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Perfect 4-Colorings of the 3-Regular Graphs of Order 10

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Abstract The perfect m-coloring with matrix $A = [a_{ij}]_{i,j\in\{1,\ldots,m\}}$ of a graph $G = (V, E)$ with $\{1, \ldots, m\}$ color is a vertices coloring of G with m-color so that number of vertex in color j adjacent to a fixed vertex in color i is a_{ij}, independent of the choice of vertex in color i. The matrix $A = [a_{ij}]_{i,j\in\{1,\ldots,m\}}$ is called the parameter matrix.

We study the perfect 4-colorings of the 3-regular graphs of order 10, that is, we determine a list of all color parameter matrices corresponding to perfect colorings of 3-regular graphs of order 10.

Keywords Perfect coloring · Parameter matrices · Cubic graph · Equitable partition

Mathematics Subject Classification (2010) 03E02 · 05C15 · 68R05

1 Introduction

The concept of a perfect m-coloring plays an important role in graph theory, algebraic combinatorics, and coding theory (Completely regular codes). There is another term for this concept in the literature as ”equitable partition” (see [9]).

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The existence of completely regular codes in graphs is a historical problem in mathematics. Completely regular codes are a generalization of perfect codes. In 1973, Delsarte conjectured the non-existence of perfect codes in Johnson graphs. Therefore, some effort has been done on enumerating the parameter matrices of some Johnson graphs, including $J(6,3)$, $J(7,3)$, $J(8,3)$, $J(8,4)$, and $J(n,3)$ (n odd) (see [3,4,8]).

Fon-Der-Flaass enumerated the parameter matrices of n-dimensional hypercube Q_n for $n < 24$. He also obtained some constructions and a necessary condition for the existence of perfect 2-colorings of the n-dimensional hypercube with a given parameter matrix (see [5–7]).

In [2] all perfect 3-colorings of the cubic graphs of order 10 were described and in [10] all Perfect 4-colorings of the 3-regular graphs of order at most 8 were described.

In this paper we enumerate the parameter matrices of all perfect 4-colorings of the 3-regular graphs of order 10.

2 Preliminaries

In this section we use the following definition.

Definition 1 For each graph G and each integer m, a mapping $T : V(G) \to \{1, \ldots, m\}$ is called a perfect m-coloring with matrix $A = [a_{ij}]_{i,j \in \{1,\ldots,m\}}$, if it is surjective and for all i, j for every vertex of color i , the number of its neighbors of color j is equal to a_{ij}. The matrix A is called the parameter matrix of a perfect coloring.

The spectrum of a matrix A, denoted by $\sigma(A)$ is the set of all eigenvalues of A. The set of eigenvalues of the adjacency matrix of graph G is called the spectrum of G.

We denoted $M_r(4)$ for all parameter matrices of the perfect 4-colorings of r-regular graphs. Note that if $A \in M_r(4)$, then the total number of entries for each row is equal to r.

If $A = [a_{ij}]_{n\times n}$ is a perfect 4-colorings matrix for a 3-regular graph $G = (V, E)$, then $\sum\limits_{j=1}^{4} a_{ij} = 3$ for all $1 \leq i \leq 4$. So there are 20 different models for each row of matrices. Hence there are 20^4 matrices.

Let $A = [a_{ij}]_{4\times 4}$ be a 4-color parameter matrix for a graph $G = (V, E)$. The first observation says A must possess a weak form of symmetry, described in the following lemma:

Lemma 1 *Suppose $A = [a_{ij}]_{n\times n}$ is a parameter matrix for a graph $G = (V, E)$. Then, $a_{ij} = 0$ if and only if $a_{ji} = 0$ for $1 \leq i, j \leq n$.*

Definition 2 Let A and B are two parameter matrices of the perfect 4-colorings of graph G. We define A and B are equivalent if A transformed to B by a permutation on colors.

We have the obvious lemmas:

Lemma 2 *Let* $A = [a_{ij}]_{4\times 4}$ *and* $A \in M_3(4)$ *and* $\sigma \in S_4$ *(where* S_4 *is the symmetric group of degree 4). Then* $[a_{ij}]_{4\times 4} \sim [a_{i\sigma(j)}]_{4\times 4}$.

Lemma 3 *Let* $A = [a_{ij}]_{4\times 4} \in M_3(4)$. *Then the following cases do not happen:*

1) $a_{14} = 0,\ a_{13} = 0,\ a_{12} = 0$;
2) $a_{24} = 0,\ a_{23} = 0,\ a_{21} = 0$;
3) $a_{34} = 0,\ a_{32} = 0,\ a_{31} = 0$;
4) $a_{43} = 0,\ a_{42} = 0,\ a_{41} = 0$.

