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Review article

Exact Solutions, Convergence, and Stability Analysis of Fractional Diffusion Models with Nonlocal Interactions

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Abstract

In this paper, a fractional integro-differential model involving the Caputo time-fractional derivative and the Riesz space-fractional operator is proposed and analyzed. The model incorporates both nonlinear reaction terms and nonlocal integral interactions, allowing an accurate description of anomalous diffusion processes with memory and spatial long-range effects. By applying the Fourier transform with respect to the spatial variables and the Laplace transform with respect to time, the governing equation is transformed into an algebraic equation in the transform domain, leading to an explicit representation of the solution in terms of Mittag–Leffler functions. The existence, convergence, and stability of the mild solution are established by means of an iterative scheme combined with fixed-point arguments and a fractional Gronwall inequality. It is shown that the approximate solutions converge uniformly to the unique mild solution and that the solution depends continuously on the initial data. To illustrate the theoretical results, three representative examples are presented, including a pure fractional diffusion model, a reaction–diffusion model, and a multi-mode system with nonzero integral kernels. The obtained exact solutions demonstrate the significant influence of the fractional orders on the temporal decay rate and spatial behavior of the solution. The proposed framework provides a mathematically rigorous and physically meaningful tool for modeling and analyzing fractional-order transport phenomena arising in engineering and industrial applications such as heat conduction in heterogeneous materials, diffusion in porous media, and dynamic processes in complex systems.

Keywords: Fractional integro-differential equation, Riesz fractional operator, Fourier and Laplace transforms, Mittag–Leffler function, Convergence and stability.

Mathematics Subject Classification (2020): 26A33, 35R11, 44A10, 80A20, 35B40

1 Introduction

Fractional models [8, 11, 20, 22] extend classical integer–order differential equations by allowing derivatives of non–integer order. From a physical and engineering viewpoint, this extension introduces a natural way to represent memory and hereditary effects that are frequently observed in real materials and processes. Unlike classical models, where the system state depends only on its current value and a finite number of derivatives, fractional models incorporate the entire past evolution of the system through convolution-type operators.

In materials science and solid mechanics, fractional differential equations provide accurate descriptions of viscoelastic materials. In polymers, biological tissues, and composite materials, the stress does not depend solely on the instantaneous strain but also on the history of deformation. Fractional constitutive laws, such as fractional Kelvin–Voigt and fractional Maxwell models, successfully reproduce experimentally observed stress relaxation and creep phenomena using fewer parameters than classical models. In transport and diffusion phenomena, fractional models are widely used to describe anomalous diffusion. Classical diffusion equations assume that the mean square displacement of particles grows linearly with time. However, many real systems deviate from this behavior. For example, in porous media, groundwater transport, and heterogeneous geological formations, particles may become trapped for long periods, leading to subdiffusion. Conversely, in turbulent flows, plasma transport, and atmospheric dynamics, long jumps in particle trajectories produce superdiffusion. Fractional diffusion equations naturally capture these effects through nonlocal temporal or spatial operators [12, 13, 27, 28]. Fractional operators also arise in models with long-range spatial interactions. In heat conduction in heterogeneous media, wave propagation in complex structures, and electromagnetic processes in fractal materials, the influence of distant points cannot be neglected. Fractional spatial derivatives provide a compact mathematical framework to describe such nonlocal behavior. In engineering control systems, fractional-order controllers, such as the fractional PID (Proportional–Integral–Derivative) controller, offer improved robustness and tuning flexibility compared to classical integer-order controllers. By introducing fractional integration and differentiation orders, engineers can better shape the dynamic response of systems, improving stability margins and disturbance rejection in applications such as robotics, power electronics, and process control. Another important engineering application appears in electrochemical systems and energy storage devices. Fractional models accurately represent the impedance characteristics of batteries, fuel cells, and supercapacitors, where charge transport and diffusion processes exhibit memory effects. Similarly, in geophysics and seismology, fractional wave and diffusion equations are used to model wave attenuation and energy dissipation in complex geological media. Overall, fractional models provide a physically meaningful framework for describing systems with memory, nonlocal interactions, and multi-scale dynamics. They bridge microscopic mechanisms and macroscopic observations, leading to more realistic mathematical descriptions of engineering and physical processes. Consequently, the study of fractional differential equations not only improves modeling accuracy but also motivates the development of new analytical and numerical techniques for solving complex real-world problems.

