



The Solution for Systems of High-Order Linear Volterra-Fredholm Integro-Differential Equations

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Abstract

In this research, we consider the system of linear Volterra-Fredholm integro-differential equations (SVFIDEs). The main aim of this research is to approximate the integral by Gauss-Kronrod-Legendre quadrature rules and then using quintic B-spline as the bases function. The unknown coefficients in combination determine by collocation method. The arising system of algebraic linear can be solved via iterative method. Error analysis is investigated theoretically. Numerical text problems are considered to justify the applicability and efficient nature of our approach, comparison of the results justify the considerable accuracy and efficiency proposed methods. The extended parameter in valued in the spline can be chosen in such a way to improve the accuracy also.

Keywords: System of linear Volterra-Fredholm integro-differential equations, Quintic B-spline method, Gauss-Kronrod-Legendre quadrature rules, Error analysis

Mathematics Subject Classification (2020): 45D05, 45G15, 65D05, 65R20

1 Introduction

It is well known that the system of linear SVFIDEs are of the form

$$g_k(\xi) + \int_{\alpha}^{\xi} \sum_{j=0}^n K_{k,j}(\xi, y) U_j(y) dy + \int_{\alpha}^{\beta} \sum_{j=0}^n L_{k,j}(\xi, y) U_j(y) dy = \sum_{j=0}^n \sum_{i=0}^m P_{i,k,j}(\xi) U_j^{(i)}(\xi), \quad (1)$$
$$\alpha \leq y, \xi \leq \beta, \quad 1 \leq m \leq 4, \quad k = 1, 2, \dots, n,$$

where $K_{k,j}$, $L_{k,j}$, g_k and $P_{i,k,j}$ the known and continuous functions and $P_{i,k,k} \neq 0$ for all $\xi \in [\alpha, \beta]$ and satisfies a uniform Lipschitz condition, and $U_j^{(i)}$ is unknown function. The linear SVFIDEs arise in varies filed of science and dynamics such as spread of epidemics, and semi-conductor devices [1-3]. For solution of the SVFIDEs (1), several numerical approaches have been proposed such as, the Homotopy-perturbation method [4-6], the wavelet basis [7], the quadrature Rules [8-10], the converting to optimization problem [11], an approach based on Lipschitz-continuity [12], the collocation iterated method [13, 14], the Taylor-series expansion methods [15], the Newton-Kantorovich-quadrature method [16], the Fibonacci polynomials [17], the improved cuckoo optimization algorithm [18], the natural Runge-Kutta methods [19], the Bernstein polynomial [20, 21]. We develop a collocation by using quintic B-spline to approximate in SVFIDEs (1).



2 Quintic B-Spline Collocation Method

We introduce the quintic B-spline space and basis functions to construct an interpolant S to be used in formulation of quintic B-spline collocation method.

Let $\Delta_M : \{\alpha = y_0 < y_1 < \dots < y_M = \beta\}$ be a uniform partition of the interval $[\alpha, \beta]$ with step size $h = \frac{\beta - \alpha}{M}$. The quintic B-spline space denote by

$$S_5(\Delta_M) = \{S \in C^4[\alpha, \beta]; s|_{[y_r, y_{r+1}]} \in P_5, r = 0, 1, \dots, M - 1\},$$

where P_5 is the class of quintic polynomials.

The construction of the quintic B-spline interpolate S to the analytical solution U for (1), can be performed with the help of the ten additional knots such that

$$y_{-5} < y_{-4} < y_{-3} < y_{-2} < y_{-1} \quad \text{and} \quad y_{M+1} < y_{M+2} < y_{M+3} < y_{M+4} < y_{M+5}.$$

Following [22, 23] we consider a quintic B-spline $S(y)$ of the form

$$S_j(y) = \sum_{r=-2}^{M+2} t_{r,j} B_r^5(y), \quad j = 0, 1, \dots, n, \tag{2}$$