Lemma 4 *Suppose* $A \in M_3(4)$. *Then there is not* $\sigma \in S_4$ *such that* $[a_{i\sigma(j)}] =$
$$\begin{bmatrix} * & * & 0 & 0 \\ * & * & 0 & 0 \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{bmatrix}$$

Proof It is clear with connectivity.

Remark 1 Suppose $A \in M_3(4)$ is a parameter matrix for a 3-regular graph G. If there is $\sigma \in S_4$ such that $A = [a_{i\sigma(j)}] = \begin{bmatrix} 0 & 0 & * & * \\ 0 & 0 & * & * \\ * & * & 0 & 0 \\ * & * & 0 & 0 \end{bmatrix}$, then G is bipartite.

To see this V is the set of vertices of G. Divided V in to two independent sets V_1 and V_2 with color numbers 3, 4 and 1, 2 respective. Therefore G is a bipartite graph.

It is easy to see that each perfect coloring on a graph G, create an equitable partition. So, we have the following lemma.

Lemma 5 *Suppose* $A \in M_3(4)$ *is a coloring matrix for graph G. Then the spectrum of A is a subset of the spectrum of G.*

Lemma 6 *If* $A \in M_3(4)$, *then all of the eigenvalues of A are real.*

Proof By symmetry of adjancy matrices of G is obvious.

Proposition 1 *Let* $A = \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{bmatrix}$ *be a color incidence matrix of some connected graph* $G = (V, E)$, *and suppose that* $|v|$ *denote the number of vertices of G and* v_i *denote color i;* $(1 \leq i \leq 4)$.

1) If $b \neq 0$, $c \neq 0$ *and* $d \neq 0$, *then*

$$v_1 = \frac{|v|}{1 + \frac{b}{e} + \frac{c}{i} + \frac{d}{m}}, v_2 = \frac{|v|}{\frac{e}{b} + 1 + \frac{ec}{bi} + \frac{ed}{bm}},$$

$$v_3=\frac{|v|}{\frac{i}{c}+\frac{ib}{ce}+1+\frac{id}{cm}},v_4=\frac{|v|}{\frac{m}{d}+\frac{mb}{de}+\frac{mc}{di}+1}.$$

2) If $b\neq 0$, $c\neq 0$ and $h\neq 0$, then

$$v_1=\frac{|v|}{1+\frac{b}{e}+\frac{c}{i}+\frac{bh}{en}},v_2=\frac{|v|}{\frac{e}{b}+1+\frac{ec}{bi}+\frac{h}{n}},$$

$$v_3=\frac{|v|}{\frac{i}{c}+\frac{ib}{ce}+1+\frac{ibh}{cen}},v_4=\frac{|v|}{\frac{ne}{hb}+\frac{n}{h}+\frac{nec}{hbi}+1}.$$

3) If $b\neq 0$, $c\neq 0$ and $l\neq 0$, then

$$v_1=\frac{|v|}{1+\frac{b}{e}+\frac{c}{i}+\frac{cl}{io}},v_2=\frac{|v|}{\frac{e}{b}+1+\frac{ec}{bi}+\frac{ecl}{bio}},$$

$$v_3=\frac{|v|}{\frac{i}{c}+\frac{ib}{ce}+1+\frac{l}{o}},v_4=\frac{|v|}{\frac{oi}{lc}+\frac{oib}{lce}+\frac{o}{l}+1}.$$

4) If $b\neq 0$, $d\neq 0$ and $g\neq 0$, then

$$v_1=\frac{|v|}{1+\frac{b}{e}+\frac{bg}{ej}+\frac{d}{m}},v_2=\frac{|v|}{\frac{e}{b}+1+\frac{g}{j}+\frac{ed}{bm}},$$

$$v_3=\frac{|v|}{\frac{je}{gb}+\frac{j}{g}+1+\frac{jed}{gbm}},v_4=\frac{|v|}{\frac{m}{d}+\frac{mb}{de}+\frac{mbg}{dej}+1}.$$

5) If $b\neq 0$, $d\neq 0$ and $l\neq 0$, then

$$v_1=\frac{|v|}{1+\frac{b}{e}+\frac{do}{ml}+\frac{d}{m}},v_2=\frac{|v|}{\frac{e}{b}+1+\frac{edo}{bml}+\frac{ed}{bm}},$$

$$v_3=\frac{|v|}{\frac{lm}{od}+\frac{lmb}{ode}+1+\frac{l}{o}},v_4=\frac{|v|}{\frac{m}{d}+\frac{mb}{de}+\frac{o}{l}+1}.$$