We consider the following nonlinear fractional integro-differential equation on a finite spatial interval (a,b):

$$ {}^{C}D_t^{\alpha} u(x,t) = \kappa \frac{\partial^{\beta} u(x,t)}{\partial |x|^{\beta}} + f(u(x,t)) + \int_a^b K_1(x,\xi)\, u(\xi,t)\, d\xi + \int_a^b K_2(x,\xi)\, g(u(\xi,t))\, d\xi, \quad a < x < b,\ t > 0, \tag{1} $$

with the initial condition

$$ u(x,0) = \phi(x), \quad a \le x \le b, \tag{2} $$

and homogeneous boundary conditions

$$ u(a,t) = 0, \qquad u(b,t) = 0, \quad t \ge 0. \tag{3} $$

The Caputo fractional derivative [5, 26] of order $0 < \alpha < 1$ with respect to time is defined by

$$ {}^{C}D_t^{\alpha} u(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(x,\tau)}{\partial \tau} \frac{d\tau}{(t-\tau)^{\alpha}}. \tag{4} $$

The Riesz fractional derivative [2, 19, 21, 33] of order $1 < \beta \le 2$ on (a,b), expressed via left- and right-sided Riemann-Liouville derivatives, is defined as

$$ \frac{\partial^{\beta} u(x,t)}{\partial |x|^{\beta}} = -\frac{1}{2\cos(\beta\pi/2)} \left({}_aD_x^{\beta} u(x,t) + {}_xD_b^{\beta} u(x,t) \right), \tag{5} $$

where the left-sided and right-sided Riemann-Liouville derivatives are

$$ {}_aD_x^{\beta} u(x,t) = \frac{1}{\Gamma(2-\beta)} \frac{d^2}{dx^2} \int_a^x \frac{u(\xi,t)}{(x-\xi)^{\beta-1}}\, d\xi, \tag{6} $$

$$ {}_xD_b^{\beta} u(x,t) = \frac{1}{\Gamma(2-\beta)} \frac{d^2}{dx^2} \int_x^b \frac{u(\xi,t)}{(\xi-x)^{\beta-1}}\, d\xi. \tag{7} $$

The proposed fractional integro-differential model is chosen because it captures both memory effects in time via the Caputo derivative and long-range spatial interactions via the Riesz derivative, which classical integer-order models cannot. The inclusion of Fredholm integral terms allows modeling of nonlocal interactions and system-wide feedbacks. Its nonlinear term enables realistic simulation of reactions or growth phenomena in complex systems. Compared to standard diffusion or reaction-diffusion models, this model accounts for anomalous transport, hereditary effects, and spatial nonlocality, providing a more accurate description of physical, biological, and engineering processes. Studying it helps understand phenomena like subdiffusion, superdiffusion, and viscoelastic behavior that classical models oversimplify. Its flexibility allows application across materials science, geophysics, biology, and finance. Numerically, it provides a framework for developing robust algorithms for complex systems. Physically, it bridges micro- and macro-scale dynamics. Overall, this model is more predictive, adaptable, and physically realistic than standard integer-order approaches.

Fractional integro-differential equations have attracted significant attention due to their ability to model memory and hereditary properties in various physical and engineering systems. Abdo and Panchal [1] explored fractional integro-differential equations involving the ψ-Hilfer fractional derivative, providing a rigorous framework for their analytical treatment. Similarly, Aghajani et al. [3] studied the existence of solutions for a broad class of fractional integro-differential equations, establishing conditions for well-posedness. In the numerical context, Arikoglu and Ozkol [6] proposed a fractional differential transform method to efficiently approximate solutions to these equations, demonstrating the method's effectiveness for practical computations. The theoretical investigation of Caputo-Fabrizio type fractional integro-differential equations was advanced by Baleanu et al. [9], who provided existence results for certain infinite coefficient-symmetric systems, highlighting the versatility of non-singular kernels in fractional calculus. More recently, Gunasekar and Raghavendran [14] introduced the Mohand transform as a powerful tool for solving fractional integro-differential equations, emphasizing both analytical and computational advantages. In terms of uniqueness results, Hussain et al. [16] analyzed new conditions under which solutions of fractional integro-differential equations remain unique, offering insights into stability and reliability of these systems. Ravichandran et al. [30] extended these studies to the framework of AtanganaBaleanu derivatives, presenting existence results for multi-term fractional integro-differential equations and highlighting the applicability of such operators in modeling complex phenomena. Finally, Zhang et al. [35] focused on nonlinear fractional integro-differential equations defined on unbounded domains, providing numerical methods that ensure convergence and stability, which are crucial for real-world simulations. Collectively, these studies demonstrate the richness of fractional integro-differential equations and the variety of analytical and numerical methods developed to investigate them.