where

$$B_r^5(y) = \frac{1}{h^5 5!} \begin{cases} (y - y_{r-3})^5, & \text{if } y \in [y_{r-3}, y_{r-2}) \\ (y - y_{r-3})^5 - 6(y - y_{r-2})^5, & \text{if } y \in [y_{r-2}, y_{r-1}) \\ (y - y_{r-3})^5 - 6(y - y_{r-2})^5 + 15(y - y_{r-1})^5, & \text{if } y \in [y_{r-1}, y_r) \\ (y - y_{r-3})^5 - 6(y - y_{r-2})^5 + 15(y - y_{r-1})^5 - 20(y - y_r)^5, & \text{if } y \in [y_r, y_{r+1}) \\ (y - y_{r-3})^5 - 6(y - y_{r-2})^5 + 15(y - y_{r-1})^5 - 20(y - y_r)^5 + 15(y - y_{r+1})^5, & \text{if } y \in [y_{r+1}, y_{r+2}) \\ (y - y_{r-3})^5 - 6(y - y_{r-2})^5 + 15(y - y_{r-1})^5 - 20(y - y_r)^5 + 15(y - y_{r+1})^5 - 6(y - y_{r+2})^5, & \text{if } y \in [y_{r+2}, y_{r+3}) \\ 0, & y < y_{r-3}, y_{r+3} < y, \end{cases} \tag{3}$$

satisfying the following interpolatory conditions

$$S_j(y_r) = U_j(y_r), \quad 0 \leq r \leq M, \quad 0 \leq j \leq n$$

and the end conditions

$$\begin{aligned} D^i S_j(y_0) &= D^i U_j(y_0), \quad D^i S_j(y_M) = D^i U_j(y_M), \quad i = 1, 2, \\ \text{or} \\ D^i S_j(y_0) &= D^i S_j(y_M), \quad i = 1, 2, 3, 4, \\ \text{or} \\ D^i S_j(y_0) &= 0, \quad D^i S_j(y_M) = 0, \quad i = 3, 4. \end{aligned} \tag{4}$$

3 On Quadrature Rule of the Gauss-Kronrod-Legendre

The Gauss-Kronrod-Legendre quadrature rules is

$$\int_{\mathbb{R}} f(x) d\mu(x) = \sum_{k=1}^{2\gamma+1} \bar{\delta}_k f(\bar{\tau}_k) = \sum_{v=1}^{\gamma} \delta_v f(\tau_v) + \sum_{\rho=1}^{\gamma+1} \delta_{\rho}^* f(\tau_{\rho}^*) + R_{2\gamma+1}(f),$$

where τ_v are the Gaussian nodes, i.e., the zeros of the orthogonal polynomial P_{γ} with respect to μ and the nodes τ_{ρ}^* and weights $\delta_v, \delta_{\rho}^*$ are chosen so that they maximize the polynomial degree of exactness of the quadrature formula. It turns out that

$$R_{\gamma}(f) = 0 \quad \text{for all } f \in P_{3\gamma+1},$$

(see [24–26]). Here we use the Legendre polynomial of degree 2 and $w(x) = 1$, i.e. $d\mu(x) = dx, [-1, 1]$ and we can obtain the weights δ_ν and δ_ρ^* by solving the system,

$$\int_{-1}^1 x^j dx = \sum_{\nu=1}^2 \delta_\nu x^j(\tau_\nu) + \sum_{\rho=1}^3 \delta_\rho^* x^j(\tau_\rho^*) \quad j = 0, \dots, 7.$$

Finally we obtain

$$\begin{cases} \tau_1 = \frac{\sqrt{3}}{3}, \tau_2 = \frac{-\sqrt{3}}{3}, \tau_1^* = -\sqrt{\frac{6}{7}}, \tau_2^* = 0, \tau_3^* = \sqrt{\frac{6}{7}}, \\ \delta_1 = \frac{27}{55}, \delta_2 = \frac{27}{55}, \delta_1^* = \frac{98}{495}, \delta_2^* = \frac{28}{45}, \delta_3^* = \frac{98}{495}. \end{cases} \tag{5}$$

4 System of Linear Volterra-Fredholm Integro-Differential Equation

In the given SVFIDEs (1), we can approximate the unknown function by quintic B-spline Eq. (2), we have:

$$\begin{aligned} g_k(\xi) + \int_\alpha^\xi \sum_{j=0}^n K_{k,j}(\xi, y) S_j(y) dy + \int_\alpha^\beta \sum_{j=0}^n L_{k,j}(\xi, y) S_j(y) dy &= \sum_{j=0}^n \sum_{i=0}^m P_{i,k,j}(\xi) S_j^{(i)}(\xi), \\ \alpha \leq y, \xi \leq \beta, \quad 1 \leq m \leq 4, \quad k &= 1, 2, \dots, n. \end{aligned} \tag{6}$$

