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Research article

Convergence Quantum Analysis of Positive Solution for Caputo-Hadamard Fractional q-Differential Equations

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Abstract

In this research we obtain new results of approximate solution for nonlinear singular p-Laplacian Caputo-Hadamard fractional q-differential equation under infinite-point boundary conditions. The existence of unique iterative positive, error estimation, and convergence rate of approximate solution are obtained based on the description of Green function with special properties and employing appropriate substitution and appropriate cone. A few applications are demonstrated the validity of our achievements.

Keywords: Caputo-Hadamard q-fractional derivative, Positive solutions, Nonlinear analysis.

Mathematics Subject Classification (2020): 34A08, 26A33

1 Introduction

The systems of differential equations ($\mathbb{DE}$s) involving fractional derivative have proven to be more precise and genuine compared to integer-order models, and they also serve as a remarkable tool to characterize the hereditary characteristics of materials and processes, especially in viscoelasticity, electrochemistry, porous media, and beyond [1–5].

Guo *et al.* have also made some achievements in this regard, for example, [6]. In terms of dynamics, we also participated in some work [7–10] and later prepared to build fractional-order dynamics models and establish the existence theorem of solutions. There are many methods to deal with the solution of fractional differential equations ($\mathbb{FDE}$s) such as spectral analysis [11], Guo-Krasnoselskii's fixed point (F.P) theorem [12], mixed monotone operator method [13], Bohnenblust-Karlin F.P approaches [14], the mountain pass theorem [15], Mawhin's continuation theorem [16], and so on. Boutiara *et al.* in [17] studied $\mathbb{FDE}$ involving the Caputo-Hadamard (CH) derivative operator, of the form:

$$\begin{cases} {}^{\mathrm{CH}}\mathscr{D}_1^{\vartheta}\Bbbk(z) = \varpi\big(z, \Bbbk(z)\big), & z \in \mathrm{J}_1 = [1,T],\ \vartheta \in (0,1], \\ \lambda_1 \Bbbk(1) + \lambda_2 \Bbbk(T) = \lambda_3\, {}^{\mathrm{H}}\mathscr{I}_1^{\xi}\Bbbk(r) + \lambda_4, & 0 < \xi \leq 1,\ 1 < r < T, \end{cases}$$

where $\varpi \in C\big(\mathrm{J} \times \mathbb{R}\big)$, $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbb{R}$, and ${}^{\mathrm{H}}\mathscr{I}_1^{\xi}$denotes fractional integral in the sense of Hadamard, based on the uniqueness results by means of Boyd and Wong's and Banach's F.P theorems. Ardjouni in [18], discussed the existence and uniqueness of positive solutions of the nonlinear $\mathbb{FDE}$ with integral boundary conditions,

$$ {}^{\mathrm{CH}}\mathscr{D}_1^{\vartheta}\Bbbk(z) = \varpi\big(z,\Bbbk(z)\big), \qquad \Bbbk(1) = v\int_1^T \Bbbk(s)\,\mathrm{d}s + c, $$

for $z \in \mathrm{J}_1$, $T = \exp(1)$, where $0 < \vartheta \leq 1$, $v \geq 0$, $c > 0$, and $\varpi \in C\big(\mathrm{J}_1 \times \mathbb{R}_{\geq 0}, \mathbb{R}_{\geq 0}\big)$. Derbazi investigated t the following CH $\mathbb{FDE}$:

$$ \begin{cases} {}^{\mathrm{CH}}\mathscr{D}_{1+}^{\vartheta}\Bbbk(z) = \varpi\big(z,\Bbbk(z)\big), & z \in \mathrm{J}_1, \\ m_1\Bbbk(1) + n_1\,{}^{\mathrm{CH}}\mathscr{D}_{1+}^{\gamma}\Bbbk(1) = \xi_1\,{}^{\mathrm{H}}\mathscr{I}_{1+}^{\sigma_1}\Bbbk(\eta_1), & 1 < \eta_1 < T,\ \sigma_1 > 0, \\ m_2\Bbbk(T) + n_2\,{}^{\mathrm{CH}}\mathscr{D}_{\mathsf{q}}^{\gamma}\Bbbk(T) = \xi_2\,{}^{\mathrm{H}}\mathscr{I}_{1+}^{\sigma_2}\Bbbk(\eta_2), & 1 < \eta_1 < T,\ \sigma_2 > 0, \end{cases} $$

where $1 < \vartheta \leq 2$, $0 < \gamma \leq 1$, ${}^{\mathrm{H}}\mathscr{I}_1^{\mu}$ is the fractional Hadamard integral of order $\mu > 0, \mu \in \{\sigma_1,\ \sigma_2\}$, $\varpi \in C(\mathrm{J}_1 \times \mathbb{R})$, and $m_i, n_i, \xi_i, i = 1,2$, are suitably chosen real constants [19]. In [20], Makhlouf considered the following CH $\mathbb{FDE}$:

$$ {}^{\mathrm{CH}}\mathscr{D}_1^{\vartheta}\Bbbk(z) = \varpi_1\big(z,\ \Bbbk(z)\big) + \varpi_2\big(z,\Bbbk(z)\big)\frac{\mathrm{d}\Bbbk(z)}{\mathrm{d}z}, $$

with $\Bbbk(1) = v$, where $\varpi_1 : \mathrm{J}_1 \times \mathbb{R}^d \to \mathbb{R}^d$ is measurable. Motivated by the mentioned works and [21–24], this paper considers the infinite-point singular p-Laplacian q-$\mathbb{FDE}$ involving double Hadamard and CH operators as follows

$$ {}^{\mathrm{H}}\mathscr{D}_{\mathsf{q}}^{\vartheta}\left(\mathsf{L}_p\left({}^{\mathrm{CH}}\mathscr{D}_{\mathsf{q}}^{\xi}\Bbbk\right)\right)(z) + \varpi\big(z,\Bbbk(z),\Bbbk'(z)\big) = 0, \qquad z \in \mathrm{J}_1 \setminus \{1,T\},\ T = \exp(1), \tag{1} $$

under infinite-point boundary condition

$$ \begin{cases} \Bbbk(1) = \Bbbk'(1) = \cdots = \Bbbk^{(\mathring{\imath}-1)}(1) = \Bbbk^{(\mathring{\imath}+1)}(1) = \cdots = \Bbbk^{(\aleph-1)}(1) = 0, \\ \Bbbk^{(\mathring{\imath})}(1) = \sum_{j=1}^{\infty}\lambda_{1j}\Bbbk(r_j), \quad {}^{\mathrm{CH}}\mathscr{D}_{\mathsf{q}}^{\xi}\Bbbk(1) = 0, \\ \mathsf{L}_p\left({}^{\mathrm{CH}}\mathscr{D}_{\mathsf{q}}^{\xi}\Bbbk(T)\right) = \sum_{j=1}^{\infty}\lambda_{2j}\mathsf{L}_p\left({}^{\mathrm{CH}}\mathscr{D}_{\mathsf{q}}^{\xi}\Bbbk(r_j)\right), \end{cases} \tag{2} $$

with p-Laplacian operator $\mathsf{L}_p(z) = |z|^{p-2}z$, $p, \tilde{p} > 1$, $p + \tilde{p} = p\,\tilde{p}$, and derivative orders $1 < \vartheta \leq 2$ and $\aleph - 1 < \xi \leq \aleph$ $(\aleph \geq 3)$, $\xi > \mathring{\imath}$, where $0 < \lambda_{1j}, \lambda_{2j} < 1$, $r_j \in \mathrm{J}_1 \setminus \{1,T\}$, $j \in \mathbb{N}$, $\varpi \in C\big((\mathrm{J}_1 \setminus \{1,T\}) \times \mathbb{R}^2_{\geq 0}, \mathbb{R}_{\geq 0}\big)$, such that it is singular at $z = 1, T$. By using the iterative pattern, we show the unique solution of the problem is converged and more, the error analysis and the rate of convergence to approximate solution are established.

2 Auxiliary Notations and Lemmas

In this section, we recall some definitions of q-calculus, $0 < \mathsf{q} < 1$. The objects q-number $[\vartheta]_{\mathsf{q}}$, q-shifted factorial $(\zeta;\mathsf{q})_k$, and q-analogue of the power function $(z - \mathsf{q}r)^{(j)}$, are expressed for $z, \grave{z} \in \mathbb{R}$, by $\frac{1-\mathsf{q}^{\vartheta}}{1-\mathsf{q}}$,

$$ (z;\mathsf{q})_j = \begin{cases} 1, & j = 0, \\ \prod_{k=0}^{j-1}\left(1 - \mathsf{q}^k z\right), & j \in \mathbb{N}, \end{cases} \qquad (z - \mathsf{q}\grave{z})^{(j)} = \begin{cases} 1, & j = 0, \\ \prod_{k=0}^{j-1}\left(z - \mathsf{q}^k\grave{z}\right), & j \in \mathbb{N}, \end{cases} $$

respectively [25,26]. Since, $q \in (0,1)$, these finite products will be convergent by four decimal places when n becomes large enough, for example $n = 100$ [27,28]. One can use the Algorithm 1 to estimate of the power function. Clearly,

$$ (1 - \mathsf{q})^{(j)} = \prod_{k=0}^{j-1}\left(1 - \mathsf{q}^{l+1}\right), \qquad j \in \mathbb{N}, $$

whenever $z = 1$ and $\grave{z} = \mathsf{q}$. For check related algorithms, see [29]. Further, for case $j = \vartheta \in \mathbb{R}_{\geq 0}$,

$$ \left(z - \mathsf{q}\grave{z}\right)^{(\vartheta)} = z^{\vartheta}\prod_{k=0}^{\infty}\frac{z - \mathsf{q}^k\grave{z}}{z - \mathsf{q}^{\vartheta+k}\grave{z}}, \qquad z, \grave{z} \in \mathbb{R}, $$

Algorithm 1: MATLAB lines for the power function $(z-\mathrm{q}r)^{(j)}$.

```
Input: z, ž, q, j
Output: H
if j = 0 then
    H = 1;
else
    totalout=1;
    for k = 0 : j − 1 do
        totalout = totalout×(z − žq^k);
    H =totalout;
    end;
```

and $z^{(\vartheta)} = z^{\vartheta}$ whenever $\check{z} = 0$ [30]. The fractional q-integral and q-derivative of RL type for the function $\mathbb{k} \in C(\mathbb{R}_{>0})$ are expressed as [27,31],

$$\mathscr{I}_{\mathrm{q}}^{\vartheta}\,\mathbb{k}(z) = \begin{cases} \int_0^z (z-\mathrm{q}\varsigma)^{(\vartheta-1)} \frac{\mathbb{k}(\varsigma)}{\Gamma_{\mathrm{q}}(\vartheta)}\,\mathrm{d}_{\mathrm{q}}\varsigma, & \vartheta > 0, \\ \mathbb{k}(z), & \vartheta = 0, \end{cases}$$

and

$$\mathscr{D}_{\mathrm{q}}^{\vartheta}\,\mathbb{k}(z) = \int_0^z (z-\mathrm{q}\varsigma)^{(\aleph-\vartheta-1)} \frac{\mathbb{k}(\varsigma)}{\Gamma_{\mathrm{q}}(\aleph-\vartheta)}\,\mathrm{d}_{\mathrm{q}}\varsigma, \qquad \aleph - 1 < \vartheta < \aleph,\ \aleph = [\wp],$$

respectively, along with the famous q-Gamma function, $\Gamma_{\mathrm{q}}(\vartheta) = \frac{(1-\mathrm{q})^{(\vartheta-1)}}{(1-\mathrm{q})^{\vartheta-1}}$, for $\vartheta \in \mathbb{R} \setminus \{\cdots,-2,-1,0\}$, and the relation

$$\Gamma_{\mathrm{q}}(\vartheta+1) = [\vartheta]_{\mathrm{q}}\Gamma_{\mathrm{q}}(\vartheta),$$

holds [25,30]. The ϑth fractional Caputo q-derivative for an absolute function $\mathbb{b} \in C(\mathrm{J}_{\circ})$ is described by

$${}^{\mathrm{C}}\mathscr{D}_{\mathrm{q}}^{\vartheta}\,\mathbb{k}(z) = {}^{\mathrm{RL}}\mathscr{I}_{\mathrm{q}}^{[\vartheta]-\vartheta} D_{\mathrm{q}}^{[\vartheta]}\,\mathbb{k}(z), \qquad \vartheta > 0,$$

where the structure of jth q-derivative and q-integral for function $\mathbb{k}$ are expressed as

$$D_{\mathrm{q}}^{j}\,\mathbb{k}(z) = \begin{cases} \frac{\mathbb{k}(\mathrm{q}z)-\mathbb{k}(z)}{(1-\mathrm{q})z}, & j = 1, \\ D_{\mathrm{q}}\big(D_{\mathrm{q}}^{j-1}\mathbb{k}\big)(z), & j > 1, \end{cases} \qquad I_{\mathrm{q}}^{j}\,\mathbb{k}(z) = \begin{cases} \int_0^z \mathbb{k}(\varsigma)\,\mathrm{d}_{\mathrm{q}}\varsigma = z(1-\mathrm{q})\sum_{k=0}^{\infty}\mathrm{q}^k\mathbb{k}(\mathrm{q}^k z), & j = 1, \\ I_{\mathrm{q}}\big(I_{\mathrm{q}}^{j-1}\mathbb{b}\big)(z), & j > 1, \end{cases}$$

with $[\vartheta]$ denotes the integer part of $\vartheta > 0$ [31]. Algorithm 2 shows pseudo-code for $I_{\mathrm{q}}^{j}\,\mathbb{k}(z)$. Let $\mathbb{k}$ is a continuous function at $z = 0$. Then $I_{\mathrm{q}} D_{\mathrm{q}}\,\mathbb{k}(z) = \mathbb{k}(z) - \mathbb{k}(0)$ [32]. One can check that for $\vartheta, \vartheta_0 > 0$, ${}^{\mathrm{RL}}\mathscr{I}_{\mathrm{q}}^{\vartheta}\,{}^{\mathrm{RL}}\mathscr{I}_{\mathrm{q}}^{\vartheta_0}\,\mathbb{k}(z) = {}^{\mathrm{RL}}\mathscr{I}_{\mathrm{q}}^{\vartheta+\vartheta_0}\,\mathbb{k}(z)$ [33]. Also, for $l > -1$, we have

$${}^{\mathrm{RL}}\mathscr{I}_{\mathrm{q}}^{\vartheta} z^{\nu} = \frac{\Gamma_{\mathrm{q}}(\nu+1)}{\Gamma_{\mathrm{q}}(\vartheta+\nu+1)} z^{\vartheta+l}, \qquad \left({}^{\mathrm{RL}}\mathscr{I}_{\mathrm{q}}^{\vartheta} 1\right)(z) = \frac{1}{\Gamma_{\mathrm{q}}(\vartheta+1)} z^{\vartheta},\ (z > 0).$$