In recent years, significant progress has been made in solving fractional integro-differential equations using various analytical and numerical techniques. Jassim [17] proposed analytical solutions for Volterra integro-differential equations within the framework of local fractional operators using the Yang-Laplace transform, providing a foundation for subsequent numerical methods. Arising from these developments, Ozkan and Kurt [25] introduced a conformable fractional double Laplace transform approach and demonstrated its applications to fractional partial integro-differential equations. Rahimkhani and Ordokhani [29] combined Hahn wavelet collocation methods with Laplace transform techniques to solve fractional integro-differential equations, highlighting the benefits of wavelet-based approximations. Shah et al. [31] developed a meshless method leveraging the Laplace transform for two-dimensional multi-term time-fractional partial integro-differential equations, offering an efficient computational framework for multi-dimensional problems. Similarly, Mohammadi-Firouzjaei et al. [23] applied a combination of Laplace transform and local discontinuous Galerkin methods to address fourth-order time-fractional partial integro-differential equations with weakly singular kernels, achieving high accuracy in the numerical solution. Mishra and Rani [24] utilized the Bernstein operational matrix of integration to invert Laplace transforms, demonstrating applicability to both differential and integral equations. The study of fractional equations with generalized fractional derivatives has also been advanced. Zada et al. [34] analyzed solutions of integro-differential equations with the generalized LiouvilleCaputo fractional derivative using the ρ-Laplace transform, providing a flexible tool for a wide class of problems. Guo et al. [15] investigated implicit coupled Hadamard fractional differential equations with generalized Hadamard integro-differential boundary conditions, highlighting the versatility of Hadamard-type operators. Al-Khaled et al. [4] addressed evolutionary time-fractional partial integro-differential equations with singular memory kernels, offering both analytical and numerical treatment to handle the nonlocality of fractional operators. Finally, Durdiev et al. [10] provided explicit formulas for solutions of multi-dimensional wave equations with fractional derivatives, which are valuable benchmarks for numerical methods. Additionally, Yisa et al. [32] proposed a homotopy analysis integral transform method for fractional integro-differential equations, contributing a robust semi-analytical framework for solution approximation. Together, these studies demonstrate the growing effectiveness of combining Laplace transforms, wavelet collocation, meshless methods, and fractional calculus operators to tackle complex fractional integro-differential equations in both one- and multi-dimensional domains. To derive the exact solutions of the proposed fractional integro-differential model, we employed the Fourier transform with respect to the spatial variable and the Laplace transform with respect to the temporal variable. The Fourier transform is particularly suitable for handling the Riesz fractional derivative because it converts this nonlocal spatial operator into a simple algebraic multiplier in the frequency domain, namely $|\xi|^{\beta}$. This property significantly simplifies the treatment of the fractional diffusion term and avoids the need for complicated integral representations in physical space. On the other hand, the Laplace transform in time is an efficient tool for fractional-order derivatives since it naturally incorporates initial conditions and transforms the Caputo fractional time derivative into an algebraic expression involving s^{α}. As a result, the original fractional partial integro-differential equation is reduced to an algebraic equation in the transform domain, which can be solved explicitly. The combined use of Fourier and Laplace transforms provides a systematic and transparent framework for obtaining closed-form solutions in terms of Mittag–Leffler functions. This approach offers several advantages compared with many existing methods in the literature, such as numerical discretization, perturbation techniques, or purely variational approaches. First, it yields analytical expressions that reveal the explicit influence of the fractional orders α and β as well as the kernel parameters on the solution behavior. Second, it avoids linearization or small-parameter assumptions, which are often required in approximate analytical methods. Third, the transform-based technique allows a unified treatment of diffusion, reaction, and integral interaction terms within the same formulation. Compared with related works, which usually focus either on time-fractional or space-fractional models separately, the present approach simultaneously handles both fractional effects and integral kernels in a single framework. Moreover, while many studies rely on numerical simulations to illustrate solution behavior, our method provides exact solutions that can be used as benchmark results for validating numerical algorithms. Therefore, the proposed transform technique not only enhances analytical tractability but also improves the reliability and interpretability of the obtained results, offering a clear advantage over existing methods in terms of accuracy, generality, and physical insight.

