leq C\delta_t^{3-\alpha}$. Multiplying (24) by $\delta_t E^{n-\frac{1}{2}} + E^{n-\frac{1}{2}}$ and integrating over $(0,1)$, we obtain

$$a_0\left\{b_0\left\|\delta_t E^{n-\frac{1}{2}} + E^{n-\frac{1}{2}}\right\|^2 - \sum_{k=1}^{n-1}(b_{n-k-1} - b_{n-k})\left\|\delta_t E^{k-\frac{1}{2}} + E^{k-\frac{1}{2}}\right\|\right\}\left\|\delta_t E^{n-\frac{1}{2}} + E^{n-\frac{1}{2}}\right\| + \frac{1}{2\delta_t}\left(\|E^n\|^2 - \|E^{n-1}\|^2\right)$$
$$+ \|E^{n-\frac{1}{2}}\|^2 = -\frac{1}{2\delta_t}\left(\|\nabla E^n\|^2 - \|\nabla E^{n-1}\|^2\right) - \|\nabla E^{n-\frac{1}{2}}\|^2 + \left(T^{n-\frac{1}{2}}, \delta_t E^{n-\frac{1}{2}} + E^{n-\frac{1}{2}}\right).$$

Performing a summation of the above relation across all n from 1 to m results in

$$\sum_{n=1}^{m}\frac{1}{\delta_t\,\Gamma(2-\alpha)}\left\{b_0\left\|\delta_t E^{n-\frac{1}{2}} + E^{n-\frac{1}{2}}\right\|^2 - \sum_{k=1}^{n-1}(b_{n-k-1} - b_{n-k})\left\|\delta_t E^{k-\frac{1}{2}} + E^{k-\frac{1}{2}}\right\|\right\}\left\|\delta_t E^{n-\frac{1}{2}} + E^{n-\frac{1}{2}}\right\|$$
$$+ \frac{1}{2\delta_t}\left(\|E^m\|^2 - \|E^0\|^2\right) \leq -\frac{1}{2\delta_t}\left(\|\nabla E^m\|^2 - \|\nabla E^0\|^2\right) - \sum_{n=1}^{m}\|\nabla E^{n-\frac{1}{2}}\|^2 + \sum_{n=1}^{m}\|T^{n-\frac{1}{2}}\|\,\left\|\delta_t E^{n-\frac{1}{2}} + E^{n-\frac{1}{2}}\right\|.$$

Now from [30], it follows that

$$\begin{aligned}\frac{t_m^{1-\alpha}}{2\delta_t\,\Gamma(2-\alpha)}\delta_t\sum_{n=1}^{m}\left\|\delta_t E^{n-\frac{1}{2}} + E^{n-\frac{1}{2}}\right\|^2 + \frac{1}{2\delta_t}\|E^m\|^2 + \sum_{n=1}^{m}\|E^{n-\frac{1}{2}}\|^2 + \frac{1}{2\delta_t}\|\nabla E^m\|^2 \\ + \sum_{n=1}^{m}\|\nabla E^{n-\frac{1}{2}}\|^2 \leq \sum_{n=1}^{m}\|T^{n-\frac{1}{2}}\|\,\left\|\delta_t E^{n-\frac{1}{2}} + E^{n-\frac{1}{2}}\right\|.\end{aligned} \tag{25}$$

Using $|xy| \leq \dfrac{x^2}{2\theta} + \dfrac{\theta}{2}y^2$ with $\theta = \dfrac{t_m^{1-\alpha}}{\Gamma(2-\alpha)}$ for the last term in (25), and by applying (25), we have