In the sense of Caputo, the fractional q-derivative is given for the function $\mathbb{k}$, as [31],

$${}^{\mathrm{C}}\mathscr{D}_{\mathrm{q}}^{\vartheta}\,\mathbb{k}(z) = \begin{cases} I_{\mathrm{q}}^{[\vartheta]-\vartheta}\left(D_{\mathrm{q}}^{[\vartheta]}\,\mathbb{k}(z)\right), & \vartheta > 0, \\ \mathbb{k}(z), & \vartheta = 0. \end{cases} \tag{3}$$

Lemma 1 ([27,31])**.** *For a given function* $\mathbb{b}$ *and* $\vartheta, \vartheta_0 \in \mathbb{R}_{\geq 0}$, *we have* ${}^{\mathrm{C}}\mathscr{D}_{\mathrm{q}}^{\vartheta}\left(\mathscr{I}_{\mathrm{q}}^{\vartheta}\,\mathbb{k}(z)\right) = \mathbb{k}(z)$, $\mathscr{I}_{\mathrm{q}}^{\vartheta}\left(\mathscr{I}_{\mathrm{q}}^{\vartheta_0}\,\mathbb{k}(z)\right) = \mathscr{I}_{\mathrm{q}}^{\vartheta+\vartheta_0}\,\mathbb{k}(z)$, *and*

$$\mathscr{I}_{\mathrm{q}}^{\vartheta}\left({}^{\mathrm{C}}\mathscr{D}_{\mathrm{q}}^{\vartheta}\,\mathbb{k}(z)\right) = \mathbb{k}(z) - \sum_{k=0}^{[\vartheta]-1} \frac{z^k}{\Gamma_{\mathrm{q}}(k+1)} D_{\mathrm{q}}^{k}\,\mathbb{k}(z)(0),$$

$$\mathscr{I}_{\mathrm{q}}^{\vartheta}\left(\mathscr{D}_{\mathrm{q}}^{\vartheta_0}\,\mathbb{k}(z)\right) = \mathscr{D}_{\mathrm{q}}^{\vartheta_0}\left(\mathscr{I}_{\mathrm{q}}^{\vartheta}\,\mathbb{k}(z)\right) - \sum_{k=0}^{[\vartheta_0]-1} \frac{z^{\vartheta-\vartheta_0+k}}{\Gamma_{\mathrm{q}}(\vartheta+k-\vartheta_0+1)} D_{\mathrm{q}}^{k}\,\mathbb{k}(0). \tag{4}$$

Algorithm 2: Pseudo-code for $I_{\mathrm{q}}^{j}\Bbbk(z),\ j=1$.

Input: z, n, q, function
Output: H
totalout $=0$;
$F=0$;
while $F\neq \mathrm{round}(s,4)$ **do**
  for $k=0:n$ **do**
    totalout $=$ totalout $+\mathrm{q}^{k}\times \mathrm{eval}\left(\mathrm{subs}(\text{function}, z\times \mathrm{q}^{k})\right)$;
  totalout $=z\times(1-\mathrm{q})\times$ totalout;
  $F=\mathrm{round}(s,4)$;
  end;

The RL type fractional q-integral for function $\Bbbk$ is get by [29],

$$\int_0^z (z-\mathrm{q}\varsigma)^{(\vartheta-1)}\frac{\Bbbk(\varsigma)}{\Gamma_{\mathrm{q}}(\vartheta)}\,\mathrm{d}_{\mathrm{q}}\varsigma=\frac{z^{\varsigma}(1-\mathrm{q})}{\Gamma_{\mathrm{q}}(\vartheta)}\sum_{k=0}^{\infty}\mathrm{q}^{k}\prod_{\mathring{\imath}=0}^{\infty}\frac{1-\mathrm{q}^{k+\mathring{\imath}}}{1-\mathrm{q}^{\vartheta+k+\mathring{\imath}-1}}\Bbbk\left(\mathrm{q}^{k}z\right). \tag{5}$$

To calculate fractional q-integral (5) see [29, Algorithm 2]. For $\vartheta\geq 0$ and $k>-1$, we have

$$\left(\mathscr{D}_{\mathrm{q}}^{\vartheta}(z-\kappa)^{(\mathring{\imath})}\right)(z)=\frac{\Gamma_{\mathrm{q}}(\mathring{\imath}+1)}{\Gamma_{\mathrm{q}}(\vartheta+\mathring{\imath}+1)}(z-\kappa)^{(\vartheta+\mathring{\imath})},\qquad 0<\kappa<z,$$

and in particular $\left(\mathscr{D}_{\mathrm{q}}^{\vartheta}1\right)(z)=\frac{1}{\Gamma_{\mathrm{q}}(\vartheta+1)}z^{(\vartheta)}$ [34]. Now, we introduce the Hadamard fractional q-integral, q-derivative operators, and CH fractional q-derivative based on the [35, 36].

Definition 1. *Let $b>0$, then the Hadamard-type fractional q-integral and q-derivative of a real-value function $\Bbbk\in L^{1}[b,\infty)$ and $z^{\aleph-1}\Bbbk^{(\aleph-1)}(z)\in AC[b,\infty)$, respectively, are expressed by*

$$^{\mathrm{H}}\mathscr{I}_{\mathrm{q}}^{\vartheta}\Bbbk(z)=\int_b^z\frac{(\ln z-\mathrm{q}\ln\varsigma)^{(\vartheta-1)}}{\varsigma\Gamma_{\mathrm{q}}(\vartheta)}\Bbbk(\varsigma)\,\mathrm{d}_{\mathrm{q}}s,\qquad \vartheta>0,$$
$$^{\mathrm{H}}\mathscr{D}_{\mathrm{q}}^{\vartheta}\Bbbk(z)=\left(z\frac{\mathrm{d}}{\mathrm{d}z}\right)^{\aleph}\int_b^z\frac{(\ln z-\mathrm{q}\ln\varsigma)^{(\aleph-\vartheta-1)}}{\varsigma\Gamma_{\mathrm{q}}(\aleph-\vartheta)}\Bbbk(\varsigma)\,\mathrm{d}_{\mathrm{q}}s,\qquad \aleph-1<\vartheta<\aleph,\ \aleph\in\mathbb{N}.$$

Definition 2. *The* CH *fractional q-derivative of a given function $\Bbbk\in AC(\mathrm{J}_1)$ is formulated by*

$$^{\mathrm{CH}}\mathscr{D}_{\mathrm{q}}^{\vartheta}\Bbbk(z)=\int_1^z\frac{(\ln z-\mathrm{q}\ln\varsigma)^{(\vartheta-\aleph+1)}}{\varsigma\Gamma_{\mathrm{q}}(\aleph-\vartheta)}\left(\varsigma\frac{\mathrm{d}}{\mathrm{d}\varsigma}\right)^{\aleph}\Bbbk(\varsigma)\,\mathrm{d}_{\mathrm{q}}\varsigma,\qquad \vartheta>0,$$

where $\aleph=[\vartheta]+1$, provided that the right-hand side is pointwise defined on $\mathbb{R}_{>0}$.

Lemma 2 ([36]). *Let $\aleph-1<\vartheta\leq\aleph$, $\aleph\in\mathbb{N}$, and $\Bbbk\in C^{\aleph}(\mathrm{J}_1)$. Then,*

$$\left({}^{\mathrm{CH}}\mathscr{D}_{\mathrm{q}}^{\vartheta}\,{}^{\mathrm{H}}\mathscr{I}_{\mathrm{q}}^{\vartheta}\Bbbk\right)(z)=\Bbbk(z),\qquad \left({}^{\mathrm{H}}\mathscr{I}_{\mathrm{q}}^{\vartheta}\,{}^{\mathrm{CH}}\mathscr{D}_{\mathrm{q}}^{\vartheta}\Bbbk\right)(z)=\Bbbk(z)+\sum_{k=0}^{\aleph-1}c_k(\ln z)^{k}.$$

Lemma 3 ([36]). *Let $\hbar\in L[b,\infty)$. Then, the solution of $\mathbb{FE}$, ${}^{\mathrm{H}}\mathscr{D}_{\mathrm{q}}^{\vartheta}\Bbbk(z)+\hbar(z)=0$, for $z>b$, with $\aleph-1<\vartheta<\aleph$, $\aleph\in\mathbb{N}$, is expressed as,*

$$\Bbbk(z)=\sum_{k=1}^{\aleph}\mu_k(\ln z-\mathrm{q}\ln b)^{(\vartheta-k)}-\int_b^z\frac{(\ln z-\mathrm{q}\ln\varsigma)^{(\vartheta-1)}}{\varsigma\Gamma_{\mathrm{q}}(\vartheta)}\hbar(\varsigma)\,\mathrm{d}_{\mathrm{q}}\varsigma,\quad \mu_k\in\mathbb{R}.$$

3 Main Results

Let $\mathbb{r}(z)\in C(\mathrm{J}_1)$, $T=\exp(1)$. In view of the p-Laplacian q-$\mathbb{FDE}$ (1)-(2), one can rewrite it to the following form,

$$^{\mathrm{H}}\mathscr{D}_{\mathrm{q}}^{\vartheta}\left(\mathsf{L}_p\left({}^{\mathrm{CH}}\mathscr{D}_{\mathrm{q}}^{\xi-1}\mathbb{r}(z)\right)\right)+\varpi\left(z,\Bbbk(z),\mathbb{r}(z)\right)=0,\qquad z\in\mathrm{J}_1, \tag{6}$$

under infinite-point boundary condition,

$$\begin{cases} \mathbb{r}(1) = \mathbb{r}'(1) = \cdots = \mathbb{r}^{(\mathring{\imath}-2)}(1) = \mathbb{r}^{(\mathring{\imath})}(1) = \cdots = \mathbb{r}^{(\aleph-2)}(1) = 0, \\ \mathbb{r}^{(\mathring{\imath}-1)}(T) = \sum_{j=1}^{\infty} \lambda_{1j}\mathbb{r}(r_j), \quad {}^{\mathrm{CH}}\mathscr{D}_{\mathrm{q}}^{\xi-1}\mathbb{r}(1) = 0, \\ \mathsf{L}_p\left({}^{\mathrm{CH}}\mathscr{D}_{\mathrm{q}}^{\xi-1}\mathbb{r}(T)\right) = \sum_{j=1}^{\infty} \lambda_{2j}\mathsf{L}_p\left({}^{\mathrm{CH}}\mathscr{D}_{\mathrm{q}}^{\xi-1}\mathbb{r}(r_j)\right), \end{cases} \tag{7}$$

by substitution

$$\Bbbk(z) = I_{\mathrm{q}}\left(\tfrac{1}{z}\mathbb{r}(z)\right) = \int_1^z \frac{\mathbb{r}(\varsigma)}{\varsigma}\,\mathrm{d}_{\mathrm{q}}\varsigma. \tag{8}$$

Lemma 4. *The solution of the* q-$\mathbb{FDE}$ *involving* CH *derivative operator, of the form,*

$${}^{\mathrm{CH}}\mathscr{D}_{\mathrm{q}}^{\xi-1}\mathbb{r}(z) + \hbar(z) = 0, \qquad z \in \mathrm{J}_1, \hbar \in C(\mathrm{J}_1 \setminus \{1, T\}) \cap L^1(\mathrm{J}_1), \tag{9}$$

under boundary condition $\mathbb{r}^{(\mathring{\imath}-1)}(T) = \sum_{j=1}^{\infty} \lambda_{1j}\mathbb{r}(r_j)$ *has the following expression,*

$$\mathbb{r}(z) = \int_{\mathrm{J}_1} \frac{\varphi_{\mathrm{q}}(z,\varsigma)}{\varsigma}\hbar(\varsigma)\,\mathrm{d}_{\mathrm{q}}\varsigma, \qquad \forall z \in \mathrm{J}_1, \tag{10}$$

with the following Green function, for $\varsigma, z \in \mathrm{J}_1$,

$$\varphi_{\mathrm{q}}(z,\varsigma) = \frac{1}{\Delta\Gamma_{\mathrm{q}}(\xi-1)} \begin{cases} (\ln z)^{\mathring{\imath}-1}\,\Gamma_{\mathrm{q}}(\xi-1)\gimel_{\mathrm{q}}(\varsigma)\,(\ln T - \mathrm{q}\ln\varsigma)^{(\xi-\mathring{\imath}-1)} - \Delta(\ln z - \ln\varsigma)^{\xi-2}, & \varsigma \le z, \\ (\ln z)^{\mathring{\imath}-1}\,\Gamma_{\mathrm{q}}(\xi-1)\gimel_{\mathrm{q}}(\varsigma)\,(\ln T - \mathrm{q}\ln\varsigma)^{(\xi-\mathring{\imath}-1)}, & z \le \varsigma, \end{cases} \tag{11}$$

where $\Delta = (\mathring{\imath}-1)! - \sum_{j=1}^{\infty} \lambda_{1j}\left(\ln r_j\right)^{\mathring{\imath}-1}$ *and*

$$\gimel_{\mathrm{q}}(z) = \frac{1}{\Gamma_{\mathrm{q}}(\mathring{\imath}-i)} - \frac{1}{\Gamma_{\mathrm{q}}(\xi-1)} \sum_{z \le r_j} \lambda_{1j}\left(\ln\left(\frac{r_j}{\mathrm{q}z}\right) \Big/ \ln\left(\frac{T}{\mathrm{q}z}\right)\right)^{\xi-2} (\ln T - \mathrm{q}\ln z)^{(\mathring{\imath}-1)}. \tag{12}$$