Now collocated Eq. (6), for a fixed ξ in $\alpha \leq \xi \leq \beta$ at the points $\xi_r = \alpha + rh, h = \frac{\beta-\alpha}{M}, r = 0, 1, \dots, M$, we obtain

$$\begin{aligned} g_k(\xi_r) + \int_\alpha^{\xi_r} \sum_{j=0}^n K_{k,j}(\xi_r, y) S_j(y) dy + \int_\alpha^\beta \sum_{j=0}^n L_{k,j}(\xi_r, y) S_j(y) dy &= \sum_{j=0}^n \sum_{i=0}^m P_{i,k,j}(\xi_r) S_j^{(i)}(\xi_r), \\ k &= 1, 2, \dots, n, \quad r = 0, 1, \dots, M. \end{aligned} \tag{7}$$

By partitioning the intervals $[\alpha, \beta]$ and $[\alpha, \xi]$ to M equal subintervals we obtain

$$g_k(\xi_r) + \sum_{\rho=0}^{r-1} \int_{\xi_\rho}^{\xi_{\rho+1}} \sum_{j=0}^n K_{k,j}(\xi_r, y) S_j(y) dy + \sum_{\rho=0}^{M-1} \int_{\xi_\rho}^{\xi_{\rho+1}} \sum_{j=0}^n L_{k,j}(\xi_r, y) S_j(y) dy = \sum_{j=0}^n \sum_{i=0}^m P_{i,k,j}(\xi_r) S_j^{(i)}(\xi_r). \tag{8}$$

For using the Gauss-Kronrod-Legendre rules we need to change each subinterval $[\xi_\rho, \xi_{\rho+1}]$ to the interval $[-1, 1]$. Then by the following change of variable, we have

$$y = \frac{(\xi_{\rho+1} - \xi_\rho)u + (\xi_{\rho+1} + \xi_\rho)}{2}, \quad dy = \frac{\xi_{\rho+1} - \xi_\rho}{2} du = \frac{h}{2} du.$$

To approximate the integral Eq. (8), we can use the Gauss-Kronrod-Legendre quadrature rules in the case $\gamma = 2$, we get the following $(Mn + n) \times (Mn + 5n)$, linear system, we need $4n$ more equations to obtained the unique solutions for equations Eq. (8), we impose the end conditions Eq. (4). Hence by associating equations Eq. (8) with Eq. (4) we have the following linear system $(Mn + 5n) \times (Mn + 5n)$,

$$\begin{cases} \frac{h}{2} (\sum_{j=0}^n \sum_{\rho=0}^{r-1} \sum_{\tau=1}^5 \bar{\delta}_\tau K_{k,j}(\xi_r, \zeta_{\rho\tau}) S_j(\zeta_{\rho\tau}) + \sum_{j=0}^n \sum_{\rho=0}^{M-1} \sum_{\tau=1}^5 \bar{\delta}_\tau L_{k,j}(\xi_r, \zeta_{\rho\tau}) S_j(\zeta_{\rho\tau})) \\ = g_k(\xi_r) + \sum_{j=0}^n \sum_{i=0}^m P_{i,k,j}(\xi_r) S_j^{(i)}(\xi_r), \quad k = 1, 2, \dots, n, \quad r = 0, 1, \dots, M, \\ D^i S_j(y_0) = D^i S_j(y_M), \quad 1 \leq i \leq 4, \quad 0 \leq j \leq n, \end{cases} \tag{9}$$

where

$$\zeta_{\rho\tau} = \frac{(\xi_{\rho+1} - \xi_\rho)v_\tau + (\xi_{\rho+1} + \xi_\rho)}{2},$$

we have the nodes v_τ and weights $\bar{\delta}_\tau$ of previous section. By solving the above linear system via iterative method we determine the coefficients $t_{r,j}, r = -2, \dots, M + 2, j = 0, 1, \dots, n$ by setting $t_{r,j}$ in Eq. (2), we obtain the approximate solution for SVFIDEs (1).

5 Error Analysis

To obtain the error estimation of our approach, the first of all we recall the following definition and theorem in [22–28].

Definition 1. The Gauss-Kronrod quadrature formula with multiple nodes,

$$\int_{-1}^1 f(t)w(t)dt = \sum_{v=1}^{2\gamma+1} \bar{\delta}_v f(\bar{\tau}_v) + R_{2\gamma+1}(f), \tag{10}$$

is exact for all polynomials of degree at most $3\gamma + 1$, that is, $R_\gamma(f) = 0, \forall f \in P_{3\gamma+1}$.

Theorem 1. Let $U_j(\xi) \in C^4[\alpha, \beta]$, Δ be the partition of $[\alpha, \beta]$ and $S_j(\xi)$ be the spline interpolation function $U_j(\xi)$, we have

$$\|D^r(S_j - U_j)\|_\infty \leq \phi_r h^{4-r}, \quad r = 0, \dots, 4, \quad 0 \leq j \leq n. \tag{11}$$

For the proof [27].