The remainder of this manuscript is organized as follows. In Section 2, we present the Laplace and Fourier transform approach for the proposed fractional integro-differential model, which allows us to obtain exact analytical solutions in terms of Mittag–Leffler functions. Section 3 is devoted to the existence and uniqueness analysis of the fractional model, where the mild solution is rigorously constructed using fixed-point arguments in appropriate Banach spaces. In Section 4, we establish the convergence of iterative approximations to the exact solution and provide detailed estimates demonstrating uniform convergence. Section 5 addresses the stability analysis, showing that the solution depends continuously on initial data and perturbations, and highlighting the role of the Riesz fractional derivative and the integral kernel operators in controlling the solution behavior. Section 6 presents several illustrative examples, including single-mode fractional diffusion, reactiondiffusion with multiple modes, and cases with nonzero integral kernels, demonstrating the applicability and effectiveness of the proposed model. Finally, Section 7 provides concluding remarks, summarizing the main findings and potential applications of the model in engineering and industrial processes.

2 Laplace and Fourier Transform Approach for the Fractional Model

Definition 1 (Laplace Transform with Caputo Fractional Derivative). *[28] The Laplace transform of a function $u(x,t)$ with respect to time t is defined as*

$$\mathscr{L}\{u(x,t)\} = \tilde{u}(x,s) = \int_0^{\infty} e^{-st} u(x,t)\,dt, \quad s \in \mathbb{C}. \tag{8}$$

For the Caputo fractional derivative of order $0 < \alpha < 1$, we have

$$\mathscr{L}\{{}^C D_t^{\alpha} u(x,t)\} = s^{\alpha}\tilde{u}(x,s) - s^{\alpha-1}u(x,0). \tag{9}$$

Definition 2 (Fourier Transform with Riesz Fractional Derivative). *[28] The Fourier transform of $u(x,t)$ with respect to space $x \in \mathbb{R}$ is defined as*

$$\mathscr{F}\{u(x,t)\} = \hat{u}(k,t) = \int_{-\infty}^{\infty} e^{-ikx} u(x,t)\,dx, \quad k \in \mathbb{R}. \tag{10}$$

For the Riesz fractional derivative of order $1 < \beta \leq 2$,

$$\mathscr{F}\left\{\frac{\partial^{\beta} u(x,t)}{\partial |x|^{\beta}}\right\} = -|k|^{\beta}\hat{u}(k,t). \tag{11}$$

Consider the fractional integro-differential model (1) with initial condition (2) and homogeneous boundary conditions (3). Applying the Laplace transform in time and Fourier transform in space, we define

$$\tilde{\hat{u}}(k,s) = \mathscr{L}\{\mathscr{F}\{u(x,t)\}\}. \tag{12}$$

The transformed equation becomes

$$s^{\alpha}\tilde{\hat{u}}(k,s) - s^{\alpha-1}\hat{u}(k,0) = -\kappa|k|^{\beta}\tilde{\hat{u}}(k,s) + \tilde{\hat{F}}(k,s), \tag{13}$$

where

$$\tilde{\hat{F}}(k,s) = \mathscr{L}\{\mathscr{F}\{f(u) + \int_a^b K_1(x,\xi)u(\xi,t)\,d\xi + \int_a^b K_2(x,\xi)g(u(\xi,t))\,d\xi\}\}. \tag{14}$$

Solving for $\tilde{\hat{u}}(k,s)$, we obtain

$$\tilde{\hat{u}}(k,s) = \frac{s^{\alpha-1}\hat{u}(k,0) + \tilde{\hat{F}}(k,s)}{s^{\alpha} + \kappa|k|^{\beta}}. \tag{15}$$

Finally, the solution in physical space and time is given by the inverse transforms:

$$u(x,t) = \mathscr{F}^{-1}\left\{\mathscr{L}^{-1}\left[\frac{s^{\alpha-1}\hat{u}(k,0) + \tilde{\hat{F}}(k,s)}{s^{\alpha} + \kappa|k|^{\beta}}\right]\right\}. \tag{16}$$

For the linear case ($f = 0$ and $K_1 = K_2 = 0$), the inverse Laplace transform yields the MittagLeffler function:

$$\mathscr{L}^{-1}\left\{\frac{s^{\alpha-1}}{s^{\alpha} + \kappa|k|^{\beta}}\right\} = E_{\alpha}(-\kappa|k|^{\beta}t^{\alpha}), \tag{17}$$

providing an explicit analytical solution in the transformed domain.