6) If $b\neq 0$, $g\neq 0$ and $h\neq 0$, then

$$v_1=\frac{|v|}{1+\frac{b}{e}+\frac{bg}{ej}+\frac{bh}{en}},v_2=\frac{|v|}{\frac{e}{b}+1+\frac{g}{j}+\frac{h}{n}},$$

$$v_3=\frac{|v|}{\frac{je}{gb}+\frac{j}{g}+1+\frac{jh}{gn}},v_4=\frac{|v|}{\frac{ne}{hb}+\frac{n}{h}+\frac{ng}{hj}+1}.$$

7) If $b \neq 0$, $g \neq 0$ and $l \neq 0$, then

$$v_1 = \frac{|v|}{1+\frac{b}{e}+\frac{bg}{ej}+\frac{bgl}{ejo}}, v_2 = \frac{|v|}{\frac{e}{b}+1+\frac{g}{j}+\frac{gl}{jo}},$$

$$v_3 = \frac{|v|}{\frac{je}{gb}+\frac{j}{g}+1+\frac{l}{o}}, v_4 = \frac{|v|}{\frac{oje}{lgb}+\frac{oj}{lg}+\frac{o}{l}+1}.$$

8) If $b \neq 0$, $h \neq 0$ and $l \neq 0$, then

$$v_1 = \frac{|v|}{1+\frac{b}{e}+\frac{bho}{enl}+\frac{bh}{en}}, v_2 = \frac{|v|}{\frac{e}{b}+1+\frac{ho}{nl}+\frac{h}{n}},$$

$$v_3 = \frac{|v|}{\frac{lne}{ohb}+\frac{ln}{oh}+1+\frac{l}{o}}, v_4 = \frac{|v|}{\frac{ne}{hb}+\frac{n}{h}+\frac{o}{l}+1}.$$

9) If $c \neq 0$, $d \neq 0$ and $g \neq 0$, then

$$v_1 = \frac{|v|}{1+\frac{cj}{ig}+\frac{c}{i}+\frac{d}{m}}, v_2 = \frac{|v|}{\frac{gi}{cj}+1+\frac{g}{j}+\frac{gid}{jcm}},$$

$$v_3 = \frac{|v|}{\frac{i}{c}+\frac{j}{g}+1+\frac{id}{cm}}, v_4 = \frac{|v|}{\frac{m}{d}+\frac{mcj}{dig}+\frac{mc}{di}+1}.$$

10) If $c \neq 0$, $d \neq 0$ and $h \neq 0$, then

$$v_1 = \frac{|v|}{1+\frac{dn}{mh}+\frac{c}{i}+\frac{d}{m}}, v_2 = \frac{|v|}{\frac{hm}{dn}+1+\frac{hmc}{ndi}+\frac{h}{n}},$$

$$v_3 = \frac{|v|}{\frac{i}{c}+\frac{idn}{cmh}+1+\frac{id}{cm}}, v_4 = \frac{|v|}{\frac{m}{d}+\frac{n}{h}+\frac{mc}{di}+1}.$$

11) If $c \neq 0$, $g \neq 0$ and $h \neq 0$, then

$$v_1 = \frac{|v|}{1+\frac{cj}{ig}+\frac{c}{i}+\frac{cjh}{ign}}, v_2 = \frac{|v|}{\frac{gi}{jc}+1+\frac{g}{j}+\frac{h}{n}},$$

$$v_3 = \frac{|v|}{\frac{i}{c}+\frac{j}{g}+1+\frac{jh}{gn}}, v_4 = \frac{|v|}{\frac{ngi}{hjc}+\frac{n}{h}+\frac{ng}{hj}+1}.$$

12) If $c \neq 0$, $g \neq 0$ and $l \neq 0$, then

$$v_1 = \frac{|v|}{1+\frac{cj}{ig}+\frac{c}{i}+\frac{cl}{io}}, v_2 = \frac{|v|}{\frac{gi}{jc}+1+\frac{g}{j}+\frac{gl}{jo}},$$

$$v_3 = \frac{|v|}{\frac{i}{c}+\frac{j}{g}+1+\frac{l}{o}}, v_4 = \frac{|v|}{\frac{oi}{lc}+\frac{oj}{lg}+\frac{o}{l}+1}.$$