Proof. First, applying Lemma (2), one can describe Eq. (9) to an equivalent integral equation ($\mathbb{IE}$),

$$\mathbb{r}(z) = -{}^{\mathrm{H}}\mathscr{I}_{\mathrm{q}}^{\xi-1}\hbar(z) + c_1 + c_2(\ln z) + \cdots + c_{\mathring{\imath}-1}(\ln z)^{\mathring{\imath}-2} + c_{\mathring{\imath}}(\ln z)^{\mathring{\imath}-1} + c_{\mathring{\imath}+1}(\ln z)^{\mathring{\imath}} + \cdots + c_{\aleph}(\ln z)^{\aleph-2}, \qquad c_{\mathring{\imath}} \in \mathbb{R}, \mathring{\imath} = 1, 2, \ldots, \aleph.$$

The condition $\mathbb{r}(1) = 0$, implies that $c_1 = 0$. By taking the derivative, we get

$$\mathbb{r}'(z) = -{}^{\mathrm{H}}\mathscr{I}_{\mathrm{q}}^{\xi-2}\hbar(z) + c_2\frac{1}{z} + \cdots + c_{\mathring{\imath}-1}(\mathring{\imath}-2)(\ln z)^{\mathring{\imath}-3}\frac{1}{z} + (\mathring{\imath}-1)c_{\mathring{\imath}}\frac{1}{z}(\ln z)^{\mathring{\imath}-2} + \cdots + c_{\aleph}\frac{1}{z}(\aleph-2)(\ln z)^{\aleph-3}.$$

Again, condition $\mathbb{r}'(1) = 0$, implies that $c_2 = 0$. Now, taking the derivative step by step with

$$\mathbb{r}''(1) = \cdots = \mathbb{r}^{(\mathring{\imath}-2)}(1) = \mathbb{r}^{(\mathring{\imath})}(1) = \cdots = \mathbb{r}^{(\aleph-2)}(1) = 0,$$

we get $c_j = 0$, $j \ge 3$, $j \ne \mathring{\imath}$. Hence, $\mathbb{r}(z) = c_{\mathring{\imath}}(\ln z)^{\mathring{\imath}-1} - {}^{\mathrm{H}}\mathscr{I}_{\mathrm{q}}^{\xi-1}\hbar(z)$. Indeed,

$$\mathbb{r}^{(\mathring{\imath}-1)}(T) = c_{\mathring{\imath}} \times (\mathring{\imath}-1)! - {}^{\mathrm{H}}\mathscr{I}_{\mathrm{q}}^{\xi-\mathring{\imath}}\hbar(T). \tag{13}$$

Additionally, combining $\mathbb{r}^{(\mathring{\imath}-1)}(T) = \sum_{j=1}^{\infty} \lambda_{1j}\mathbb{r}(r_j)$ with (13), we get

$$c_{\mathring{\imath}} = \int_{\mathrm{J}_1} \frac{(\ln T - \mathrm{q}\ln\varsigma)^{(\xi-\mathring{\imath}-1)}}{\Delta\varsigma\Gamma_{\mathrm{q}}(\xi-\mathring{\imath})}\hbar(\varsigma)\,\mathrm{d}_{\mathrm{q}}\varsigma - \sum_{j=1}^{\infty} \lambda_{1j}\int_1^{r_j} \frac{(\ln r_j - \mathrm{q}\ln\varsigma)^{(\xi-2)}}{\Delta\varsigma\Gamma_{\mathrm{q}}(\xi-1)}\hbar(\varsigma)\,\mathrm{d}_{\mathrm{q}}\varsigma = \int_{\mathrm{J}_1} \frac{(\ln T - \mathrm{q}\ln\varsigma)^{(\xi-\mathring{\imath}-1)}}{\Delta\varsigma}\gimel_{\mathrm{q}}(\varsigma)\hbar(\varsigma)\,\mathrm{d}_{\mathrm{q}}\varsigma.$$

It means that,

$$\mathbb{r}(z) = c_{\mathring{\imath}+1}(\ln z)^{\mathring{\imath}-1} - {}^{\mathrm{H}}\mathscr{I}_{\mathrm{q}}^{\xi-1}\hbar(z) = -\int_1^z \frac{(\ln z - \mathrm{q}\ln\varsigma)^{(\xi-2)}}{\varsigma\Gamma_{\mathrm{q}}(\xi-1)}\hbar(\varsigma)\,\mathrm{d}_{\mathrm{q}}\varsigma + \int_{\mathrm{J}_1} \frac{(\ln T - \mathrm{q}\ln\varsigma)^{(\xi-\mathring{\imath}-1)}(\ln z)^{\mathring{\imath}-1}}{\Delta\varsigma}\gimel_{\mathrm{q}}(\varsigma)\hbar(\varsigma)\,\mathrm{d}_{\mathrm{q}}\varsigma,$$

which is $\varphi_{\mathrm{q}}(z,\varsigma)$ in Eq. (10), accordingly. □

Lemma 5. *The p-Laplacian* q-$\mathbb{FDE}$ (6)-(7) *has a unique solution as follows,*

$$\mathbb{r}(z) = \int_{\mathrm{J}_1} \frac{\varphi_{\mathsf{q}}(z,\varsigma)}{\varsigma} \mathsf{L}_{\tilde{p}}\left(\int_{\mathrm{J}_1} \frac{\mathscr{Y}_{\mathsf{q}}(\varsigma,s)}{s} \varpi\big(s, I_{\mathsf{q}}\mathbb{r}(s), \mathbb{r}(s)\big)\, \mathrm{d}_{\mathsf{q}}s \right) \mathrm{d}_{\mathsf{q}}\varsigma, \tag{14}$$

for $\varpi \in C\big((\mathrm{J}_1 \setminus \{T\}) \times \mathbb{R}^2_{>0}, \mathbb{R}_{\geq 0}\big)$, *with the following Green function for* $z,\varsigma \in \mathrm{J}_1$,

$$\mathscr{Y}_{\mathsf{q}}(z,\varsigma) = \frac{1}{\Delta_1 \Gamma_{\mathsf{q}}(\vartheta)} \begin{cases} \Gamma_{\mathsf{q}}(\vartheta)\gimel_{\mathsf{q};1}(\varsigma)(\ln z)^{\vartheta-1}\left(\ln T - \mathsf{q}\ln\varsigma\right)^{(\vartheta-1)} - \Delta\big(\ln z - \ln\varsigma\big)^{\vartheta-1}, & \varsigma \leq z, \\ \Gamma_{\mathsf{q}}(\vartheta)\gimel_{\mathsf{q};1}(\varsigma)(\ln z)^{\vartheta-1}\left(\ln T - \mathsf{q}\ln\varsigma\right)^{(\vartheta-1)}, & z \leq \varsigma, \end{cases} \tag{15}$$

where $\Delta_1 = 1 - \sum_{j=1}^{\infty} \lambda_{2j}\left(\ln r_j\right)^{\vartheta-1}$,

$$\gimel_{\mathsf{q};1}(z) = \frac{1}{\Gamma_{\mathsf{q}}(\vartheta)} - \frac{1}{\Gamma_{\mathsf{q}}(\vartheta)} \sum_{z \leq r_j} \lambda_{2j}\left(\ln\left(\frac{r_j}{\mathsf{q}z}\right) \big/ \ln\left(\frac{T}{\mathsf{q}z}\right)\right)^{\xi-2}. \tag{16}$$

Proof. Consider the following p-Laplacian q-$\mathbb{FDE}$,

$$\begin{cases} {}^{\mathrm{H}}\mathscr{D}^{\vartheta}_{\mathsf{q}}\left(\mathsf{L}_p\left({}^{\mathrm{CH}}\mathscr{D}^{\xi-1}_{\mathsf{q}}\mathbb{r}(z)\right)\right) + \hbar_1(z), & z \in \mathrm{J}_1, \\ \mathsf{L}_p\left({}^{\mathrm{CH}}\mathscr{D}^{\xi-1}_{\mathsf{q}}\mathbb{r}(1)\right) = 0, \quad \mathbb{r}(T) = \sum_{j=1}^{\infty} \lambda_{2j}\mathsf{L}_p\left({}^{\mathrm{CH}}\mathscr{D}^{\xi-1}_{\mathsf{q}}\mathbb{r}(r_j)\right), \end{cases} \tag{17}$$

for $\hbar_1 \in C(\mathrm{J}_1)$. According to Lemma (3), we reduce (17) to the following integral equation,

$$\mathsf{L}_p\left({}^{\mathrm{CH}}\mathscr{D}^{\xi-1}_{\mathsf{q}}\mathbb{r}(z)\right) = -{}^{\mathrm{H}}\mathscr{I}^{\vartheta}_{\mathsf{q}}\hbar_1(z) + \tilde{c}_1(\ln z)^{\vartheta-1} + \tilde{c}_2(\ln z)^{\vartheta-2}.$$

The first condition in (17), yields $\tilde{c}_2 = 0$ and consequently,

$$\mathsf{L}_p\left({}^{\mathrm{CH}}\mathscr{D}^{\xi-1}_{\mathsf{q}}\mathbb{r}(z)\right) = \tilde{c}_1(\ln z)^{\vartheta-1} - {}^{\mathrm{H}}\mathscr{I}^{\vartheta}_{\mathsf{q}}\hbar_1(z). \tag{18}$$

Further, using the second condition in (17), and combining with (18), we obtain

$$\tilde{c}_1 = \int_{\mathrm{J}_1} \frac{(\ln T - \mathsf{q}\ln\varsigma)^{(\vartheta-1)}}{\Delta_1\, \varsigma\, \Gamma_{\mathsf{q}}(\vartheta)} \hbar_1(\varsigma)\, \mathrm{d}_{\mathsf{q}}\varsigma - \sum_{j=1}^{\infty} \lambda_{2j} \int_1^{r_j} \frac{(\ln r_j - \mathsf{q}\ln\varsigma)^{(\vartheta-1)}}{\Delta_1\, \varsigma\, \Gamma_{\mathsf{q}}(\vartheta)} \hbar_1(\varsigma)\, \mathrm{d}_{\mathsf{q}}\varsigma = \int_0^1 \frac{(\ln T - \mathsf{q}\ln\varsigma)^{(\vartheta-1)}}{\Delta_1\, \varsigma} \gimel_{\mathsf{q};1}(\varsigma)\hbar(\varsigma)\, \mathrm{d}\varsigma.$$

Indeed,

$$\mathsf{L}_p\left({}^{\mathrm{CH}}\mathscr{D}^{\xi-1}_{\mathsf{q}}\mathbb{r}(z)\right) = \tilde{c}_1(\ln z)^{\vartheta-1} - {}^{\mathrm{H}}\mathscr{I}^{\vartheta}_{\mathsf{q}}\hbar_1(z) = -\int_1^z \frac{(\ln z - \mathsf{q}\ln\varsigma)^{(\vartheta-1)}}{\Gamma_{\mathsf{q}}(\vartheta)\Delta_1} \hbar_1(\varsigma)\, \mathrm{d}_{\mathsf{q}}\varsigma + \int_{\mathrm{J}_1} \frac{(\ln T - \mathsf{q}\ln\varsigma)^{(\vartheta-1)}}{\Delta_1} \gimel_{\mathsf{q};1}(\varsigma)\hbar_1(\varsigma)\, \mathrm{d}_{\mathsf{q}}\varsigma.$$

This is $\mathscr{Y}_{\mathsf{q}}(z,\varsigma)$, as in (14), accordingly. □

Lemma 6. *The Green functions* $\varphi_{\mathsf{q}}(z,\varsigma)$ *and* $\mathscr{Y}_{\mathsf{q}}(z,\varsigma)$ *in Eqs.* (10) *and* (15), *satisfying the following properties.*

i) *By considering* $\sigma_{\mathsf{q}}(z) = \left(\ln T - \mathsf{q}\ln z\right)^{(\xi-\mathring{\imath}-1)}\left[1 - \left(\ln T - \mathsf{q}\ln z\right)^{(\mathring{\imath}-1)}\right]$, $\aleph - 1 < \xi \leq \aleph$, $(\aleph \geq 3)$, $\xi > \mathring{\imath}$ *we have,*

$$\Delta(\ln z)^{\mathring{\imath}-1}\sigma_{\mathsf{q}}(z) \leq \varphi_{\mathsf{q}}(z,\varsigma) \leq \frac{(\ln z)^{\mathring{\imath}-1}}{\Delta\Gamma_{\mathsf{q}}(\xi-\mathring{\imath})}, \qquad z,\varsigma \in \mathrm{J}_1. \tag{19}$$

ii) $\mathscr{Y}_{\mathsf{q}} \in C(\mathrm{J}_1^2, \mathbb{R}_{\geq 0})$ *has positive value, and by choosing*

$$\sigma_{\mathsf{q};1}(z) = \frac{(\ln T - \mathsf{q}\ln z)^{(\vartheta-1)}}{\Delta_1}\left[\gimel_{\mathsf{q};1}(z) - \frac{\Delta_1}{\Gamma_{\mathsf{q}}(\vartheta)}\right], \tag{20}$$

for $z,\varsigma \in \mathrm{J}_1 \setminus \{1,T\}$, *we have,* $(\ln z)^{\vartheta-1}\sigma_{\mathsf{q};1}(\varsigma) \leq \mathscr{Y}_{\mathsf{q}}(z,\varsigma) \leq \sigma_{\mathsf{q};1}(\varsigma)$.