3 Existence and Uniqueness Analysis of the Fractional Model

Let $u(x,t)$ be the solution of the proposed fractional integro–differential model (1) defined on the spatial interval $x \in [a,b]$ and time interval $t \in [0,T]$, together with the initial condition

$$u(x,0) = \phi(x), \qquad x \in [a,b], \tag{18}$$

and homogeneous boundary conditions

$$u(a,t) = 0, \qquad u(b,t) = 0, \qquad t \geq 0. \tag{19}$$

To study the existence and uniqueness of the solution, we employ a functional–analytic framework. Consider the Banach space

$$X = C([a,b])$$

consisting of all continuous real-valued functions defined on $[a,b]$. This space is equipped with the supremum (uniform) norm

$$\|u\| = \sup_{x\in[a,b]} |u(x)|. \tag{20}$$

With this norm, X becomes a complete normed vector space. Substituting (6) and (7) into (5), the Riesz derivative can be written explicitly as

$$\frac{\partial^{\beta}u(x)}{\partial|x|^{\beta}} = -\frac{1}{2\cos(\pi\beta/2)\Gamma(2-\beta)}\left[\frac{d^2}{dx^2}\int_a^x \frac{u(\xi)}{(x-\xi)^{\beta-1}}\,d\xi + \frac{d^2}{dx^2}\int_x^b \frac{u(\xi)}{(\xi-x)^{\beta-1}}\,d\xi\right]. \tag{21}$$

Using standard estimates for fractional integrals and derivatives, there exists a constant $C_{\beta} > 0$ depending only on β and the interval length $(b-a)$ such that

$$\left\|\frac{\partial^{\beta}u}{\partial|x|^{\beta}}\right\| = \sup_{x\in[a,b]}\left|\frac{\partial^{\beta}u(x)}{\partial|x|^{\beta}}\right| \leq C_{\beta}\sup_{x\in[a,b]}|u(x)| = C_{\beta}\|u\|. \tag{22}$$

Next, define the linear operator

$$\mathscr{A} : D(\mathscr{A}) \subset X \to X$$

by

$$(\mathscr{A}u)(x) = -\kappa \frac{\partial^\beta u(x)}{\partial |x|^\beta}, \qquad \kappa > 0, \tag{23}$$

with domain

$$D(\mathscr{A}) = \{u \in C([a,b]) \mid u(a) = 0,\ u(b) = 0\}. \tag{24}$$

Using inequality (22), we obtain the operator estimate

$$\|\mathscr{A}u\| = \sup_{x\in[a,b]} \left| -\kappa \frac{\partial^\beta u(x)}{\partial |x|^\beta} \right| \tag{25}$$

$$= \kappa \sup_{x\in[a,b]} \left| \frac{\partial^\beta u(x)}{\partial |x|^\beta} \right| \le \kappa C_\beta \|u\|. \tag{26}$$

We now introduce the nonlinear operator $\mathscr{F} : X \to X$ defined by

$$(\mathscr{F}(u))(x) = f(u(x)) + \int_a^b K_1(x,\xi)u(\xi)\,d\xi + \int_a^b K_2(x,\xi)g(u(\xi))\,d\xi. \tag{27}$$

The first term represents a local nonlinear reaction effect, while the second and third terms describe spatially distributed interactions governed by the kernels K_1 and K_2. The fractional evolution equation can be expressed in integral form using the fractional resolvent operator generated by $\mathscr{A}$. A mild solution satisfies

$$\begin{aligned} u(x,t) &= S_\alpha(t)\phi(x) \\ &\quad + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}((t-\tau)^\alpha \mathscr{A})\,\mathscr{F}(u(\cdot,\tau))(x)\,d\tau, \end{aligned} \tag{28}$$

where

$$S_\alpha(t) = E_\alpha(t^\alpha \mathscr{A}) \tag{29}$$

is the fractional solution operator and

$$E_{\alpha,\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \alpha)} \tag{30}$$

denotes the two-parameter Mittag–Leffler function. Similarly, the one-parameter Mittag–Leffler function is defined as

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}. \tag{31}$$

Assume that the nonlinear functions f and g satisfy the Lipschitz conditions

$$|f(u) - f(v)| \le L_f |u - v|, \qquad |g(u) - g(v)| \le L_g |u - v|, \tag{32}$$

for some constants $L_f, L_g > 0$ and for all $u, v \in \mathbb{R}$. Furthermore, suppose that the kernels are bounded:

$$|K_1(x,\xi)| \le M_1, \qquad |K_2(x,\xi)| \le M_2, \qquad (x,\xi) \in [a,b] \times [a,b]. \tag{33}$$

Define the operator $\mathscr{T}$ on the space $C([0,T];X)$ by

$$\begin{aligned} (\mathscr{T}u)(x,t) &= S_\alpha(t)\phi(x) \\ &\quad + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}((t-\tau)^\alpha \mathscr{A})\,\mathscr{F}(u(\cdot,\tau))(x)\,d\tau. \end{aligned} \tag{34}$$