Next we will prove the following theorem for convergence of our method in Eq. (9).

Theorem 2. The approximate method Eq. (9),

$$\begin{aligned} \frac{h}{2} \left(\sum_{j=0}^n \sum_{\rho=0}^{r-1} \sum_{\tau=1}^5 \bar{\delta}_\tau K_{k,j}(\xi_r, \zeta_{\rho\tau}) S_j(\zeta_{\rho\tau}) + \sum_{j=0}^n \sum_{\rho=0}^{M-1} \sum_{\tau=1}^5 \bar{\delta}_\tau L_{k,j}(\xi_r, \zeta_{\rho\tau}) S_j(\zeta_{\rho\tau}) \right) &= g_k(\xi_r) + \sum_{j=0}^n \sum_{i=0}^m P_{i,k,j}(\xi_r) S_j^{(i)}(\xi_r), \\ k = 1, 2, \dots, n, \quad r = 0, 1, \dots, M, \end{aligned} \tag{12}$$

for solution of the linear SVFIDE (1), is converge and the error bounded is

$$\mathbf{E}^{(m)} \leq \mathbf{P}^{-1} [\mathbf{M}_1 + \frac{h}{2} (\mathbf{M}_2 + \mathbf{M}_3)],$$

where \mathbf{P} is the band matrix with $|P_{m,k,k}(\xi)| \neq 0$ for all $\xi \in [\alpha, \beta]$, \mathbf{M}_1 is the matrix with $|S_{jr}^{(i)} - U_{jr}^{(i)}|$ and $\mathbf{M}_2, \mathbf{M}_3$ are the matrices with

$$|S_{j\rho\tau} - U_{j\rho\tau}|, \quad r = 0, 1, \dots, M, \quad j = 0, \dots, n, \quad \rho = 0, \dots, r-1, \quad i = 0, \dots, m-1.$$

Proof. : We suppose that for a fixted ξ_f in $\alpha < \xi_f \leq \beta$ at the points $\xi_r = \alpha + rh, h = \frac{\xi_f - \alpha}{M}, r = 0, 1, \dots, M$, the corresponding approximation method for system of linear SVFIDE (1), is Eq. (12).

We solve the problem by discretizing SVFIDE (1) and approximating the integral using the Gauss-Kronrod-Legendre quadrature rules, we can obtain

$$\begin{aligned} \frac{h}{2} \left(\sum_{j=0}^n \sum_{\rho=0}^{r-1} \sum_{\tau=1}^5 \bar{\delta}_\tau K_{k,j}(\xi_r, \zeta_{\rho\tau}) U_j(\zeta_{\rho\tau}) + \sum_{j=0}^n \sum_{\rho=0}^{M-1} \sum_{\tau=1}^5 \bar{\delta}_\tau L_{k,j}(\xi_r, \zeta_{\rho\tau}) U_j(\zeta_{\rho\tau}) \right) &= g_k(\xi_r) + \sum_{j=0}^n \sum_{i=0}^m P_{i,k,j}(\xi_r) U_j^{(i)}(\xi_r), \\ k = 1, 2, \dots, n, \quad r = 0, 1, \dots, M. \end{aligned} \tag{13}$$

By subtracting Eq. (13) from Eq. (12) and using interpolatory condition of quintic spline, we get

$$\begin{aligned} \frac{h}{2} \left(\sum_{j=0}^n \sum_{\rho=0}^{r-1} \sum_{\tau=1}^5 \bar{\delta}_\tau [K_{k,j}(\xi_r, \zeta_{\rho\tau}) S_j(\zeta_{\rho\tau}) - K_{k,j}(\xi_r, \zeta_{\rho\tau}) U_j(\zeta_{\rho\tau})] + \sum_{j=0}^n \sum_{\rho=0}^{M-1} \sum_{\tau=1}^5 \bar{\delta}_\tau [L_{k,j}(\xi_r, \zeta_{\rho\tau}) S_j(\zeta_{\rho\tau}) \right. \\ \left. - L_{k,j}(\xi_r, \zeta_{\rho\tau}) U_j(\zeta_{\rho\tau})] \right) = \sum_{j=0}^n \sum_{i=0}^m P_{i,k,j}(\xi_r) [S_j^{(i)}(\xi_r) - U_j^{(i)}(\xi_r)], \\ k = 1, 2, \dots, n, \quad r = 0, 1, \dots, M, \end{aligned} \tag{14}$$