13) If $c \neq 0$, $h \neq 0$ and $l \neq 0$, then

$$v_1 = \frac{|v|}{1+\frac{cln}{ioh}+\frac{c}{i}+\frac{cl}{io}}, v_2 = \frac{|v|}{\frac{hoi}{nlc}+1+\frac{ho}{nl}+\frac{h}{n}},$$

$$v_3 = \frac{|v|}{\frac{i}{c}+\frac{ln}{oh}+1+\frac{l}{o}}, v_4 = \frac{|v|}{\frac{oi}{lc}+\frac{n}{h}+\frac{o}{l}+1}.$$

14) If $d \neq 0$, $g \neq 0$ and $h \neq 0$, then

$$v_1 = \frac{|v|}{1+\frac{dn}{mh}+\frac{dng}{mhj}+\frac{d}{m}}, v_2 = \frac{|v|}{\frac{hm}{nd}+1+\frac{g}{j}+\frac{h}{n}},$$

$$v_3 = \frac{|v|}{\frac{jhm}{gnd}+\frac{j}{g}+1+\frac{jh}{gn}}, v_4 = \frac{|v|}{\frac{m}{d}+\frac{n}{h}+\frac{ng}{hj}+1}.$$

15) If $d \neq 0$, $g \neq 0$ and $l \neq 0$, then

$$v_1 = \frac{|v|}{1+\frac{doj}{mlg}+\frac{do}{ml}+\frac{d}{m}}, v_2 = \frac{|v|}{\frac{glm}{jod}+1+\frac{g}{j}+\frac{gl}{jo}},$$

$$v_3 = \frac{|v|}{\frac{lm}{od}+\frac{j}{g}+1+\frac{l}{o}}, v_4 = \frac{|v|}{\frac{m}{d}+\frac{oj}{lg}+\frac{o}{l}+1}.$$

16) If $d \neq 0$, $h \neq 0$ and $l \neq 0$, then

$$v_1 = \frac{|v|}{1+\frac{dn}{mh}+\frac{do}{ml}+\frac{d}{m}}, v_2 = \frac{|v|}{\frac{hm}{nd}+1+\frac{ho}{nl}+\frac{h}{n}},$$

$$v_3 = \frac{|v|}{\frac{lm}{od}+\frac{ln}{oh}+1+\frac{l}{o}}, v_4 = \frac{|v|}{\frac{m}{d}+\frac{n}{h}+\frac{o}{l}+1}.$$

By using above proposition and lemmas, for n=10 we only have the following matrices, which we have shown with $M_1, \ldots, M_{28}$.

$$M_1 = \begin{bmatrix} 0&1&1&1 \\ 1&1&0&1 \\ 3&0&0&0 \\ 1&1&0&1 \end{bmatrix}, M_2 = \begin{bmatrix} 1&0&0&2 \\ 0&1&0&2 \\ 0&0&1&2 \\ 1&1&1&0 \end{bmatrix}, M_3 = \begin{bmatrix} 2&0&0&1 \\ 0&1&1&1 \\ 0&3&0&0 \\ 1&1&0&1 \end{bmatrix}, M_4 = \begin{bmatrix} 2&0&0&1 \\ 0&2&0&1 \\ 0&0&0&3 \\ 1&1&1&0 \end{bmatrix},$$

$$M_5 = \begin{bmatrix} 0&0&2&1 \\ 0&0&3&0 \\ 1&1&0&1 \\ 1&0&1&1 \end{bmatrix}, M_6 = \begin{bmatrix} 0&0&2&1 \\ 0&0&0&3 \\ 3&0&0&0 \\ 1&2&0&0 \end{bmatrix}, M_7 = \begin{bmatrix} 0&0&2&1 \\ 0&0&0&3 \\ 1&0&1&1 \\ 1&1&1&0 \end{bmatrix}, M_8 = \begin{bmatrix} 0&0&2&1 \\ 0&0&0&3 \\ 2&0&1&0 \\ 1&1&0&1 \end{bmatrix},$$

$$M_9 = \begin{bmatrix} 0&0&2&1 \\ 0&0&0&3 \\ 2&0&0&1 \\ 1&1&1&0 \end{bmatrix}, M_{10} = \begin{bmatrix} 0&0&2&1 \\ 0&1&1&1 \\ 1&1&1&0 \\ 1&1&0&1 \end{bmatrix}, M_{11} = \begin{bmatrix} 0&0&2&1 \\ 0&1&1&1 \\ 1&1&1&0 \\ 2&1&0&0 \end{bmatrix}, M_{12} = \begin{bmatrix} 0&0&2&1 \\ 0&1&1&1 \\ 1&1&0&1 \\ 1&1&1&0 \end{bmatrix},$$