Proof. i) First, we find that

$$\Delta(\ln z)^{\mathring{\imath}-1}\big(\ln T-\mathrm{q}\ln\varsigma\big)^{(\xi-\mathring{\imath}-1)}\left[1-\big(\ln T-\mathrm{q}\ln\varsigma\big)^{(\mathring{\imath}-1)}\right]\leq\Delta\Gamma_{\mathrm{q}}(\xi-1)\varphi_{\mathrm{q}}(z,\varsigma)\leq(\ln z)^{\mathring{\imath}-1}\Gamma_{\mathrm{q}}(\xi-1)\gimel_{\mathrm{q}}(\varsigma)\big(\ln T-\mathrm{q}\ln\varsigma\big)^{(\xi-\mathring{\imath}-1)}. \tag{21}$$

Obviously, $\gimel_{\mathrm{q}}'(z)\geq 0$, $z\in \mathrm{J}_1$, and more, $\gimel_{\mathrm{q}}$ is nondecreasing with respect to z. Now, we get

$$\begin{aligned}\Gamma_{\mathrm{q}}(\lambda-1)\gimel_{\mathrm{q}}(z)&=(\xi-2)(\xi-3)\cdots(\xi-\mathring{\imath})-\sum_{\varsigma\leq r_j}\lambda_{1j}\left(\ln\left(\frac{r_j}{\mathrm{q}z}\right)\big/\ln\left(\frac{T}{\mathrm{q}z}\right)\right)^{\xi-1}\big(\ln T-\mathrm{q}\ln z\big)^{(\mathring{\imath}-1)}\\&\geq\Gamma_{\mathrm{q}}(\xi-1)\gimel_{\mathrm{q}}(1)=(\xi-2)(\xi-3)\cdots(\xi-\mathring{\imath})-\sum_{j=1}^{\infty}\lambda_j(\ln r_j)^{\xi-1}\\&\geq(\mathring{\imath}-1)!-\sum_{j=1}^{\infty}\lambda_{1j}(\ln r_j)^{\mathring{\imath}}=\Delta.\quad z\in\mathrm{J}_1,\ \xi-1>\mathring{\imath}.\end{aligned} \tag{22}$$

Clearly, the right inequality of (21) holds. We know that for $z,\varsigma\in\mathrm{J}_1$, with $\varsigma\leq z$, $\ln z-\ln\varsigma\leq\ln z-\ln z\ln\varsigma=(1-\ln\varsigma)\ln z$. Hence, in this case,

$$\big(\ln z-\mathrm{q}\ln\varsigma\big)^{(\xi-2)}\leq\big(\ln T-\mathrm{q}\ln\varsigma\big)^{(\xi-2)}(\ln z)^{\xi-2},\qquad \mathrm{q}\in(0,1).$$

Then, from $\xi-1>\mathring{\imath}$ and by (22), one has

$$\begin{aligned}\Delta\Gamma_{\mathrm{q}}(\xi-1)\varphi_{\mathrm{q}}(z,\varsigma)&=(\ln z)^{\mathring{\imath}-1}\gimel_{\mathrm{q}}(\varsigma)\Gamma_{\mathrm{q}}(\xi-1)\big(\ln T-\mathrm{q}\ln\varsigma\big)^{(\xi-\mathring{\imath}-1)}-\Delta\big(\ln z-\mathrm{q}\ln\varsigma\big)^{(\xi-2)}\\&\geq\Delta\left[(\ln z)^{\mathring{\imath}-1}\big(\ln T-\mathrm{q}\ln\varsigma\big)^{(\xi-\mathring{\imath}-1)}-\Delta\big(\ln z-\mathrm{q}\ln\varsigma\big)^{(\xi-2)}\right]\\&\geq\Delta\left[(\ln z)^{\mathring{\imath}-1}\big(\ln T-\mathrm{q}\ln\varsigma\big)^{(\xi-\mathring{\imath}-1)}-\big(\ln T-\mathrm{q}\ln\varsigma\big)^{(\xi-2)}(\ln z)^{\xi-2}\right]\\&\geq\Delta(\ln z)^{\mathring{\imath}-1}\big(\ln T-\mathrm{q}\ln\varsigma\big)^{(\xi-\mathring{\imath}-1)}\left[1-\big(\ln T-\mathrm{q}\ln\varsigma\big)^{(\mathring{\imath}-1)}\right].\end{aligned}$$

Now, in the second case $z\leq\varsigma$, by (22), we obtain,

$$\begin{aligned}\Delta\Gamma_{\mathrm{q}}(\xi-1)\varphi_{\mathrm{q}}(z,\varsigma)&=(\ln z)^{\mathring{\imath}-1}\Gamma_{\mathrm{q}}(\xi-1)\gimel_{\mathrm{q}}(\varsigma)\big(\ln T-\mathrm{q}\ln\varsigma\big)^{(\xi-\mathring{\imath}-1)}\\&\geq\Delta\left[(\ln z)^{\mathring{\imath}-1}\big(\ln T-\mathrm{q}\ln\varsigma\big)^{(\xi-\mathring{\imath}-1)}-\Delta\big(\ln z-\mathrm{q}\ln\varsigma\big)^{(\xi-2)}\right]\\&\geq\Delta\left[(\ln z)^{\mathring{\imath}-1}\big(\ln T-\mathrm{q}\ln\varsigma\big)^{(\xi-\mathring{\imath}-1)}-\big(\ln T-\mathrm{q}\ln\varsigma\big)^{(\xi-2)}(\ln z)^{\xi-2}\right]\\&\geq\Delta(\ln z)^{\mathring{\imath}-1}\big(\ln T-\mathrm{q}\ln\varsigma\big)^{(\xi-\mathring{\imath}-1)}\left[1-\big(\ln T-\mathrm{q}\ln\varsigma\big)^{(\mathring{\imath}-1)}\right].\end{aligned}$$

So the left inequality of (21) is fulfilled. Further, Lemma (6), $\gimel_{\mathrm{q}}(z)\leq\frac{1}{\Gamma_{\mathrm{q}}(\xi-\mathring{\imath})}$.
ii) The proof is similar to [37, Lemma 3]. □

Denote the Banach space $\mathfrak{Y}=\big\{\Bbbk\in C(\mathrm{J}_1):\Bbbk'\in C(\mathrm{J}_1)\big\}$ endowed the norm $\|\Bbbk\|=\max\big\{\Bbbk(z),\Bbbk'(z)\big\}$, a normal cone $P=\big\{\Bbbk\in\mathfrak{Y}:\Bbbk(z)\geq 0\big\}$ on $C(\mathrm{J}_1)$ with normality constants 1 in $\mathfrak{Y}$, and consider the subset

$$\mathsf{K}=\Big\{\Bbbk(z)\in P:\exists\,{}_{\Bbbk}\mathrm{x}_m,{}_{\Bbbk}\mathrm{x}_M>0,\ {}_{\Bbbk}\mathrm{x}_m<1<{}_{\Bbbk}\mathrm{x}_M\ \text{s.t}\ {}_{\Bbbk}\mathrm{x}_m(\ln z)^{\mathring{\imath}-1}\leq\Bbbk(z)\leq{}_{\Bbbk}\mathrm{x}_M(\ln z)^{\mathring{\imath}-1}\Big\}.$$

Thus, thanks to the Lemma 5 and Eq. (14), Problem (6)-(7) has a positive solution if and only if $\mathbb{r}^*$ is a F.P of operator $\mathscr{O}$ as follows,

$$\mathscr{O}\mathbb{r}(z)=\int_{\mathrm{J}_1}\frac{\varphi_{\mathrm{q}}(z,\varsigma)}{\varsigma}\,\mathsf{L}_{\tilde{p}}\left(\int_{\mathrm{J}_1}\frac{\mathscr{Y}_{\mathrm{q}}(\varsigma,s)}{s}\,\varpi\big(s,I_{\mathrm{q}}\mathbb{r}(s),\mathbb{r}(s)\big)\,\mathrm{d}_{\mathrm{q}}s\right)\mathrm{d}_{\mathrm{q}}\varsigma, \tag{23}$$

in K, that is, p-Laplacian q-$\mathbb{FDE}$ (1)-(2) has a positive solution $\Bbbk(z)=I_{\mathrm{q}}\mathbb{r}(z)$.

Theorem 1. *Suppose conditions* (A1) *and* (A2) *hold.*

A1) ϖ *is nondecreasing in 2th and 3th variables, and there exists* $0<\nu<1$ *such that*

$$\forall(z,\Bbbk,\grave{\Bbbk})\in\mathrm{J}_1\times\mathbb{R}^2_{\geq 0},\quad\varpi\big(z,\eta_0\Bbbk,\eta_0\grave{\Bbbk}\big)\begin{cases}\geq\eta_0^{\nu}\varpi\big(z,\Bbbk,\grave{\Bbbk}\big), & \eta_0<1,\\ \leq\eta_0^{\nu}\varpi\big(z,\Bbbk,\grave{\Bbbk}\big), & \eta_0\geq 1.\end{cases} \tag{24}$$

A2) *According to Lemmas 4 and 5, we have* $\Delta, \Delta_1 > 0$.

Then, Problem (6)-(7) *has a positive solution* $\mathfrak{r}^* \in P$*, whenever,*

$$\int_{\mathrm{J}_1} \frac{\varpi\left(\varsigma,(\ln\varsigma)^{\mathring{i}},(\ln\varsigma)^{\mathring{i}-1}\right)}{\varsigma}\, \mathrm{d}_{\mathrm{q}}\varsigma \in (0,\infty). \tag{25}$$

Proof. First, we prove that operator $\mathscr{O}$ in (23) is well defined such that $\mathscr{O}(\mathsf{K}) \subset \mathsf{K}$. In this regard, for any $\mathfrak{r} \in \mathsf{K}$, there exist ${}_{\mathfrak{r}}\mathrm{x}_m, {}_{\mathfrak{r}}\mathrm{x}_M > 0$, ${}_{\mathfrak{r}}\mathrm{x}_m < 1 < {}_{\mathfrak{r}}\mathrm{x}_M$ such that ${}_{\mathfrak{r}}\mathrm{x}_m(\ln z)^{\mathring{i}-1} \le \mathfrak{r}(z) \le {}_{\mathfrak{r}}\mathrm{x}_M(\ln z)^{\mathring{i}-1}$, and so,

$$\begin{aligned}\int_1^z {}_{\mathfrak{r}}\mathrm{x}_M \frac{(\ln\varsigma)^{\mathring{i}-1}}{\varsigma}\, \mathrm{d}_{\mathrm{q}}\varsigma &= {}_{\mathfrak{r}}\mathrm{x}_M \int_1^z (\ln\varsigma)^{\mathring{i}-1}\, \mathrm{d}_{\mathrm{q}}(\ln\varsigma) = {}_{\mathfrak{r}}\mathrm{x}_M \frac{(\ln z)^{\mathring{i}}}{\mathring{i}} \ge \mathcal{I}_{\mathrm{q}}\mathfrak{r}(z) = \int_1^z \frac{\mathfrak{r}(\varsigma)}{\varsigma}\, \mathrm{d}_{\mathrm{q}}\varsigma \\ &\ge \int_1^z {}_{\mathfrak{r}}\mathrm{x}_m \frac{(\ln\varsigma)^{\mathring{i}-1}}{\varsigma}\, \mathrm{d}_{\mathrm{q}}\varsigma = {}_{\mathfrak{r}}\mathrm{x}_m \int_1^z (\ln\varsigma)^{\mathring{i}-1}\, \mathrm{d}_{\mathrm{q}}(\ln\varsigma) = {}_{\mathfrak{r}}\mathrm{x}_m \frac{(\ln z)^{\mathring{i}}}{\mathring{i}}.\end{aligned} \tag{26}$$