we suppose that, $U_j(\zeta_{\rho\tau}) = U_{j\rho\tau}, S_j(\zeta_{\rho\tau}) = S_{j\rho\tau}, S_j^{(i)}(\xi_r) = S_{jr}^{(i)}, U_j^{(i)}(\xi_r) = U_{jr}^{(i)}, K_{k,j}(\xi_r, \zeta_{\rho\tau}) = K_{kjr\rho\tau}$ and $L_{k,j}(\xi_r, \zeta_{\rho\tau}) = L_{kjr\rho\tau}$. We get

$$\sum_{j=0}^n |P_{mkjr}| |S_{jr}^{(m)} - U_{jr}^{(m)}| \leq \sum_{j=0}^n \sum_{i=0}^{m-1} |P_{ikjr}| |S_{jr}^{(i)} - U_{jr}^{(i)}| + \frac{h}{2} \left(\sum_{j=0}^n \sum_{\rho=0}^{r-1} \sum_{\tau=1}^5 |K_{kjr\rho\tau}| |\bar{\delta}_v| |S_{j\rho\tau} - U_{j\rho\tau}| + \sum_{j=0}^n \sum_{\rho=0}^{M-1} \sum_{\tau=1}^5 |L_{kjr\rho\tau}| |\bar{\delta}_v| |S_{j\rho\tau} - U_{j\rho\tau}| \right). \tag{15}$$

Then the system of Eq. (15) can be expressed in a matrix form

$$\mathbf{PE}^{(m)} \leq \mathbf{M}_1 + \frac{h}{2}(\mathbf{M}_2 + \mathbf{M}_3), \tag{16}$$

where for $k = 1, 2, \dots, n, r = 0, 1, \dots, M,$

$$\begin{aligned} \mathbf{E}_{((M+1)n \times 1)}^{(m)} &= \sum_{j=0}^n |S_{jr}^{(m)} - U_{jr}^{(m)}|, & \mathbf{P}_{((M+1)n \times (M+1)n)} &= \sum_{j=0}^n |P_{mkjr}|, \\ \mathbf{M}_{1((M+1)n \times (M+1)n)} &= \sum_{j=0}^n \sum_{i=0}^{m-1} |P_{ikjr}| |S_{jr}^{(i)} - U_{jr}^{(i)}|, & \mathbf{M}_{2((M+1)n \times (M+1)n)} &= \sum_{j=0}^n \sum_{\rho=0}^{r-1} \sum_{\tau=1}^5 |K_{kjr\rho\tau}| |\bar{\delta}_v| |S_{j\rho\tau} - U_{j\rho\tau}|, \\ \mathbf{M}_{3((M+1)n \times (M+1)n)} &= \sum_{j=0}^n \sum_{\rho=0}^{M-1} \sum_{\tau=1}^5 |L_{kjr\rho\tau}| |\bar{\delta}_v| |S_{j\rho\tau} - U_{j\rho\tau}|. \end{aligned}$$

Since that $P_{m,k,k}(\xi) \neq 0$ for all $\xi \in [\alpha, \beta]$ and $\mathbf{P}_{(n \times (M+1)n)}$ denote the $((M + 1)n \times (M + 1)n)$ band matrix, of bandwidth n therefore the inverse \mathbf{P} exists [28].

We get

$$\mathbf{E}^{(m)} \leq \mathbf{P}^{-1}[\mathbf{M}_1 + \frac{h}{2}(\mathbf{M}_2 + \mathbf{M}_3)].$$

When $h \rightarrow 0$ then the above terms are zero and also these terms are due to interpolating of $U_j^{(i)}(\xi)$ by quintic spline (in Theorem 1),

$$E^{(m)} \rightarrow 0 \text{ as } h \rightarrow 0, m = 0, \dots, 4.$$

□

6 Numerical Examples

We consider text problems of the linear SVFIDEs (1). Our numerical results are compared with methods in [29]. All numerical computations for the two examples are carried out using Mathematica, the running time is also reported in seconds (CPU time).