$$\begin{aligned}\frac{t_m^{1-\alpha}}{2\Gamma(2-\alpha)}\sum_{n=1}^{m}\left\|\delta_t E^{n-\frac{1}{2}} + E^{n-\frac{1}{2}}\right\|^2 + \frac{1}{2\delta_t}\|E^m\|^2 + \sum_{n=1}^{m}\|E^{n-\frac{1}{2}}\|^2 + \frac{1}{2\delta_t}\|\nabla E^m\|^2 \\ + \sum_{n=1}^{m}\|\nabla E^{n-\frac{1}{2}}\|^2 \leq \frac{\Gamma(2-\alpha)}{2t_m^{1-\alpha}}\sum_{n=1}^{m}\|T^{n-\frac{1}{2}}\| + \frac{t_m^{1-\alpha}}{2\Gamma(2-\alpha)}\sum_{n=1}^{\infty}\left\|\delta_t E^{n-\frac{1}{2}} + E^{n-\frac{1}{2}}\right\|,\end{aligned} \tag{26}$$

With some calculations, we have

$$\|E^n\|^2 + 2\,\delta_t\sum_{k=1}^{n}\left\|E^{k-\frac{1}{2}}\right\|^2 + \left\|\nabla E^n\right\|^2 + 2\,\delta_t\sum_{k=1}^{n}\left\|\nabla E^{k-\frac{1}{2}}\right\|^2 \leq \delta_t\Gamma(2-\alpha)t_n^{\alpha-1}\sum_{j=1}^{n}\|T^{j-\frac{1}{2}}\|^2 \leq n\delta_t\Gamma(2-\alpha)t_n^{\alpha-1}\max_{1\leq j\leq n}\|T^{j-\frac{1}{2}}\|^2 \tag{27}$$

Using the estimate $\|T^{j-\frac{1}{2}}\| \leq C\delta_t^{3-\alpha}$ and the fact that $t_n = n\delta_t \leq T$, we obtain

$$\|E^n\|^2 \leq n\,\delta_t\,\Gamma(2-\alpha)\,t_n^{\alpha-1}\max_{1\leq j\leq n}\|T^{j-\frac{1}{2}}\|^2 \leq T^2\,\Gamma(2-\alpha)\,C^2\,\delta_t^{2(3-\alpha)}.$$

Therefore we have

$$\|E^n\| \leq C^*\,\delta_t^{3-\alpha},$$

where $C^* = \sqrt{T^2\Gamma(2-\alpha)C^2}$, the proof is hereby concluded. □

4 Collocation Approach for Approximating Spatial Discretization

The spatial discretization is implemented using a local meshless collocation approach. The computational domain Ω is populated with a set of M scattered nodes. Around each node x_i, a compact local support domain Ω_i is defined such that the union of all subdomains covers Ω. For a given node $x_i \in \Omega_i$, its support domain includes its m_i nearest neighboring nodes, labeled $\{x_{i_1}, x_{i_2}, \ldots, x_{i_{m_i}}\}$, within each support domain Ω_i, the numerical solution $u(x,t_n)$ at time level t_n is expanded in terms of both radial and polynomial basis functions according to the expression

$$\tilde{u}^{(i)}(x,t_n) = \sum_{j=1}^{m_i} \lambda_j^{(n,i)} \phi\left(\|x - x_{i,j}\|\right) + \sum_{k=1}^{l} \gamma_k^{(n,i)} p_k(x),$$

where ϕ is a chosen radial basis function depending on the Euclidean norm $\|\cdot\|$, and $\{p_k\}_{k=1}^{l}$ provide a basis for the collection of d-dimensional polynomials whose total degree does not exceed $m-1$. The unknown expansion coefficients $\{\lambda_j^{(n,i)}\}_{j=1}^{m_i}$ and $\{\gamma_k^{(n,i)}\}_{k=1}^{l}$ are determined locally for each support domain at the n-th time level.