Applying Lemma (6), Eqs. (25)-(26), and (A1), we get

$$\begin{aligned}\int_{\mathrm{J}_1} \frac{\varphi_{\mathrm{q}}(z,\varsigma)}{\varsigma} \mathsf{L}_{\tilde{p}}\left(\int_{\mathrm{J}_1} \frac{\mathscr{Y}_{\mathrm{q}}(\varsigma,s)}{s} \varpi\left(s, \mathcal{I}_{\mathrm{q}}\mathfrak{r}(s), \mathfrak{r}(s)\right) \mathrm{d}_{\mathrm{q}}s\right) \mathrm{d}_{\mathrm{q}}\varsigma &\le \int_{\mathrm{J}_1} \frac{(\ln z)^{\mathring{i}-1}}{\Delta\varsigma\Gamma_{\mathrm{q}}(\xi-\mathring{i})} \mathsf{L}_{\tilde{p}}\left(\int_{\mathrm{J}_1} \frac{\sigma_{\mathrm{q};1}(s)}{s} \varpi\left(s, {}_{\mathfrak{r}}\mathrm{x}_M \frac{(\ln s)^{\mathring{i}}}{\mathring{i}}, {}_{\mathfrak{r}}\mathrm{x}_M(\ln z)^{\mathring{i}-1}\right) \mathrm{d}_{\mathrm{q}}s\right) \mathrm{d}_{\mathrm{q}}\varsigma \\ &\le \frac{1}{\Delta\Gamma_{\mathrm{q}}(\xi-\mathring{i})} \int_{\mathrm{J}_1} \frac{1}{\varsigma} \left[\frac{{}_{\mathfrak{r}}\mathrm{x}_M^{\nu}}{\Delta_1\Gamma_{\mathrm{q}}(\vartheta)}\right]^{\tilde{p}-1} \left[\int_{\mathrm{J}_1} \frac{\varpi\left(s,(\ln s)^{\mathring{i}},(\ln s)^{\mathring{i}-1}\right)}{s} \mathrm{d}_{\mathrm{q}}s\right]^{\tilde{p}-1} \mathrm{d}_{\mathrm{q}}\varsigma < \infty,\end{aligned}$$

and

$$\begin{aligned}&\int_{\mathrm{J}_1} \frac{\varphi_{\mathrm{q}}(z,\varsigma)}{\varsigma} \mathsf{L}_{\tilde{p}}\left(\int_{\mathrm{J}_1} \frac{\mathscr{Y}_{\mathrm{q}}(\varsigma,s)}{s} \varpi\left(s, \mathcal{I}_{\mathrm{q}}\mathfrak{r}(s), \mathfrak{r}(s)\right) \mathrm{d}_{\mathrm{q}}s\right) \mathrm{d}_{\mathrm{q}}\varsigma \\ &\ge \int_{\mathrm{J}_1} \frac{\Delta(\ln z)^{\mathring{i}-1}\sigma_{\mathrm{q}}(\varsigma)}{\varsigma} \mathsf{L}_{\tilde{p}}\left(\int_{\mathrm{J}_1} \frac{(\ln\varsigma)^{\vartheta-1}\gimel_{\mathrm{q};1}(s)}{s} \varpi\left(s, \mathcal{I}_{\mathrm{q}}\mathfrak{r}(s), \mathfrak{r}(s)\right) \mathrm{d}_{\mathrm{q}}s\right) \mathrm{d}_{\mathrm{q}}\varsigma \\ &= \int_{\mathrm{J}_1} \frac{\Delta(\ln z)^{\mathring{i}-1}\sigma_{\mathrm{q}}(\varsigma)(\ln\varsigma)^{(\tilde{p}-1)(\vartheta-1)}}{\varsigma} \mathsf{L}_{\tilde{p}}\left(\int_{\mathrm{J}_1} \frac{\gimel_{\mathrm{q};1}(s)}{s} \varpi\left(s, {}_{\mathfrak{r}}\mathrm{x}_m \frac{(\ln s)^{\mathring{i}}}{\mathring{i}}, {}_{\mathfrak{r}}\mathrm{x}_m(\ln s)^{\mathring{i}-1}\right) \mathrm{d}_{\mathrm{q}}s\right) \mathrm{d}_{\mathrm{q}}\varsigma \\ &= \int_{\mathrm{J}_1} \frac{\Delta(\ln z)^{\mathring{i}-1}\sigma_{\mathrm{q}}(\varsigma)(\ln\varsigma)^{(\tilde{p}-1)(\vartheta-1)}}{\varsigma} \left[\frac{{}_{\mathfrak{r}}\mathrm{x}_m^{\nu}}{\mathring{i}}\right]^{\tilde{p}-1} \left[\int_{\mathrm{J}_1} \frac{\gimel_{\mathrm{q};1}(s)}{s} \varpi\left(s, (\ln s)^{\mathring{i}}, (\ln s)^{\mathring{i}-1}\right) \mathrm{d}_{\mathrm{q}}s\right]^{\tilde{p}-1} \mathrm{d}_{\mathrm{q}}\varsigma \\ &= (\ln z)^{\mathring{i}-1} \int_{\mathrm{J}_1} \frac{\Delta\sigma_{\mathrm{q}}(\varsigma)(\ln\varsigma)^{(\tilde{p}-1)(\vartheta-1)}}{\varsigma} \left[\frac{{}_{\mathfrak{r}}\mathrm{x}_m^{\nu}}{\mathring{i}}\right]^{\tilde{p}-1} \left[\int_{\mathrm{J}_1} \frac{\gimel_{\mathrm{q};1}(s)}{s} \varpi\left(s, (\ln s)^{\mathring{i}}, (\ln s)^{\mathring{i}-1}\right) \mathrm{d}_{\mathrm{q}}s\right]^{\tilde{p}-1} \mathrm{d}_{\mathrm{q}}\varsigma.\end{aligned}$$

These results yield that the operator $\mathscr{O}$ is well defined, and $\mathscr{O}(\mathsf{K}) \subset \mathsf{K}$. Thus, for an arbitrary element $\mathfrak{r}_0 \in \mathsf{K}$, we find that

$$\left.\begin{aligned}&{}_{\mathfrak{r}_0}\mathrm{x}_m(\ln z)^{\mathring{i}-1} \le \mathfrak{r}_0(z) \le {}_{\mathfrak{r}_0}\mathrm{x}_M(\ln z)^{\mathring{i}-1}, \\ &{}_{\mathscr{O}\mathfrak{r}_0}\mathrm{x}_m(\ln z)^{\mathring{i}-1} \le (\mathscr{O}\mathfrak{r}_0)(z) \le {}_{\mathscr{O}\mathfrak{r}_0}\mathrm{x}_M(\ln z)^{\mathring{i}-1}\end{aligned}\right\} \Rightarrow \frac{{}_{\mathscr{O}\mathfrak{r}_0}\mathrm{x}_m}{{}_{\mathfrak{r}_0}\mathrm{x}_M}\mathfrak{r}_0(z) \le (\mathscr{O}\mathfrak{r}_0)(z) \le \frac{{}_{\mathscr{O}\mathfrak{r}_0}\mathrm{x}_M}{{}_{\mathfrak{r}_0}\mathrm{x}_m}\mathfrak{r}_0(z).$$

Now, by choosing $z_0 \in \mathrm{J}_1$ such that,

$$1 < z_0 \le \min\left\{\exp\left(\left[\frac{{}_{\mathscr{O}\mathfrak{r}_0}\mathrm{x}_m}{{}_{\mathfrak{r}_0}\mathrm{x}_M}\right]^{1/(1-\nu(\tilde{p}-1))}\right), \exp\left(\left[\frac{{}_{\mathscr{O}\mathfrak{r}_0}\mathrm{x}_M}{{}_{\mathfrak{r}_0}\mathrm{x}_m}\right]^{1/(1-\nu(\tilde{p}-1))}\right)\right\}, \tag{27}$$

in which $\nu < \frac{1}{\tilde{p}-1}$, we get

$$(\ln z_0)^{1-\nu(\tilde{p}-1)}\mathfrak{r}_0(z) \le (\mathscr{O}\mathfrak{r}_0)(z) \le \left[\frac{1}{\ln z_0}\right]^{1-\nu(\tilde{p}-1)} \mathfrak{r}_0(z), \qquad z \in \mathrm{J}_1. \tag{28}$$

In this level, we consider the following two sequences based on the iterative pattern for $n \in \mathbb{N}$,

$$\begin{cases} \tilde{s}_0 = \mathfrak{r}_0(z_0)\ln z_0, \\ \tilde{s}_n = \mathscr{O}\tilde{s}_{n-1}, \end{cases} \qquad \begin{cases} \tilde{t}_0 = \frac{\mathfrak{r}_0(z_0)}{\ln z_0}, \\ \tilde{t}_n = \mathscr{O}\tilde{t}_{n-1}. \end{cases} \tag{29}$$

Clearly, $\tilde{s}_0 \leq \tilde{t}_0$. Since ϖ is nondecreasing and $\mathscr{O}$ is also an increasing operator in $\mathbb{r}$, for two cases $0 < \rho < 1$ and $\rho \geq 1$, we obtain

$$\begin{aligned}
\mathscr{O}(\rho\mathbb{r})(z) &= \int_{\mathrm{J}_1} \frac{\varphi_{\mathrm{q}}(z,\varsigma)}{\varsigma} \mathsf{L}_{\tilde{p}}\left(\int_{\mathrm{J}_1} \frac{\mathscr{Y}_{\mathrm{q}}(\varsigma,s)}{s} \varpi\big(s, I_{\mathrm{q}}(\rho\mathbb{r})(s), (\rho\mathbb{r}(s))\, \mathrm{d_q}s\right) \mathrm{d_q}\varsigma \\
&\geq \rho^{\nu(\tilde{p}-1)} \int_{\mathrm{J}_1} \frac{\varphi_{\mathrm{q}}(z,\varsigma)}{\varsigma} \mathsf{L}_{\tilde{p}}\left(\int_{\mathrm{J}_1} \frac{\mathscr{Y}_{\mathrm{q}}(\varsigma,s)}{s} \varpi\big(s, I_{\mathrm{q}}\mathbb{r}(s), \mathbb{r}(s)\big)\, \mathrm{d_q}s\right) \mathrm{d_q}\varsigma = \left[\rho^{\nu(\tilde{p}-1)} T\right] \mathbb{r}(z), \qquad 0 < \rho < 1, \\
\mathscr{O}(\rho\mathbb{r})(z) &= \int_{\mathrm{J}_1} \frac{\varphi_{\mathrm{q}}(z,\varsigma)}{\varsigma} \mathsf{L}_{\tilde{p}}\left(\int_{\mathrm{J}_1} \frac{\mathscr{Y}_{\mathrm{q}}(\varsigma,s)}{s} \varpi\big(s, I_{\mathrm{q}}(\rho\mathbb{r})(s), (\rho\mathbb{r}(s))\, \mathrm{d_q}s\right) \mathrm{d_q}\varsigma \\
&\leq \rho^{\nu(\tilde{p}-1)} \int_{\mathrm{J}_1} \frac{\varphi_{\mathrm{q}}(z,\varsigma)}{\varsigma} \mathsf{L}_{\tilde{p}}\left(\int_{\mathrm{J}_1} \frac{\mathscr{Y}_{\mathrm{q}}(\varsigma,s)}{s} \varpi\big(s, I_{\mathrm{q}}\mathbb{r}(s), \mathbb{r}(s)\big)\, \mathrm{d_q}s\right) \mathrm{d_q}\varsigma = \left[\rho^{\nu(\tilde{p}-1)} T\right] \mathbb{r}(z), \qquad \rho \geq 1.
\end{aligned}$$

These imply, for $z \in \mathrm{J}_1$,

$$\tilde{s}_0 = (\ln z_0)\mathbb{r}_0(z) \leq (\ln z_0)^{\nu(\tilde{p}-1)} (\mathscr{O}\mathbb{r}_0)(z) \leq \mathscr{O}\tilde{s}_0 = \tilde{s}_1 \leq \left[\tfrac{1}{\ln z_0}\right]^{\nu(\tilde{p}-1)} (\mathscr{O}\mathbb{r}_0)(z) \leq \tfrac{1}{\ln z_0}\mathbb{r}_0(z) = \tilde{t}_0.$$

Thus, by induction, we get

$$\tilde{s}_0 \leq \tilde{s}_1 \leq \cdots \leq \tilde{s}_n \leq \cdots \leq \tilde{t}_n \leq \tilde{t}_{n-1} \leq \cdots \leq \tilde{t}_1 \leq \tilde{t}_0, \tag{30}$$

and by substitution $\tilde{s}_0 = \tilde{t}_0 (\ln z_0)^2$, we find that, $\tilde{s}_1 = \mathscr{O}\tilde{s}_0 = \mathscr{O}\big(\tilde{t}_0(\ln z_0)^2\big) \geq (\ln z_0)^{2\nu(\tilde{p}-1)}\, \mathscr{O}(\tilde{t}_0) = \tilde{t}_1 (\ln z_0)^{2\nu(\tilde{p}-1)}$. Finally, by induction, we get $\tilde{s}_n \geq \tilde{t}_n (\ln z_0)^{2[\nu(\tilde{p}-1)]^n}$, $n \in \{0\} \cup \mathbb{N}$. The properties of the cone P and the inequality $\tilde{s}_{n+m} - \tilde{s}_n \leq \tilde{t}_n - \tilde{s}_n$, for any $m \in \mathbb{N}$, we have

$$\|\tilde{s}_m - \tilde{t}_n\| \leq \|\tilde{t}_n - \tilde{t}_n\| \leq \left[1 - (\ln z_0)^{2[\nu(\tilde{p}-1)]^n}\right] \|\tilde{t}_0\| \to 0, \tag{31}$$

as $n \to \infty$, which implies that $\{\tilde{s}_n\}$ is a Cauchy sequence, and $\tilde{s}_n$ converges to some $\mathbb{r}^* \in P$. By (31) and $\|\tilde{s}_n - \mathbb{r}^*\| \leq \|\tilde{s}_n - \tilde{t}_n\| + \|\tilde{s}_n - \mathbb{r}^*\|$, we get $\tilde{t}_n \to \mathbb{r}^*(z)$ for $z \in \mathrm{J}_1$. Indeed, $\mathbb{r}^* \in P$ is a $\mathsf{F.P}$ of $\mathscr{O}$, such that $\tilde{s} \leq \mathbb{r}^*(z) \leq \tilde{t}$. Consequently, p-Laplacian q-$\mathbb{FDE}$ (1)-(2) has a positive solution by taking $\mathbb{k}(z) = I_{\mathrm{q}}\frac{\mathbb{r}^*(z)}{z}$. Then, the proof is completed. □

Theorem 2. *For any initial value $\mathbb{r}_0 \in \mathsf{K}$, the sequence of functions defined by*

$$\mathbb{r}_n(z) = \int_{\mathrm{J}_1} \frac{\varphi_{\mathrm{q}}(z,\varsigma)}{\varsigma} \mathsf{L}_{\tilde{p}}\left(\int_{\mathrm{J}_1} \frac{\mathscr{Y}_{\mathrm{q}}(\varsigma,s)}{s} \varpi\big(s, I_{\mathrm{q}}\mathbb{r}_{n-1}(s), \mathbb{r}_{n-1}(s)\big)\, \mathrm{d_q}s\right) \mathrm{d_q}\varsigma, \qquad n \in \mathbb{N},$$

converges uniformly to $\mathbb{r}^*(z)$ on J_1 as $n \to +\infty$, the corresponding sequence $\mathbb{k}_n(z)$ converges uniformly to $\mathbb{k}^*(z) = I_{\mathrm{q}}\frac{\mathbb{r}^*(z)}{z}$.