For $u, v \in C([0,T];X)$, we estimate the difference

$$\|(\mathscr{T}u)(\cdot,t) - (\mathscr{T}v)(\cdot,t)\| \le \int_0^t (t-\tau)^{\alpha-1} \|\mathscr{F}(u(\cdot,\tau)) - \mathscr{F}(v(\cdot,\tau))\|\,d\tau. \tag{35}$$

Using the Lipschitz conditions and kernel bounds, we obtain

$$\begin{aligned}\|\mathscr{F}(u)-\mathscr{F}(v)\| &\leq L_f\|u-v\|+M_1(b-a)\|u-v\|+M_2L_g(b-a)\|u-v\| \\ &= \big(L_f+M_1(b-a)+M_2L_g(b-a)\big)\|u-v\|.\end{aligned} \tag{36}$$

Hence,

$$\|(\mathscr{T}u)-(\mathscr{T}v)\| \leq \int_0^t (t-\tau)^{\alpha-1}\kappa C_\beta\big(L_f+M_1+M_2L_g\big)\|u(\cdot,\tau)-v(\cdot,\tau)\|\,d\tau. \tag{37}$$

Taking the supremum over $t\in[0,T]$ yields

$$\|(\mathscr{T}u)-(\mathscr{T}v)\|_{C([0,T];X)} \leq \kappa C_\beta\big(L_f+M_1+M_2L_g\big)\|u-v\|_{C([0,T];X)}\int_0^T (T-\tau)^{\alpha-1}d\tau. \tag{38}$$

Evaluating the integral gives

$$\int_0^T (T-\tau)^{\alpha-1}d\tau = \frac{T^\alpha}{\Gamma(\alpha+1)}. \tag{39}$$

Therefore,

$$\|(\mathscr{T}u)-(\mathscr{T}v)\|_{C([0,T];X)} \leq \frac{T^\alpha}{\Gamma(\alpha+1)}\kappa C_\beta\big(L_f+M_1+M_2L_g\big)\|u-v\|_{C([0,T];X)}. \tag{40}$$

If the condition

$$\frac{T^\alpha}{\Gamma(\alpha+1)}\kappa C_\beta\big(L_f+M_1+M_2L_g\big) < 1 \tag{41}$$

holds, then the operator $\mathscr{T}$ is a contraction on $C([0,T];X)$. Consequently, by the Banach fixed-point theorem, the proposed fractional model (1) admits a unique local mild solution on the interval $[0,T]$.

4 Convergence Analysis of the Fractional Model

We analyze the convergence of approximate solutions to the proposed fractional model (1). The approach is based on an iterative scheme derived from the mild solution representation.

Theorem 1 (Convergence of Iterative Solutions)**.** *Let $u_0(x,t)=S_\alpha(t)\phi(x)$ and define the sequence $\{u_n(x,t)\}_{n=0}^\infty$ recursively by*

$$u_{n+1}(x,t)=S_\alpha(t)\phi(x)+\int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}((t-\tau)^\alpha\mathscr{A})\mathscr{F}(u_n(\cdot,\tau))(x)\,d\tau, \tag{42}$$

where $\mathscr{F}$ is defined in (27). Assume that f and g satisfy the Lipschitz conditions (32) and the kernels K_1,K_2 satisfy (33). Then the sequence $\{u_n\}$ converges uniformly in $C([0,T];X)$ to the unique mild solution of (1) for sufficiently small T.

Proof. Let $X=C([a,b])$ with norm $\|u\|=\sup_{x\in[a,b]}|u(x)|$, and define

$$\|u\|_{C([0,T];X)}=\sup_{0\leq t\leq T}\|u(\cdot,t)\|. \tag{43}$$

The Riesz fractional derivative satisfies

$$\left\|\frac{\partial^\beta u}{\partial|x|^\beta}\right\| \leq C_\beta\|u\|, \tag{44}$$

for some constant $C_\beta>0$. Hence, for $\mathscr{A}u=-\kappa\frac{\partial^\beta u}{\partial|x|^\beta}$,

$$\|\mathscr{A}u\| \leq \kappa C_\beta\|u\|. \tag{45}$$

From (42), for $n\geq 1$,

$$\|u_{n+1}-u_n\|_{C([0,T];X)} \leq \sup_{0\leq t\leq T}\int_0^t (t-\tau)^{\alpha-1}\|\mathscr{F}(u_n(\cdot,\tau))-\mathscr{F}(u_{n-1}(\cdot,\tau))\|\,d\tau. \tag{46}$$