These coefficients are computed by enforcing two sets of conditions:

$$\begin{cases} \tilde{u}^{(i)}(x_{i,j},t_n) = u(x_{i,j},t_n), \quad j = 1,2,\ldots,m_i, & (28) \\ \sum_{j=1}^{m_i} \lambda_j^{(n,i)} p_k(x_{i,j}) = 0, \quad k = 1,2,\ldots,l. & (29) \end{cases}$$

The first condition ensures that the interpolant matches the function values at all m_i nodes in the support domain. The second condition imposes the polynomial orthogonality constraints. Equations (28) and (29) lead to the linear system for the subdomain Ω_i:

$$\begin{bmatrix} \Phi & P \\ P^T & 0 \end{bmatrix} \begin{bmatrix} \lambda^{(i)} \\ \gamma^{(i)} \end{bmatrix} = \begin{bmatrix} \mathbf{u}_n^{(i)} \\ \mathbf{0} \end{bmatrix}. \tag{30}$$

The submatrices are defined as

$$\Phi_{jk} = \phi\left(\|x_{i,j} - x_{i,k}\|\right), \qquad 1 \le j,k \le m_i,$$
$$P_{jk} = p_k(x_{i,j}), \qquad 1 \le j \le m_i,\ 1 \le k \le l,$$

and

$$\mathbf{u}_n^{(i)} = \left[u(x_{i,1},t_n), \ldots, u(x_{i,m_i},t_n)\right]^T.$$

The solution of (30) can be written as

$$\Lambda_i = \begin{bmatrix} \lambda^{(i)} \\ \gamma^{(i)} \end{bmatrix} = A_i^{-1}\mathbf{U}_i^n.$$

4.1 Discretization of Differential Operator

To approximate a spatial differential operator D at the central node x_i, the operator is applied analytically to the local interpolant:

$$D\tilde{u}^{(i)}(x_i,t_n) = \sum_{j=1}^{m_i} \lambda_j^{(n,i)} D\phi\left(\|x_i - x_{i,j}\|\right) + \sum_{k=1}^{l} \gamma_k^{(n,i)} Dp_k(x_i). \tag{31}$$

This can be expressed as

$$D\tilde{u}^{(i)}(x_i,t_n) = [D\phi_{i,1}, \ldots, D\phi_{i,m_i}, Dp_1(x_i), \ldots, Dp_l(x_i)]$$

where

$$D\phi_{i,j} = D\phi(\|x_i - x_{i,j}\|).$$

So,

$$\Lambda_i = \mathbf{d}_i^T A_i^{-1}\mathbf{U}_i^n.$$

This process is repeated for every node $x_i \in \Omega$. When applied to the governing spatial operator $\mathscr{L}$ of the PDE, it yields a discrete equation at each node:

$$\mathbf{d}_{\mathscr{L},i}^T A_i^{-1}\mathbf{U}_i^n = b_i, \qquad i = 1,2,\ldots,M,$$

where b_i denotes the discrete right–hand side at node x_i. Assembling all M local equations into a global system results in the matrix equation

$$\mathscr{L}\mathbf{U}^n = \mathbf{b},$$

where $\mathscr{L}$ is an $M \times M$ sparse matrix. The i-th row of $\mathscr{L}$ contains the non–zero entries from the local stencil $\mathbf{d}^T_{\mathscr{L},i}A_i^{-1}$, placed in the columns corresponding to the global indices of the nodes in Ω_i.

5 Simulation Results

The present section assesses the efficacy of the proposed numerical scheme through a benchmark problem. To quantify the precision achieved, we employ two conventional error metrics: the L_∞ norm, which measures the maximum absolute deviation, and the L_{rms} norm, representing the root mean square error, defined as follows:

$$L_\infty = \max_{1\le i\le M}\left|u(x_i,T) - U(x_i,T)\right|, \qquad L_{rms} = \left(\frac{1}{M}\sum_{i=1}^{M}\left(u(x_i,T) - U(x_i,T)\right)^2\right)^{1/2},$$

where $u(x_i,T)$ and $U(x_i,T)$ correspond to the analytical and discrete solutions, respectively, and M indicates the number of collocation points distributed throughout . Selecting a suitable shape parameter c poses a significant challenge in the context of RBF approximations. To address this concern, we adopt the second-order spline RBF for this particular test case. The discretization employs the same number of collocation points, $m_i = M$, along each dimension. The following numerical test case is designed to showcase the performance of the proposed methodology. All computations were performed using MATLAB R2016a on a personal laptop equipped with an Intel Core i7 processor and 8 GB RAM, running the Windows 10 operating system.