Proof. For any initial value $\mathbb{r}_0 \in P$, it follow from $\tilde{s}_0 \leq \mathbb{r}_0(z) \leq \tilde{t}_0$, for $z \in \mathrm{J}_1$, in proof of Theorem 1, we have $\tilde{s}_n \leq \mathbb{r}_n(z) \leq \tilde{t}_n$, $n \in \mathbb{N}$. Hence,

$$\|\mathbb{r}_n - \mathbb{r}^*\| \leq \|\mathbb{r}_n - \tilde{s}_n\| + \|\tilde{s}_n - \mathbb{r}^*\| \leq 2\|\tilde{t}_n - \tilde{s}_n\| \leq 2\left[1 - (\ln z_0)^{2[\nu(\tilde{p}-1)]^n}\right] \|\tilde{t}_0\|,$$

which implies that the formulated sequence of functions as follows,

$$\mathbb{r}_n(z) = \int_{\mathrm{J}_1} \frac{\varphi_{\mathrm{q}}(z,\varsigma)}{\varsigma} \mathsf{L}_{\tilde{p}}\left(\int_{\mathrm{J}_1} \frac{\mathscr{Y}_{\mathrm{q}}(\varsigma,s)}{s} \varpi\big(s, I_{\mathrm{q}}\mathbb{r}_{n-1}(s), \mathbb{r}_{n-1}(s)\big)\, \mathrm{d_q}s\right) \mathrm{d_q}\varsigma, \qquad n \in \mathbb{N},$$

converges uniformly to the positive solution $\mathbb{r}^*(z)$ of p-Laplacian q-$\mathbb{FDE}$ (6)-(7) on J_1 as $n \to \infty$. By considering the expression $\mathbb{k}_n(z) = I_{\mathrm{q}}\frac{\mathbb{r}_n(z)}{z}$, to establish the iteration sequence (29), we see that $\mathbb{k}_n \to +\infty$ as $n \to +\infty$. Thus, the proof is completed. □

Theorem 3. *There exists an error estimation*

$$\|\mathbb{r}_n - \mathbb{r}^*\| \leq \tfrac{2}{\sqrt{\varepsilon}}\left[1 - \varepsilon^{(\tilde{p}-1)^n}\right] \|\mathbb{r}_0\| \Rightarrow \|\mathbb{k}_n - \mathbb{k}^*\| \leq \tfrac{2}{\sqrt{\varepsilon}}\left[1 - \varepsilon^{(\tilde{p}-1)^n}\right] \|\mathbb{k}_0\|,$$

with $\mathbb{k}_0(z) = I_{\mathrm{q}}\frac{\mathbb{r}_0(z)}{z}$, which has the rate of convergence $\|\mathbb{r}_n - \mathbb{r}^*\| = o\big(1 - \varepsilon^{(\tilde{p}-1)^n}\big)$ as $n \to +\infty$, that is $\|\mathbb{k}_n - \mathbb{k}^*\| = o\big(1 - \varepsilon^{(\tilde{p}-1)^n}\big)$ as $n \to +\infty$, where $0 < \varepsilon < 1$ which is determined by the initial value $\mathbb{r}_0$.

Proof. We have the error estimation,

$$\|\mathbb{r}_n - \mathbb{r}^*\| \leq \tfrac{2}{\sqrt{\varepsilon}}\left[1-(\ln z_0)^{2[\nu(\tilde{p}-1)]^n}\right]\|\mathbb{r}_0\|,$$

which has the rate of convergence

$$\|\mathbb{r}_n - \mathbb{r}^*\| \leq o\left(1-(\ln z_0)^{2[\nu(\tilde{p}-1)]^n}\right)\|\mathbb{r}_0\|, \qquad 0<\varepsilon=(\ln z_0)^2<1.$$

Further, there exists an error estimation $\|\mathbb{k}_n - \mathbb{k}^*\| \leq \frac{2}{\sqrt{\varepsilon}}\left(1-(\ln z_0)^{2[\nu(\tilde{p}-1)]^n}\right)\|\mathbb{k}_0\|$, when $\mathbb{k}_n(z) = I_{\mathsf{q}}\frac{\mathbb{r}_n(z)}{z}$ and $\mathbb{k}_0(z) = I_{\mathsf{q}}\frac{\mathbb{r}_0(z)}{z}$, are considered for $\mathbb{r}_n \in P$, $n \in \mathbb{N}$, which has the rate of convergence

$$\|\mathbb{k}_n - \mathbb{k}^*\| = o\left(1-(\ln z_0)^{2[\nu(\tilde{p}-1)]^n}\right),$$

and is determined by the initial value $\mathbb{r}_0$. This shows that the claim holds. □

Theorem 4. *The positive solution $\mathbb{r}^* \in P$ of Problem (6)-(7) is unique.*

Proof. Suppose $\mathbb{r}_1^*, \mathbb{r}_2^* \in P$ are F.P of $\mathscr{O}$. Let $z_1 = \sup\{z>0 : \mathbb{r}_2^*(z) \geq z\mathbb{r}_1^*(z)\}$. We have $1<z_1<+\infty$. If $z_1<T$, by choosing $1<z<T$, we get

$$\mathbb{r}_2^*(z) = (\mathscr{O}\mathbb{r}_2^*)(z) \geq (\ln z_1)^{\nu(\tilde{p}-1)}\left(\mathscr{O}(z\mathbb{r}_1^*)\right)(z) = (\ln z_1)^{\nu(\tilde{p}-1)}(\mathscr{O}\mathbb{r}_1^*)(z) = (\ln t_1)^{\nu(\tilde{p}-1)}\mathbb{r}_1^*(z).$$

Since $1<\tilde{p}<2$, we get $(\ln z_1)^{\tilde{p}-1} > \ln z_1$, which is a contradiction and so, we assume that $z_1 \geq T$ and $\mathbb{r}_2^*(z) \geq \mathbb{r}_1^*(z)$. By the same way, we get $\mathbb{r}_2^*(z) \leq \mathbb{r}_1^*(z)$ for $z \in \mathrm{J}_1$. Hence, $\mathbb{r}_1^*(z) = \mathbb{r}_2^* = (z)$ for $z \in \mathrm{J}_1$. Indeed, p-Laplacian q-$\mathbb{FDE}$ (6)-(7) has a unique positive solution. Therefore, the proof is finished. □

4 Applications

In Theorem 1, we choose the simple function $(\ln z)^{\mathring{\imath}-1}$ in the iterative sequences $\tilde{s}$ and $\tilde{t}$ which is suitable for numerical purpose. Thus, one can be chosen arbitrarily functions. In this regards, we aim to examine the proposed problem numerically.

Example 1. Consider the infinite-point 3-Laplacian q-$\mathbb{FDE}$ of the form,

$$ {}^{\mathrm{H}}\mathscr{D}_{\mathsf{q}}^{\vartheta}\left(\mathsf{L}_3\left({}^{\mathrm{CH}}\mathscr{D}_{\mathsf{q}}^{\xi}\,\mathbb{k}\right)\right)(z) + \tfrac{(\mathbb{k}(z))^{1/3}+(\mathbb{k}'(z))^{1/4}}{(\ln z)^{1/3}} = 0, \tag{32}$$

for $z \in \mathrm{J}_1 \setminus \{1,T\}$, $T=\exp(1)$, $0 \xrightarrow{\mathsf{q}} 1$, $p=3$, $\tilde{p}=\frac{3}{2}$, and two cases as follow,

$$\begin{aligned}
&\texttt{Case 1:} \quad \vartheta \in \left\{\tfrac{3}{2}, \tfrac{5}{3}, \tfrac{9}{5}\right\} \subset (1,2],\ \xi = \tfrac{7}{2} \geq \aleph = 3,\\
&\texttt{Case 2:} \quad \vartheta = \tfrac{5}{3} \in (1,2],\ \xi \in \left\{\tfrac{7}{2}, \tfrac{18}{5}, \tfrac{31}{8}\right\} \geq \aleph = 3,
\end{aligned}$$

under $\mathring{\imath} = 2$,

$$\begin{cases}
\mathbb{k}(1) = \mathbb{k}'(1) = \mathbb{k}''(1) = 0,\\
\mathbb{k}''(1) = \sum_{j=1}^{\infty} \frac{1}{2j^2}\,\mathbb{k}\left(\left(\exp(1)\right)^{1/j^2}\right), \quad {}^{\mathrm{CH}}\mathscr{D}_{\mathsf{q}}^{\xi}\,\mathbb{k}(1) = 0,\\
\mathsf{L}_3\left({}^{\mathrm{CH}}\mathscr{D}_{\mathsf{q}}^{\xi}\,\mathbb{k}(T)\right) = \sum_{j=1}^{\infty} \frac{1}{3j^3}\,\mathsf{L}_3\left({}^{\mathrm{CH}}\mathscr{D}_{\mathsf{q}}^{\xi}\,\mathbb{k}\left(\left(\exp(1)\right)^{1/j^2}\right)\right),
\end{cases} \tag{33}$$

$\lambda_{1j} = \frac{1}{2j^2}$, $\lambda_{2j} = \frac{1}{3j^3}$, $r_j = \left(\exp(1)\right)^{1/j^2}$, $j \in \mathbb{N}$. we define $\varpi(z, \mathbb{s}, \grave{\mathbb{s}}) = \frac{\mathbb{s}^{1/3} + \grave{\mathbb{s}}^{1/4}}{(\ln z)^{1/3}}$. Clearly, $\varpi \in C\left((\mathrm{J}_1 \setminus \{1,T\}) \times \mathbb{R}^2_{\geq 0}, \mathbb{R}_{\geq 0}\right)$ and for any fixed $1<z<T$, $\varpi(z, \mathbb{s}, \grave{\mathbb{s}})$ is nondecreasing in $\mathbb{s}$ and $\grave{\mathbb{s}}$. Let $\nu = \frac{1}{4} \in (0,1)$. For $\eta_0 \in \mathrm{J}_1$ and any $(z, \mathbb{s}, \grave{\mathbb{s}}) \in \left(\mathrm{J}_1 \setminus \{1,T\}\right) \times \mathbb{R}^2_{\geq 0}$, we get

$$\varpi(z, \eta_0\mathbb{s}, \eta_0\grave{\mathbb{s}}) = \tfrac{(\eta_0\mathbb{s})^{1/3} + (\eta_0\grave{\mathbb{s}})^{1/4}}{(\ln z)^{1/3}} \geq \eta_0^{1/4}\,\tfrac{\mathbb{s}^{1/3} + \grave{\mathbb{s}}^{1/4}}{(\ln z)^{1/3}} = \eta_0^{1/4}\varpi(z, \mathbb{s}, \grave{\mathbb{s}}). \tag{34}$$

Thus, (A1) is satisfied. In the sequel, for different predicted cases, we review the proposed problem analytically.

`Case 1:` From 3-Laplacian q-$\mathbb{FDE}$, additionally,

$$\Delta = (\mathring{\imath}-1)! - \sum_{j=1}^{\infty} \lambda_{1j} \left(\ln r_j\right)^{\mathring{\imath}-1} = 1 - \sum_{j=1}^{\infty} \tfrac{1}{2j^2} \ln\left(\left(\exp(1)\right)^{1/j^2}\right) \approx 0.4588 > 0,$$

$$\Delta_1 = 1 - \sum_{j=1}^{\infty} \lambda_{2j} \left(\ln r_j\right)^{\vartheta-1} = 1 - \sum_{j=1}^{\infty} \tfrac{1}{3j^3} \left(\ln\left(\left(\exp(1)\right)^{1/j^2}\right)\right)^{\vartheta-1} \approx \left\{ \begin{array}{ll} 0.8271, & \vartheta = 3/2, \\ 0.7695, & \vartheta = 5/3, \\ 0.7234, & \vartheta = 9/5, \end{array} \right\} > 0.$$

Therefore, condition (A2) is also fulfilled. To check Ineq. (25), we have,

$$\int_{\mathrm{J}_1} \frac{\varpi\left(\varsigma,(\ln\varsigma)^{\mathring{\imath}},(\ln\varsigma)^{\mathring{\imath}-1}\right)}{\varsigma}\, \mathrm{d}_{\mathrm{q}}\varsigma = \int_{\mathrm{J}_1} \frac{\varpi\left(\varsigma,(\ln\varsigma)^{2},(\ln\varsigma)\right)}{\varsigma}\, \mathrm{d}_{\mathrm{q}}\varsigma = \int_{\mathrm{J}_1} \frac{(\ln\varsigma)^{2/3}+(\ln\varsigma)^{1/4}}{\varsigma(\ln\varsigma)^{1/3}}\, \mathrm{d}_{\mathrm{q}}\varsigma \approx 4.0714 \to 1.6798 \in (0,\infty), \qquad \forall\vartheta. \tag{35}$$

These estimated results are shown in Table 1. Fig. 1 shows the 2D-plot of Ineq. (35) for the range of $\mathrm{q} \in (0,1)$ and for different values of derivative order ϑ which are, of course, independent of its changes. Then the 3-Laplacian q-$\mathbb{FDE}$ (32)-(33) has a unique positive solution $\mathbb{k}^*(z) = I_{\mathrm{q}} r^*(z)$, $r^* \in P$ in `Case 1`. Moreover, for any initial value $r_0 \in P$, the following sequence of functions,

Table 1. Estimated variables for q-$\mathbb{FDE}$ (32)-(33) with three values of derivative order ϑ in `Case 1`.