$$M_{13} = \begin{bmatrix} 0&0&2&1 \\ 0&1&1&1 \\ 1&2&0&0 \\ 1&1&0&1 \end{bmatrix}, M_{14} = \begin{bmatrix} 0&0&2&1 \\ 0&1&2&0 \\ 1&1&1&0 \\ 1&0&0&2 \end{bmatrix}, M_{15} = \begin{bmatrix} 0&0&2&1 \\ 0&1&2&0 \\ 1&1&0&1 \\ 1&0&1&1 \end{bmatrix}, M_{16} = \begin{bmatrix} 0&0&2&1 \\ 0&1&2&0 \\ 1&1&0&1 \\ 1&0&2&0 \end{bmatrix},$$

$$M_{17} = \begin{bmatrix} 0&0&2&1 \\ 0&2&1&0 \\ 1&1&0&1 \\ 2&0&1&0 \end{bmatrix}, M_{18} = \begin{bmatrix} 0&0&2&1 \\ 0&2&1&0 \\ 2&1&0&0 \\ 3&0&0&0 \end{bmatrix}, M_{19} = \begin{bmatrix} 0&0&2&1 \\ 0&0&1&2 \\ 2&1&0&0 \\ 1&1&0&1 \end{bmatrix}, M_{20} = \begin{bmatrix} 0&0&2&1 \\ 0&2&0&1 \\ 1&0&1&1 \\ 1&1&1&0 \end{bmatrix},$$

$$M_{21} = \begin{bmatrix} 0&0&2&1 \\ 0&2&0&1 \\ 1&0&2&0 \\ 1&1&0&1 \end{bmatrix}, M_{22} = \begin{bmatrix} 0&0&2&1 \\ 0&2&0&1 \\ 1&0&0&2 \\ 1&1&1&0 \end{bmatrix}, M_{23} = \begin{bmatrix} 0&0&2&1 \\ 0&2&0&1 \\ 2&0&1&0 \\ 1&2&0&0 \end{bmatrix}, M_{24} = \begin{bmatrix} 0&0&2&1 \\ 0&0&2&1 \\ 1&1&1&0 \\ 1&1&0&1 \end{bmatrix},$$

$$M_{25} = \begin{bmatrix} 0&0&2&1 \\ 0&0&2&1 \\ 1&1&1&0 \\ 2&1&0&0 \end{bmatrix}, M_{26} = \begin{bmatrix} 0&0&2&1 \\ 0&0&2&1 \\ 1&1&0&1 \\ 1&1&1&0 \end{bmatrix}, M_{27} = \begin{bmatrix} 0&0&2&1 \\ 0&0&2&1 \\ 2&1&0&0 \\ 1&1&0&1 \end{bmatrix}, M_{28} = \begin{bmatrix} 0&0&2&1 \\ 0&0&2&1 \\ 2&1&0&0 \\ 1&2&0&0 \end{bmatrix},$$