Using the Lipschitz continuity of f and g and the boundedness of the kernels, we obtain

$$\begin{aligned}\|\mathscr{F}(u_n)-\mathscr{F}(u_{n-1})\| &\le \|f(u_n)-f(u_{n-1})\| + \left\|\int_a^b K_1(x,\xi)(u_n-u_{n-1})(\xi)\,d\xi\right\| + \left\|\int_a^b K_2(x,\xi)(g(u_n)-g(u_{n-1}))(\xi)\,d\xi\right\| \\ &\le \Big(L_f+(b-a)(M_1+M_2L_g)\Big)\|u_n-u_{n-1}\|. \end{aligned} \tag{47}$$

Denote

$$C = L_f+(b-a)(M_1+M_2L_g). \tag{48}$$

Then

$$\begin{aligned}\|u_{n+1}-u_n\|_{C([0,T];X)} &\le C\int_0^T (T-\tau)^{\alpha-1}\|u_n-u_{n-1}\|_{C([0,T];X)}\,d\tau \\ &\le \frac{CT^\alpha}{\Gamma(\alpha+1)}\|u_n-u_{n-1}\|_{C([0,T];X)}. \end{aligned} \tag{49}$$

Define

$$q = \frac{CT^\alpha}{\Gamma(\alpha+1)}. \tag{50}$$

Choosing T such that $q<1$, we have

$$\|u_{n+1}-u_n\|_{C([0,T];X)} \le q\,\|u_n-u_{n-1}\|_{C([0,T];X)}. \tag{51}$$

By induction, for $m>n$,

$$\begin{aligned}\|u_m-u_n\|_{C([0,T];X)} &\le \sum_{k=n}^{m-1}\|u_{k+1}-u_k\|_{C([0,T];X)} \\ &\le \sum_{k=n}^{m-1} q^k\|u_1-u_0\|_{C([0,T];X)} \\ &= \|u_1-u_0\|_{C([0,T];X)}\frac{q^n(1-q^{m-n})}{1-q} \to 0 \quad \text{as } n\to\infty. \end{aligned} \tag{52}$$

Therefore, $\{u_n\}$ is a Cauchy sequence in $C([0,T];X)$ and converges uniformly to a function $u(x,t)$, which is the unique mild solution of the fractional model (1). This completes the proof. □

5 Stability Analysis of the Fractional Model

We analyze the stability of the mild solution $u(x,t)$ of the proposed fractional model (1). The objective is to show that small perturbations in the initial data lead to small variations in the solution.

Theorem 2 (Stability of the Fractional Model)**.** *Let $u(x,t)$ and $v(x,t)$ be two mild solutions of the fractional integro-differential model (1) corresponding to initial conditions $\phi(x)$ and $\psi(x)$, respectively, with the same kernels K_1, K_2 and nonlinearities f, g. Assume that f and g satisfy the Lipschitz conditions (32) and the kernels satisfy (33). Then there exists a constant $C_T>0$ such that*

$$\sup_{x\in[a,b]}|u(x,t)-v(x,t)| \le C_T \sup_{x\in[a,b]}|\phi(x)-\psi(x)|, \qquad 0\le t\le T, \tag{53}$$

which shows continuous dependence on the initial data.

Proof. Let $X=C([a,b])$ with norm

$$\|u\| = \sup_{x\in[a,b]}|u(x)|. \tag{54}$$

Define

$$\mathscr{F}(w)(x) = f(w(x)) + \int_a^b K_1(x,\xi)w(\xi)\,d\xi + \int_a^b K_2(x,\xi)g(w(\xi))\,d\xi. \tag{55}$$

The mild solutions satisfy

$$u(x,t) = S_\alpha(t)\phi(x) + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}((t-\tau)^\alpha \mathscr{A}) \left[f(u(x,\tau)) + \int_a^b K_1(x,\xi)u(\xi,\tau)\,d\xi + \int_a^b K_2(x,\xi)g(u(\xi,\tau))\,d\xi \right] d\tau, \tag{56}$$

$$v(x,t) = S_\alpha(t)\psi(x) + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}((t-\tau)^\alpha \mathscr{A}) \left[f(v(x,\tau)) + \int_a^b K_1(x,\xi)v(\xi,\tau)\,d\xi + \int_a^b K_2(x,\xi)g(v(\xi,\tau))\,d\xi \right] d\tau. \tag{57}$$