$0 \xrightarrow{\mathrm{q}} 1$	Δ	Δ_1			Ineq.
		$\vartheta = 3/2$	$\vartheta = 5/3$	$\vartheta = 9/5$	(35)
0.02	0.4588	0.8272	0.7696	0.7235	4.0715
0.11	0.4588	0.8272	0.7696	0.7235	3.8881
0.20	0.4588	0.8272	0.7696	0.7235	3.6700
0.29	0.4588	0.8272	0.7696	0.7235	3.4302
0.38	0.4588	0.8272	0.7696	0.7235	3.3378
0.47	0.4588	0.8272	0.7696	0.7235	3.1245
0.57	0.4588	0.8272	0.7696	0.7235	2.8667
0.66	0.4588	0.8272	0.7696	0.7235	2.6874
0.75	0.4588	0.8272	0.7696	0.7235	2.4633
0.84	0.4588	0.8272	0.7696	0.7235	2.2356
0.93	0.4588	0.8272	0.7696	0.7235	1.6799

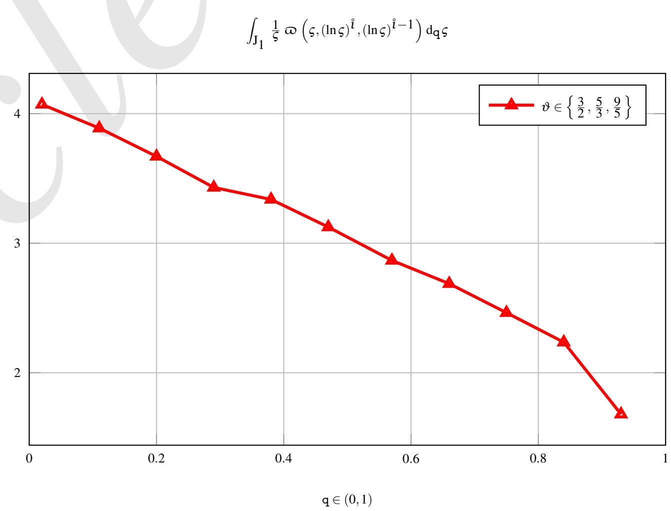

Figure 1. 2D-plots of Ineq. (35) for $\mathrm{q} \in (0,1)$ with three values of derivative order ϑ in `Case 1`.

$$r_n(z) = \int_{\mathrm{J}_1} \frac{\varphi_{\mathrm{q}}(z,\varsigma)}{\varsigma} \mathsf{L}_{3/2}\left(\int_{\mathrm{J}_1} \frac{\mathscr{Y}_{\mathrm{q}}(\varsigma,s)}{s} \left[\frac{\left(I_{\mathrm{q}} r_{n-1}(s)\right)^{1/3} + \left(r_{n-1}(s)\right)^{1/4}}{(\ln z)^{1/3}} \right] \mathrm{d}_{\mathrm{q}}s \right) \mathrm{d}_{\mathrm{q}}\varsigma, \qquad n \in \mathbb{N}, \tag{36}$$

Algorithm 3: MATLAB lines for to estimate variables in Example 1.

Input: ϑ, ξ, λ_{1j}, λ_{2j}, r_j, $\mathring{\imath}$, $\aleph$, n, $\mathtt{q}$

Output: paramsmatrix

syms v e u;
$t_0 = 1$;
$T = \exp(1)$;
column= 1;
$[x_\vartheta y_\vartheta] = \text{size}(\vartheta)$;
for $s = 1 : y_{vartheta}$ **do**
  n=1;
  $var\mathtt{q} = 0.02$;
  while $var\mathtt{q} < 1$ **do**
    paramsmatrix(n, column) $= n$;
    paramsmatrix(n, column+1) $= var\mathtt{q}$;
    $\Delta = \text{factor}(\mathring{\imath} - 1) - \text{eval(symsum(subs}(\lambda_{1j},\text{u,e}) \times \log((\text{subs}(r_j, \text{u,e}))^{(\mathring{\imath}-1)}), 1, 101))$;
    paramsmatrix(n, column+2) $= \Delta$;
    $\Delta_1 = 1 - \text{eval(symsum(subs}(\lambda_{2j}, \text{u,e}) \times \log((\text{subs}(r_j, \text{u,e}))^{(}\vartheta(s) - 1)), 1, 101))$;
    paramsmatrix(n, column+3) $= \Delta_1$;
    II= $\text{abs}(\text{Iqfixed}(var\mathtt{q}, T, k, ((\log(v))^{2/3} + (\log(v))^{1/4})/((v+0.001) \times (\log(v+0.001))^{1/3})) -$ $\text{Iqfixed}(var\mathtt{q}, t_0, k, ((\log(v))^{2/3} + (\log(v))^{1/4})/((v+0.001) \times (\log(v+0.001))^{1/3})))$;
    paramsmatrix(n, column+4) =II;
    $var\mathtt{q} = var\mathtt{q} + 1/11$;
    $n = n + 1$;
  column = column +5;

converges uniformly to $\mathbb{r}^*(z)$ on J_1 as $n \to \infty$. By taking $\mathbb{k}(z) = \int_1^z \frac{\mathbb{r}(\varsigma)}{\varsigma}\, \mathrm{d}_{\mathtt{q}}\varsigma$, $\mathbb{r}(z) \in P$, we have that $\mathtt{q}$-$\mathbb{FDE}$ (32)-(33) has a unique positive solution, and all conditions of Theorem 1 hold in `Case 1`. Further, we can consider ${}_{\mathbb{r}}\mathrm{x}_m, {}_{\mathbb{r}}\mathrm{x}_M > 0$, ${}_{\mathbb{r}}\mathrm{x}_m < 1 < {}_{\mathbb{r}}\mathrm{x}_M$ and ${}_{\mathscr{O}\mathbb{r}}\mathrm{x}_m, {}_{\mathscr{O}\mathbb{r}}\mathrm{x}_M > 0$, ${}_{\mathscr{O}\mathbb{r}}\mathrm{x}_m < 1 < {}_{\mathscr{O}\mathbb{r}}\mathrm{x}_M$, such that ${}_{\mathbb{r}}\mathrm{x}_m(\ln z)^{\mathring{\imath}-1} = {}_{\mathbb{r}}\mathrm{x}_m(\ln z) \leq \mathbb{r}(z) \leq {}_{\mathbb{r}}\mathrm{x}_M(\ln z) = {}_{\mathbb{r}}\mathrm{x}_M(\ln z)^{\mathring{\imath}-1}$, and

$$\frac{{}_{\mathscr{O}\mathbb{r}_0}\mathrm{x}_m}{{}_{\mathbb{r}_0}\mathrm{x}_M}\mathbb{r}_0(z) \leq (\mathscr{O}\mathbb{r}_0)(z) \leq \frac{{}_{\mathscr{O}\mathbb{r}_0}\mathrm{x}_M}{{}_{\mathbb{r}_0}\mathrm{x}_m}\mathbb{r}_0(z).$$

Moreover, with the given $\nu = \frac{1}{4}$, there exist an error estimation

$$\|\mathbb{k}_n - \mathbb{k}^*\| = \max_{z \in \mathrm{J}_1}\left|I_{\mathtt{q}}\big(\mathbb{r}_n(z) - \mathbb{r}^*(z)\big)\right| \leq \frac{2}{\sqrt{\varepsilon}}\left[1 - \varepsilon^{[\nu(\tilde{p}-1)]^n}\right]\|\mathbb{r}_0\| = \frac{2}{\sqrt{\varepsilon}}\left[1 - \varepsilon^{[1/4(1/2)]^n}\right]\|\mathbb{r}_0\| = \frac{2}{\sqrt{\varepsilon}}\left[1 - \varepsilon^{(1/8)^n}\right]\|\mathbb{r}_0\|.$$

in Theorem 3, which has the rate of convergence, $\|\mathbb{k}_n - \mathbb{k}^*\| = o\left(1 - \varepsilon^{(1/8)^n}\right)$, for $0 < \varepsilon < 1$, which is determined by the initial value $\mathbb{r}_0$. If $\mathbb{r}_0(z) = \ln z$ by Eq. (28), and computation, we obtain

$$(\ln z_0)^{1-\nu(\tilde{p}-1)}\mathbb{r}_0(z) = (\ln z_0)^{1-1/4(1/2)}\mathbb{r}_0(z) = (\ln z_0)^{1/8}\mathbb{r}_0(z) \leq (\mathscr{O}\mathbb{r}_0)(z) \leq \left[\frac{1}{\ln z_0}\right]^{1-\nu(\tilde{p}-1)}\mathbb{r}_0(z) = \left[\frac{1}{\ln z_0}\right]^{7/8}\mathbb{r}_0(z),$$

for $z \in \mathrm{J}_1$, we have $\frac{31}{100}\ln z \leq \mathscr{O}\mathbb{r}_0(z) \leq \frac{50}{59}\ln z$, and Theorem 2 is obtained in `Case 1`. Take

$$\begin{aligned} 1 < z_0 &\leq \min\left\{\exp\left(\left[\frac{{}_{\mathscr{O}\mathbb{r}_0}\mathrm{x}_m}{{}_{\mathbb{r}_0}\mathrm{x}_M}\right]^{1/(1-\nu(\tilde{p}-1))}\right), \exp\left(\left[\frac{{}_{\mathscr{O}\mathbb{r}_0}\mathrm{x}_M}{{}_{\mathbb{r}_0}\mathrm{x}_m}\right]^{1/(1-\nu(\tilde{p}-1))}\right)\right\} \\ &= \min\left\{\exp\left(\left[\frac{{}_{\mathscr{O}\mathbb{r}_0}\mathrm{x}_m}{{}_{\mathbb{r}_0}\mathrm{x}_M}\right]^{8/7}\right), \exp\left(\left[\frac{{}_{\mathscr{O}\mathbb{r}_0}\mathrm{x}_M}{{}_{\mathbb{r}_0}\mathrm{x}_m}\right]^{8/7}\right)\right\} = \min\{0.3584, 9.1604\}, \end{aligned}$$

then we get the error estimation with $\varepsilon = \ln z_0 = \ln(1.1442) \simeq$ as follows,

$$\begin{aligned}\|\Bbbk_n - \Bbbk^*\| &\leq \frac{2}{\sqrt{\varepsilon}}\left[1-(\ln z_0)^{2[\nu(\tilde{p}-1)]^n}\right]\|\mathbb{r}_0\| = \frac{2}{\sqrt{\varepsilon}}\left[1-(\ln 1.1442)^{2[1/4(3/2-1)]^n}\right]\|\mathbb{r}_0\| \\ &= \frac{2}{\sqrt{0.1347}}\left[1-(0.1347)^{2(1/8)^n}\right]\|\mathbb{r}_0\| \approx 5.4493\left[1-(0.0181)^{(1/8)^n}\right]\|\mathbb{r}_0\|\end{aligned}$$

and the estimate of convergence rate $\|\Bbbk_n - \Bbbk^*\| = o\left(1-0.0181^{(1/8)^n}\right)$. Then Theorem 3 is obtained in `Case 1`.

`Case 2:` By given data, we have,

$$\Delta = (\hat{\imath}-1)! - \sum_{j=1}^{\infty}\lambda_{1j}(\ln r_j)^{\hat{\imath}-1} = 1 - \sum_{j=1}^{\infty}\frac{1}{2j^2}\ln\left((\exp(1))^{1/j^2}\right) \approx 0.4588 > 0,$$

$$\Delta_1 = 1 - \sum_{j=1}^{\infty}\lambda_{2j}(\ln r_j)^{\vartheta-1} = 1 - \sum_{j=1}^{\infty}\frac{1}{3j^3}\left(\ln\left((\exp(1))^{1/j^2}\right)\right)^{\vartheta-1} \approx 0.7695 > 0.$$

Therefore, condition (A2) is also fulfilled. To check Ineq. (25), we have,

$$\int_{\mathrm{J}_1}\frac{\varpi\left(\varsigma,(\ln\varsigma)^{\hat{\imath}},(\ln\varsigma)^{\hat{\imath}-1}\right)}{\varsigma}\,\mathrm{d}_{\mathrm{q}}\varsigma = \int_{\mathrm{J}_1}\frac{\varpi\left(\varsigma,(\ln\varsigma)^2,(\ln\varsigma)\right)}{\varsigma}\,\mathrm{d}_{\mathrm{q}}\varsigma = \int_{\mathrm{J}_1}\frac{(\ln\varsigma)^{2/3}+(\ln\varsigma)^{1/4}}{\varsigma(\ln\varsigma)^{1/3}}\,\mathrm{d}_{\mathrm{q}}\varsigma \approx 4.0714 \to 1.6798 \in (0,\infty), \qquad \forall\vartheta. \tag{37}$$