Example 1. Consider the following linear SVFIDE with exact solution $U_1(\xi) = \xi^3, U_2(\xi) = \xi^2 + 1,$ [29]

$$\begin{cases} U_1^{(2)}(\xi) = 6\xi - \frac{3}{4}\xi^2 - \frac{1}{5}\xi^6 + \int_0^\xi \xi y U_1(y) dy + \int_0^1 \xi^2 y U_2(y) dy \\ U_2^{(2)}(\xi) = 2 - \frac{1}{5}\xi^5 - \frac{3}{2}\xi + \int_0^\xi \xi U_1(y) dy + \int_0^1 2\xi y U_2(y) dy. \end{cases} \tag{17}$$

We apply the presented method Eq. (9), the maximum absolute errors (MAEs) in the solutions with $M = 10$ are tabulated in Table 1.

Example 2. Consider the following linear SVFIDE with exact solution $U_1(\xi) = \cos(\xi), U_2(\xi) = \sin(\xi),$ [29]

$$\begin{cases} U_1^{(2)}(\xi) = \xi - 2\cos(\xi) - \int_0^\xi (\xi - y) U_1(y) dy - \int_0^{\frac{\pi}{2}} (\xi - y) U_2(y) dy \\ U_2^{(2)}(\xi) = \sin(\xi) - \xi - 1 + \int_0^\xi (\xi - y)^2 U_1(y) dy - \int_0^{\frac{\pi}{2}} (\xi - y) U_2(y) dy. \end{cases} \tag{18}$$

We apply the presented method Eq. (9), the maximum absolute errors(MAEs) in the solutions with $M = 10$ are tabulated in Table 2.

7 Conclusions

This paper presents method to find the solution system of linear Volterra-Fredholm integro-differential equations by using the Gauss-Kronrod-Legendre quadrature rule and collocating by quintic B-spline. These equations are converted to a system of linear algebraic equations in terms of the linear combination coefficients appearing in the representation of the solution in spline basic functions. Numerical text problems are considered to justify the applicability and efficient nature of our approach.

Table 1. The MAEs at particular points for $M = 10$, CPU time: 0.735.

| ξ | $U_1(\xi)$ | $U_2(\xi)$ |
|-------|---------------------------|---------------------------|
| 0.0 | 1.20857×10^{-16} | 1.79856×10^{-14} |
| 0.1 | 9.51930×10^{-17} | 2.88658×10^{-15} |
| 0.2 | 2.42861×10^{-16} | 5.55112×10^{-15} |
| 0.3 | 3.81639×10^{-16} | 6.43929×10^{-15} |
| 0.4 | 4.85723×10^{-16} | 3.33067×10^{-15} |
| 0.5 | 6.10623×10^{-16} | 4.21885×10^{-15} |
| 0.6 | 7.77156×10^{-16} | 4.44089×10^{-15} |
| 0.7 | 8.88178×10^{-16} | 5.10703×10^{-15} |
| 0.8 | 8.88178×10^{-16} | 5.77316×10^{-15} |
| 0.9 | 8.88178×10^{-16} | 5.77316×10^{-15} |
| 1.0 | 1.11022×10^{-16} | 6.21725×10^{-15} |

Table 2. The MAEs at particular points for $M = 10$, CPU time: 0.75.

| ξ | $U_1(\xi)$ | $U_2(\xi)$ |
|-------------------|---------------------------|---------------------------|
| 0 | 1.11022×10^{-16} | 1.45555×10^{-16} |
| $\frac{\pi}{20}$ | 2.61359×10^{-8} | 1.65285×10^{-8} |
| $\frac{2\pi}{20}$ | 4.92215×10^{-8} | 3.74049×10^{-8} |
| $\frac{3\pi}{20}$ | 4.50643×10^{-8} | 4.71814×10^{-8} |
| $\frac{4\pi}{20}$ | 2.18167×10^{-8} | 4.63787×10^{-8} |
| $\frac{5\pi}{20}$ | 2.04795×10^{-8} | 3.15365×10^{-8} |
| $\frac{6\pi}{20}$ | 7.90841×10^{-8} | 6.56791×10^{-8} |
| $\frac{7\pi}{20}$ | 1.51506×10^{-7} | 4.81734×10^{-8} |
| $\frac{8\pi}{20}$ | 2.34686×10^{-7} | 1.16353×10^{-7} |
| $\frac{9\pi}{20}$ | 3.25470×10^{-7} | 2.04853×10^{-7} |
| $\frac{\pi}{2}$ | 4.19695×10^{-7} | 3.14187×10^{-7} |

Data Availability

All data in the paper are available from the corresponding authors upon reasonable request.

Conflicts of Interest

The author declares that there is no conflict of interest.

Ethical Considerations

The author has diligently addressed ethical concerns, such as informed consent, plagiarism, data fabrication, misconduct, falsification, double publication, redundancy, submission, and other related matters.

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