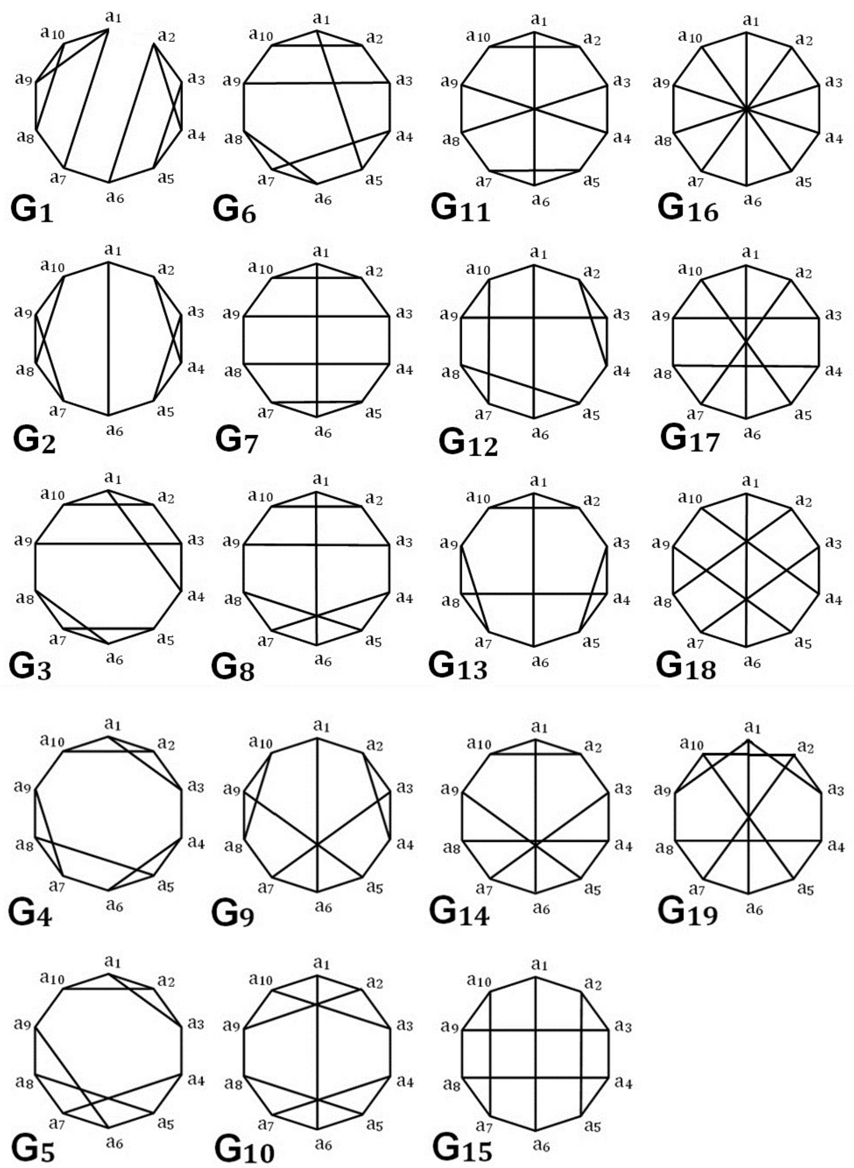

Fig. 1: Connected 3-regular graphs of order 10

3 Main Results

A cubic graph is a 3-regular graph and there are 19 none isomorphic cubic graphs of order 10 as shown below in Fig. 1.

Table 1

Graph	Matrix									
G_1	M_1	M_7	M_{16}	M_{24}	M_{26}					
G_2	M_1	M_2	M_7	M_9	M_{16}	M_{24}	M_{26}			
G_3	M_{15}									
G_4	M_{23}	M_{26}								
G_5	M_{23}	M_{26}								
G_6	M_1	M_7	M_{16}	M_{24}	M_{26}					
G_8	M_5									
G_9	M_2	M_4	M_7	M_8	M_9	M_{16}	M_{20}			
G_{10}	M_1	M_2	M_4	M_6	M_7	M_8	M_{15}	M_{20}	M_{24}	M_{26}	M_{28}
G_{12}	M_{15}	M_{18}								
G_{13}	M_9									
G_{15}	M_{19}									
G_{17}	M_{10}	M_{19}								
G_{18}	M_1	M_2	M_7	M_9	M_{16}	M_{24}	M_{26}			
G_{19}	M_2	M_9								

Note: By using Lemma 6 there are no perfect 4-colorings with the matrices $M_1, \ldots, M_{28}$ for graphs $G_7, G_{11}, G_{14}, G_{16}$, and $M_1, \ldots, M_{28}$ can be parameter matrices of order 10 for graphs G_1, G_2, G_3, G_4, G_5, G_6, G_8, G_9, G_{10}, G_{12}, G_{13}, G_{15}, G_{17}, G_{18}, G_{19} are listed in Table 1.

Theorem 1 *The parameter matrices of 3-regular graphs of order* 10 *are listed in the Table 2. (Checkmark(✓) means having perfect 4-colorings and cross(×) means not having perfect 4-colorings.)*

Table 2: The parameter matrices of 3-regular graphs of order 10

Graph	Matrix 1	2	4	5	6	7	8	9	10	15	16	18	19	20	23	24	26	28
G_1	×					×					×					×	×	
G_2	×	×				×		×			×					×	×	
G_3										×								
G_4															×		×	
G_5															×		×	
G_6	×					×					✓					✓	×	
G_8				×														
G_9		×	✓			×	×	×			✓			×				
G_{10}	×	×	×		×	×	×			×				×		×	×	×
G_{12}										×		×						
G_{13}								✓										
G_{15}													×					
G_{17}									×				×					
G_{18}	×	×				×		×			×					✓	×	
G_{19}		✓						✓										

Proof With consideration of 3-regular graphs eigenvalues and using Proposition 1, and Lemmas 5 and 6, it can be seen that the connected 3-regular graphs with 10 vertices can have perfect 4-colorings with matrices M_1, M_2, M_4, M_5, M_6, M_7, M_8, M_9, M_{10}, M_{15}, M_{16}, M_{18}, M_{19}, M_{20}, M_{23}, M_{24}, M_{26} and M_{28} which are represented by Table 2.