Example 1. We now turn our attention to the one–dimensional case of the time-fractional telegraph equation written as:

$${}_0^C D_t^\alpha u(x,t) + {}_0^C D_t^{\alpha-1} u(x,t) + u(x,t) = u_{xx}(x,t) + f(x,t),$$

on $(x,t) \in (0,1)\times(0,T]$. Based on the exact solution

$$u(x,t) = t^3\sin^2(x),$$

the appropriate initial and boundary conditions follow directly from the exact solution. Evaluating the telegraph equation provides the linear source term f(x,t) in the form

$$f(x,t) = \left(\frac{6t^{3-\alpha}}{\Gamma(4-\alpha)} + \frac{6t^{4-\alpha}}{\Gamma(5-\alpha)}\right)\sin^3(x) - 2t^3\cos(2x) + t^3\sin^2(x).$$

The values of the L_∞ and L_{rms} errors for $M = 500$ spatial points in the domain $[0,1]$, $m = 5$ and $T = 1$ are reported in Tables 1 and 2.

Table 1. The comparison of the various error measures for several choices of α and the time step size δt in the context of Example 1.

	$\alpha = 1.5$		$\alpha = 1.9$	
δt	L_∞	L_{rms}	L_∞	L_{rms}
0.1	2.388e-03	1.676e-03	1.376e-02	9.811e-03
0.05	8.605e-04	6.037e-04	6.521e-03	4.716e-03
0.025	3.083e-04	2.163e-04	3.116e-03	2.235e-03

Numerical computations for this example were performed on the interval $[0,1]$ considering different values of α and δt. Table 1 displays the absolute and root mean square errors obtained under the settings $m = 5$ at $T = 1s$. Using the formula $\frac{\log(E_1/E_2)}{\log(\delta t_1/\delta t_2)}$, where E_1 and E_2 are the errors corresponding to step sizes δt_1 and δt_2, the convergence rates for Example 1 are computed and listed in Table 2. The results indicate that the achieved convergence rates are consistent with the theoretical predictions.

Table 2. The rate of convergence for Example 1 for differenet values of α and δt

δt	$\alpha = 1.5$	$\alpha = 1.9$
0.1	—	—
0.05	1.47	1.06
0.025	1.48	1.08

6 Conclusion

In this paper a hybrid numerical strategy for the Caputo time–fractional telegraph equation has been developed, merging finite difference approximations for temporal evolution with radial basis function collocation for spatial representation. This combination yields a discrete formulation that is both computationally efficient and amenable to theoretical analysis. Stability properties and convergence behavior were examined in detail, which was subsequently corroborated through a test problem. The numerical evidence further indicates that the methodology can be effectively applied to nonlinear generalizations of the governing equation. Promising directions for subsequent work include extension to higher–dimensional domains, incorporation of variable or multiple fractional–order terms and design of fast solvers to alleviate the computational burden arising from the memory characteristics of fractional operators.

Authors' Contributions

Mitra Dabiri and Javad Damirchi jointly conceived the study, developed the numerical methodology, and carried out the theoretical analysis. Mitra Dabiri implemented the algorithms and performed the numerical experiments, while Javad Damirchi supervised the project and refined the analytical results and manuscript. Both authors read and approved the final version of the paper.

Data Availability

All data in the paper are available from the corresponding authors upon reasonable request.

Conflicts of Interest

The authors declare that there is no conflict of interest.

Ethical Considerations

The authors have diligently addressed ethical concerns, such as informed consent, plagiarism, data fabrication, misconduct, falsification, double publication, redundancy, submission, and other related matters.

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