Then the 3-Laplacian q-$\mathbb{FDE}$ (32)-(33) has a unique positive solution $\Bbbk^*(z) = I_{\mathrm{q}}\mathbb{r}^*(z)$, $\mathbb{r}^* \in P$ in `Case 2`. Moreover, for any initial value $\mathbb{r}_0 \in P$, the following sequence of functions,

$$\mathbb{r}_n(z) = \int_{\mathrm{J}_1}\frac{\varphi_{\mathrm{q}}(z,\varsigma)}{\varsigma}\,\mathrm{L}_{3/2}\left(\int_{\mathrm{J}_1}\frac{\mathscr{Y}_{\mathrm{q}}(\varsigma,s)}{s}\left[\frac{\left(I_{\mathrm{q}}\mathbb{r}_{n-1}(s)\right)^{1/3}+\left(\mathbb{r}_{n-1}(s)\right)^{1/4}}{(\ln z)^{1/3}}\right]\mathrm{d}_{\mathrm{q}}s\right)\mathrm{d}_{\mathrm{q}}\varsigma, \qquad n \in \mathbb{N}, \tag{38}$$

converges uniformly to $\mathbb{r}^*(z)$ on J_1 as $n \to \infty$. By taking $\Bbbk(z) = \int_1^z \frac{\mathbb{r}(\varsigma)}{\varsigma}\,\mathrm{d}_{\mathrm{q}}\varsigma$, $\mathbb{r}(z) \in P$, we have that q-$\mathbb{FDE}$ (32)-(33) has a unique positive solution, and all conditions of Theorem 1 hold in `Case 2`. Moreover, with the given $\nu = \frac{1}{4}$, there exist an error estimation

$$\|\Bbbk_n - \Bbbk^*\| = \max_{z\in\mathrm{J}_1}\left|I_{\mathrm{q}}\left(\mathbb{r}_n(z)-\mathbb{r}^*(z)\right)\right| \leq \frac{2}{\sqrt{\varepsilon}}\left[1-\varepsilon^{[\nu(\tilde{p}-1)]^n}\right]\|\mathbb{r}_0\| = \frac{2}{\sqrt{\varepsilon}}\left[1-\varepsilon^{[1/4(1/2)]^n}\right]\|\mathbb{r}_0\| = \frac{2}{\sqrt{\varepsilon}}\left[1-\varepsilon^{(1/8)^n}\right]\|\mathbb{r}_0\|.$$

in Theorem 3, which has the rate of convergence, $\|\Bbbk_n - \Bbbk^*\| = o\left(1-\varepsilon^{(1/8)^n}\right)$ for $0 < \varepsilon < 1$, which is determined by the initial value $\mathbb{r}_0$. If $\mathbb{r}_0(z) = \ln z$ by Eq. (28), and computation, we obtain

$$(\ln z_0)^{1-\nu(\tilde{p}-1)}\mathbb{r}_0(z) = (\ln z_0)^{1-2/3(1/2)}\mathbb{r}_0(z) = (\ln z_0)^{2/3}\mathbb{r}_0(z) \approx\leq (\mathscr{O}\mathbb{r}_0)(z) \leq \left[\frac{1}{\ln z_0}\right]^{1-\nu(\tilde{p}-1)}\mathbb{r}_0(z) = \left[\frac{1}{\ln z_0}\right]^{7/8}\mathbb{r}_0(z),$$

for $z \in \mathrm{J}_1$, we have $\frac{31}{100}\ln z \leq \mathscr{O}\mathbb{r}_0(z) \leq \frac{50}{59}\ln z$, and Theorem 2 is obtained in `Case 2`. Take

$$\begin{aligned}1 < z_0 &\leq \min\left\{\exp\left(\left[\frac{\mathscr{O}\mathbb{r}_0 \mathrm{x}_m}{\mathbb{r}_0\mathrm{x}_M}\right]^{1/(1-\nu(\tilde{p}-1))}\right), \exp\left(\left[\frac{\mathscr{O}\mathbb{r}_0 \mathrm{x}_M}{\mathbb{r}_0\mathrm{x}_m}\right]^{1/(1-\nu(\tilde{p}-1))}\right)\right\} \\ &= \min\left\{\exp\left(\left[\frac{\mathscr{O}\mathbb{r}_0 \mathrm{x}_m}{\mathbb{r}_0\mathrm{x}_M}\right]^{8/7}\right), \exp\left(\left[\frac{\mathscr{O}\mathbb{r}_0 \mathrm{x}_M}{\mathbb{r}_0\mathrm{x}_m}\right]^{8/7}\right)\right\} = \min\min\left\{0.3584, 9.1604\right\},\end{aligned}$$

then we get the error estimation with $\varepsilon = \ln z_0 = \ln(1.1442) \simeq$ as follows,

$$\begin{aligned}\|\Bbbk_n - \Bbbk^*\| &\leq \frac{2}{\sqrt{\varepsilon}}\left[1-(\ln z_0)^{2[\nu(\tilde{p}-1)]^n}\right]\|\mathbb{r}_0\| = \frac{2}{\sqrt{\varepsilon}}\left[1-(\ln 1.1442)^{2[1/4(3/2-1)]^n}\right]\|\mathbb{r}_0\| \\ &= \frac{2}{\sqrt{0.1347}}\left[1-(0.1347)^{2(1/8)^n}\right]\|\mathbb{r}_0\| \approx 5.4493\left[1-(0.0181)^{(1/8)^n}\right]\|\mathbb{r}_0\|\end{aligned}$$

and the estimate of convergence rate $\|\Bbbk_n - \Bbbk^*\| = o\left(1-0.0181^{(1/8)^n}\right)$. Then Theorem 3 is obtained in `Case 2`.

5 Disscussion and Conclusion

By deriving the expression of Green functions and some special techniques, we established appropriate cone. Since the nonlinearity contains derivative terms, we overcome the difficulty caused by derivative terms by proper substitution. Then the existence of unique iterative positive, error estimation, and convergence rate of approximate solution are obtained for singular p-Laplacian CH q-$\mathbb{FDE}$s with infinite-point boundary conditions. We choose some simple function such 0 or $(\ln z)^{\hat{\imath}-1}$, which is suitable for numerical aim. The understanding of the properties of the solution and its future application may provide convenient and theoretical guidance.

Abbreviations

CH: Caputo-Hadamard; $\mathbb{DE}$: Differential equation; F.P: Fixed point; $\mathbb{FDE}$: Fractional differential equation; q-$\mathbb{FDE}$: Quantum fractional differential equation

Authors' Contributions

MES: Actualization, methodology, formal analysis, supervisor, validation, investigation, software, simulation, initial draft, review and was a major contributor in writing the manuscript. **SN**: Actualization, formal analysis, investigation, initial draft. All authors read and approved the final manuscript.

Data Availability

The manuscript has no associated data or the data will not be deposited.

Conflicts of Interest

The authors declare that there is no conflict of interest.

Ethical Considerations

The authors have diligently addressed ethical concerns, such as informed consent, plagiarism, data fabrication, misconduct, falsification, double publication, redundancy, submission, and other related matters.

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References

[1] L. Guo, C. Li, and J. Zhao, Existence of monotone positive solutions for Caputo-Hadamard nonlinear fractional differential equation with infinite-point boundary value conditions, Symmetry, 15(5), 970, (2023).

[2] K. Miller, B. Ross, and E. Upfal, An introduction to the Fractional Calculus and Fractional Differential Equations. New York: Wiley, J and Sons, 1993.

[3] M. Jleli and B. Samet, Nonexistence results for a time-fractional biharmonic diffusion equation, Boundary Value Problems, 66(4), (2024).

[4] M. Jleli and B. Samet, On a one-dimensional time-fractionally damped wave equation with a singular potential, Physica Scripta, 99(7), (2024).

[5] X. Wang, G. Wang, Z. Chen, C. Lim, S. Li, and C. Li, Controllable flexural wave in laminated metabeam with embedded multiple resonators, Journal of Sound and Vibration, 581, (2024).

[6] L. Guo, H. Liu, C. Li, J. Zhao, and J. Xu, Existence of positive solutions for singular p-Laplacian Hadamard fractional differential equations with the derivative term contained in the nonlinear term, Nonlinear Analysis Modelling and Control, 28, 491–515, (2023).

[7] L. Guo, J. Cai, Z. Xie, and C. Li, Mechanical responses of symmetric straight and curved composite microbeams, Journal of Vibration Engineering and Technologies, 12, (2024).

[8] C. Li, C. X. Zhu, N. Zhang, S. H. Sui, and J. Zhao, Free vibration of self-powered nanoribbons subjected to thermal-mechanical-electrical fields based on a nonlocal strain gradient theory, Applied Mathematical Modelling, 110, (2022).

[9] J. P. Shen, C. Li, X. L. Fan, and C. M. Jung, Dynamics of silicon nanobeams with axial motion subjected to transverse and longitudinal loads considering nonlocal and surface effects, Smart Structures and Systems, 110, 105–113, (2017).

[10] Z. Y. Xie, L. M. Guo, C. Li, Y. T. Shi, and B. Han, Modeling the deformation of thin-walled circular tubes filled with metallic foam under two lateral loading patterns, Structures, 69, (2024).

[11] C. Li and L. M. Guo, Positive solutions and their existence of a nonlinear Hadamard fractional- order differential equation with a singular source item using spectral analysis, Fractal and Fractional, 8(7), 2827–2842, (2024).

[12] C. Li and L. M. Guo, Positive solution pairs for coupled p-Laplacian Hadamard fractional differential model with singular source item on time variable, Fractal and Fractional, 8(12), 2827–2842, (2024).

[13] K. Muangchoo and S. Phiangsungnoen, Hybrid CG-like algorithm for nonlinear equations and image restoration, Carpathian Journal Mathematics, 41(1), (2025).

[14] H. A. Hammad and M. De la Sen, Existence of a mild solution and approximate controllability for fractional random integro-differential inclusions with non-instantaneous impulses, Alexandria Engineering Journal, 111, 306–328, (2025).

[15] E. F. S. Feitosa, J. Vanterler da C. Sousa, S. I. Moreira, and G. S. A. Costa, Existence and multiplicity for fractional differential equations with $m(\xi)$-Kirchhoff type-equation, Computational and Applied Mathematics, 44(1), (2025).

[16] X. Fu, J. Ni, and G. Chen, Existence of solutions for fractional p-Laplacian differential equation with multi-point boundary conditions at resonance, IAENG International Journal of Applied Mathematics, 54, 657–662, (2024).

[17] A. Boutiara, M. Benbachir, and K. Guerbati, Boundary value problem for nonlinear Caputo-Hadamard fractional differential equation with Hadamard fractional integral and anti-periodic conditions, Facta Universitatis Series Mathematics and Inform, 36(4), 735–748, (2021).

[18] A. Ardjouni and A. Djoudi, Positive solutions for nonlinear Caputo-Hadamard fractional differential equations with integral boundary conditions, Open Journal of Mathematical Analysis, 3(1), 62–69, (2019).

[19] C. Derbazi and H. Hammouche, Caputo-Hadamard fractional differential equations with nonlocal fractional integro-differential boundary conditions via topological degree theory, AIMS Mathematics, 3(1), 2694–2709, (2020).

[20] A. B. Makhlouf and L. Mchiri, Some results on the study of Caputo-Hadamard fractional stochastic differential equations, Chaos, Solitons and Fractals, 155(111757), 2694–2709, (2022).

[21] E. Yousefi, A. Akbari, and M. E. Samei, Existence and uniqueness results for mixed pantograph $\mathbb{FDE}$s equipped the p-Laplacian operator with impulsive boundary conditions, Beni-Suef University Journal of Basic and Applied Sciences, 15, 42, (2026).

[22] S. Nazari and M. E. Samei, An existence of the solution for generalized system of fractional q-differential inclusions involving p-Laplacian operator and sequential derivatives, Boundary Value Problems, 2024, 117, (2024).

[23] H. Beddani, M. Beddani, and M. E. Samei, Discussion on a new tripled system of hybrid type of fdes with p-Laplacian involving $\varphi-$Caputo derivatives, Annals of the University of Craiova, Mathematics and Computer Science Series, 52(2), 416–435, (2025).

[24] B. Belhadji, J. Alzabut, M. E. Samei, and N. Fatima, On the global behaviour of solutions for a delayed viscoelastic type petrovesky wave equation with p-Laplace and logarithmic source, Mathematics, 10, 4194, (2022).

[25] F. H. Jackson, q-difference equations, American Journal of Mathematics, 32, 305–314, (1910).

[26] C. R. Adams, The general theory of a class of linear partial $q-$difference equations, Transactions of the American Mathematical Society, 26, 283–312, (1924).

[27] M. H. Annaby and Z. S. Mansour, q–Fractional Calculus and Equations. Cambridge: Springer Heidelberg, 2012.

[28] S. N. Hajiseyedazizi, J. Samei, M. E. Alzabut, and Y. Chu, On multi-step methods for singular fractional q-integro-differential equations, Open Mathematics, 19, 1378–1405, (2021).

[29] M. E. Samei, H. Zanganeh, and S. M. Aydogan, Investigation of a class of the singular fractional integro-differential quantum equations with multi-step methods, Journal of Mathematical Extension, 15, 1–45, (2021).

[30] S. G. Georgiev, Fractional Dynamic Calculus and Fractional Dynamic Equations on Time Scales. Switzerland: Springer, 2012.

[31] P. M. Rajković, S. D. Marinković, and M. S. Stanković, On q-analogues of Caputo derivative and Mittag-Leffer function, Fractional Calculus and Applied Analysis, 10(4), 359–373, (2007).

[32] M. S. Stanković, P. M. Rajković, and S. D. Marinković, On the fractional q-derivative of Caputo type, Proceedings of the Bulgarian Academy of Sciences, 63(2), 197–204, (2010).

[33] R. A. C. Ferreira, Positive solutions for a class of boundary value problems with fractional $q-$differences, Computers & Mathematics with Applications, 61, 367–373, (2011).

[34] P. M. Rajković, S. D. Marinković, and M. S. Stanković, Fractional integrals and derivatives in q-calculus, Applicable Analysis and Discrete Mathematics, 1(1), 311–323, (2007).

[35] M. Alotaibi, M. Jleli, M. A. Ragusa, and B. Samet, On the absence of global weak solutions for a nonlinear time-fractional Schrödinger equation, Applicable Analysis, 103(1), 1–15, (2022).

[36] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, Theory and Applications of Fractional Differential Equations. Amsterdam: North-Holland Mathematics Studies, 2006.

[37] X. Zhang, Positive solutions for a class of singular fractional differential equation with innite point boundary value conditions, Applied Mathematics Letters, 39, 22–27, (2015).