Now we introduce the mappings of all graphs that have perfect 4-colorings with the parameter matrices. The graph G_6 has perfect 4-colorings with the matrices M_{16} and M_{24}.

Consider two mappings T_1 and T_2 as follows :

$T_1(a_1) = T_1(a_8) = 1$, $T_1(a_3) = T_1(a_4) = 2$,

$T_1(a_2) = T_1(a_5) = T_1(a_7) = T_1(a_9) = 3$, $T_1(a_6) = T_1(a_{10}) = 4$.

$T_2(a_5) = T_2(a_9) = 1$, $T_2(a_2) = T_2(a_7) = 2$,

$T_2(a_1) = T_2(a_6) = T_2(a_8) = T_2(a_{10}) = 3$, $T_2(a_3) = T_2(a_4) = 4$.

Can be seen that T_1 and T_2 are perfect 4-colorings with the matrices M_{16} and M_{24}, respectively.

The graph G_9 has perfect 4-colorings with the matrices M_4 and M_{16}. Consider two mappings T_1 and T_2 as follows:

$T_1(a_8) = T_1(a_9) = T_1(a_{10}) = 1$, $T_1(a_2) = T_1(a_3) = T_1(a_4) = 2$,

$T_1(a_6) = 3$, $T_1(a_1) = T_1(a_5) = T_1(a_7) = 4$.

$T_2(a_2) = T_2(a_9) = 1$, $T_2(a_6) = T_2(a_7) = 2$,

$T_2(a_1) = T_2(a_3) = T_2(a_5) = T_2(a_8) = 3$, $T_2(a_4) = T_2(a_{10}) = 4$.

Can be seen that T_1 and T_2 are perfect 4-colorings with the matrices M_4 and M_{16}, respectively.

The graph G_{13} has perfect 4-colorings with the matrix M_9. Consider the mapping T as follows:

$T(a_1) = T(a_5) = T(a_7) = 4$, $T(a_2) = T(a_4) = T(a_9) = 1$,

$T(a_3) = T(a_8) = T(a_{10}) = 3$, $T(a_6) = 2$.

Can be seen that T is a perfect 4-colorings with the matrix M_9.

The graph G_{18} has perfect 4-colorings with the matrix M_{24}. Consider the mapping T as follows:

$T(a_2) = T(a_5) = 1$, $T(a_7) = T(a_{10}) = 2$,

$T(a_3) = T(a_4) = T(a_8) = T(a_9) = 3$, $T(a_1) = T(a_6) = 4$.

Can be seen that T is a perfect 4-colorings with the matrix M_{24}.

The graph G_{19} has perfect 4-colorings with the matrices M_2 and M_9. Consider two mappings T_1 and T_2 as follows:

$T_1(a_2) = T_1(a_7) = 1$, $T_1(a_1) = T_1(a_9) = 2$,

$T_1(a_4) = T_1(a_5) = 3$, $T_1(a_3) = T_1(a_6) = T_1(a_8) = T_1(a_{10}) = 4$.

$T_2(a_1) = T_2(a_8) = T_2(a_{10}) = 4$, $T_2(a_9) = 2$,

$T_2(a_2) = T_2(a_4) = T_2(a_6) = 3$, $T_2(a_3) = T_2(a_5) = T_2(a_7) = 1$.

Can be seen that T_1 and T_2 are perfect 4-colorings with the matrices M_2 and M_9, respectively.

Here, we prove that other cases are not feasible. Some examples are as follows. The rest of the graphs in Table 2 will be demonstrated in the some order.

We show that the graph G_1 has no perfect 4-colorings with the matrices M_1, M_7, M_{16}, M_{24} and M_{26}. For example we claim that M_{26} has no perfect 4-colorings for graph G_1. Contrary to our claim, suppose that T is a perfect

4-colorings with the matrix M_{26} for graph G_1. Then according to the matrix M_{26}, by symmetry we have five cases for the color of number 1 as follows:

1. If $T(a_1) = 1$, because $m_{33} = 0$, thus none two vertices with color 3 shouldn't be adjacent. Therefore $T(a_5) = 3$, $T(a_2) = 3$ or $T(a_3) = 3$. If $T(a_2) = 3$ then $T(a_3) = 4$ therefore $T(a_4) = 2$ so $T(a_6) = 4$ but $T(a_7) = 1$ or $T(a_{10}) = 1$, which is a contradiction because we must colored two vertices adjacent with color 3. If $T(a_3) = 3$, $T(a_2) = 4$, $T(a_4) = 2$ so $T(a_6) = 4$, which is a contradiction with above.
2. $T(a_2) = 1$, it follows that $T(a_1) = T(a_4) = 3$ then $T(a_3) = 4$ because $m_{43} = 1$, which is a contradiction with the fourth row of the matrix M_{26}.
3. If $T(a_3) = 1$, it follows that $T(a_1) = T(a_4) = 3$ so $T(a_2) = 4$ because $m_{43} = 1$, which is a contradiction.
4. If $T(a_4) = 1$ then $T(a_3) = T(a_5) = 3$. It follows that $T(a_2) = 4$ then $T(a_1) = 2$ and $T(a_6) = 4$, which is a contradiction with the first case.
5. If $T(a_5) = 1$ then we have 2 cases;
   $T(a_1) = T(a_4) = 3$, it follows that $T(a_6) = 4$ therefore $T(a_4) = 3$ so $T(a_3) = 2$ and $T(a_2) = 4$ or $T(a_4) = 4$ and $T(a_2) = 3$, in both cases because $m_{43} = 1$; which is a contradiction. Therefore the graph G_1 has no perfect 4-colorings with matrix M_{26}.

Now we show that the graph G_{15} has no perfect 4-colorings with the matrix M_{19}. Contorary to our claim, suppose that T is a perfect 4-colorings with matrix M_{19} for the graph G_{15}. According to the matrix M_{19}, by symmetry we have three cases for the color of number 3 as follows:

1. If $T(a_1) = 3$, according to the matrix M_{19}, there are two situations; $T(a_2) = T(a_6) = 1$ or $T(a_2) = T(a_{10}) = 1$. For first case if $T(a_2) = T(a_6) = 1$, then $T(a_{10}) = 2$ therefore $T(a_7) = T(a_9) = 4$ then $T(a_8) = 2$. It follows that $T(a_4) = 3$ and $T(a_5) = 3$, which is a contradiction with the $m_{33} = 0$ and for the second case, if $T(a_2) = T(a_{10}) = 1$ then $T(a_6) = 2$. It follows that $T(a_5) = T(a_7) = 4$ because $T(a_2) = 1$ so $T(a_3) = 3$, similarly $T(a_9) = 3$, which is a contradiction with the $m_{33} = 0$ of the matrix M_{19}.
2. If $T(a_2) = 3$, according to the matrix M_{19}, there are three situations; $T(a_1) = T(a_3) = 1$ or $T(a_1) = T(a_5) = 1$ or $T(a_3) = T(a_5) = 1$. For first case if $T(a_1) = T(a_3) = 1$, then $T(a_5) = 2$ therefore $T(a_4) = T(a_8) = 4$, which is a contradiction with the $m_{44} = 1$ of the matrix M_{19}.
   For the second case, if $T(a_1) = T(a_5) = 1$ then $T(a_3) = 2$ therefore $T(a_4) = 4$ and $T(a_9) = 4$. It follows that $T(a_8) = 4$, which is a contradiction with the $m_{44} = 1$ of the matrix M_{19}.
   For the third case, if $T(a_3) = T(a_5) = 1$, then $T(a_1) = 2$ therefore $T(a_6) = T(a_{10}) = 4$. It follows that $T(a_7) = 4$, which is a contradiction with the $m_{44} = 1$ of the matrix M_{19}.
3. If $T(a_3) = 3$, according to the matrix M_{19}, there are three situations: $T(a_2) = T(a_4) = 1$ or $T(a_2) = T(a_9) = 1$ or $T(a_4) = T(a_9) = 1$.
   For first case if $T(a_2) = T(a_4) = 1$, then $T(a_9) = 2$ therefore $T(a_8) = T(a_{10}) = 4$. It follows that $T(a_7) = 4$, which is a contradiction with the fourth row of the matrix M_{19}.

For the second case if $T(a_2) = T(a_9) = 1$, then $T(a_4) = 2$ therefore, $T(a_5) = T(a_8) = 4$, on the otherhand $T(a_7) = 4$. It follows that $T(a_6) = 4$, which is a contradiction as above.
For the third case, if $T(a_4) = T(a_9) = 1$, then $T(a_2) = 2$, therefore $T(a_5) = T(a_1) = 4$. It follows that $T(a_6) = 4$, which is a contradiction as above. Therefore the graph G_{15} has no perfect 4-colorings with matrix M_{19}. As it is stated, the graph G_{15} has no perfect 4-colorings with matrix M_{19}.

About the other graphs of order 10, similarly we can get the same result as in Table 2 in Theorem 1.

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