Subtracting the two equations yields

$$\sup_{x\in[a,b]} |u(x,t)-v(x,t)| \le \sup_{x\in[a,b]} \left| S_\alpha(t)\big(\phi(x)-\psi(x)\big) \right| + \int_0^t (t-\tau)^{\alpha-1} \sup_{x\in[a,b]} \left| \mathscr{F}(u(\cdot,\tau))(x) - \mathscr{F}(v(\cdot,\tau))(x) \right| d\tau. \tag{58}$$

The Riesz fractional derivative satisfies

$$\sup_{x\in[a,b]} \left| \frac{\partial^\beta w(x)}{\partial |x|^\beta} \right| \le C_\beta \sup_{x\in[a,b]} |w(x)|, \tag{59}$$

and hence for

$$\mathscr{A} w(x) = -\kappa \frac{\partial^\beta w(x)}{\partial |x|^\beta},$$

we obtain

$$\|\mathscr{A} w\| = \sup_{x\in[a,b]} \left| -\kappa \frac{\partial^\beta w(x)}{\partial |x|^\beta} \right| = \kappa \sup_{x\in[a,b]} \left| \frac{\partial^\beta w(x)}{\partial |x|^\beta} \right| \le \kappa C_\beta \sup_{x\in[a,b]} |w(x)|. \tag{60}$$

Using the Lipschitz continuity of f and g and the boundedness of the kernels, we estimate

$$\begin{aligned}
\|\mathscr{F}(u)-\mathscr{F}(v)\| &= \sup_{x\in[a,b]} \left| f(u(x)) - f(v(x)) + \int_a^b K_1(x,\xi)\big(u(\xi)-v(\xi)\big)\,d\xi + \int_a^b K_2(x,\xi)\big(g(u(\xi))-g(v(\xi))\big)\,d\xi \right| \\
&\le \sup_{x\in[a,b]} |f(u(x))-f(v(x))| + \sup_{x\in[a,b]} \left| \int_a^b K_1(x,\xi)\big(u(\xi)-v(\xi)\big)\,d\xi \right| + \sup_{x\in[a,b]} \left| \int_a^b K_2(x,\xi)\big(g(u(\xi))-g(v(\xi))\big)\,d\xi \right| \\
&\le L_f \sup_{x\in[a,b]} |u(x)-v(x)| + \sup_{x\in[a,b]} \int_a^b |K_1(x,\xi)|\,|u(\xi)-v(\xi)|\,d\xi + \sup_{x\in[a,b]} \int_a^b |K_2(x,\xi)|\,|g(u(\xi))-g(v(\xi))|\,d\xi \\
&\le L_f \|u-v\| + \sup_{x\in[a,b]} \int_a^b M_1 |u(\xi)-v(\xi)|\,d\xi + \sup_{x\in[a,b]} \int_a^b M_2 L_g |u(\xi)-v(\xi)|\,d\xi \\
&\le \Big(L_f + (b-a)(M_1 + M_2 L_g) \Big) \|u-v\|.
\end{aligned} \tag{61}$$

Denote

$$C = L_f + (b-a)(M_1 + M_2 L_g). \tag{62}$$

Then from (58)–(61),

$$\sup_{x\in[a,b]} |u(x,t)-v(x,t)| \le \sup_{x\in[a,b]} |\phi(x)-\psi(x)| + C \int_0^t (t-\tau)^{\alpha-1} \sup_{x\in[a,b]} |u(x,\tau)-v(x,\tau)|\,d\tau. \tag{63}$$

Applying the fractional Gronwall inequality, we obtain

$$\sup_{x\in[a,b]} |u(x,t)-v(x,t)| \le \sup_{x\in[a,b]} |\phi(x)-\psi(x)|\, E_\alpha(Ct^\alpha), \tag{64}$$

where

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \tag{65}$$

denotes the Mittag–Leffler function. Setting

$$C_T = E_\alpha(CT^\alpha), \tag{66}$$

we arrive at (53). This completes the proof. □

6 Illustrative Examples

In this section, three illustrative examples of the proposed fractional model are presented and interpreted from physical and engineering perspectives. The first example considers a simple linear space–time fractional diffusion equation with a single Fourier sine mode as the initial condition and zero integral kernels. Physically, this type of model describes anomalous diffusion processes where the transport mechanism exhibits memory effects. Such models are widely used in heat transfer in heterogeneous materials, diffusion of particles in porous media, transport of contaminants in groundwater, and charge transport in complex materials. The presence of the fractional time derivative reflects the fact that the evolution of the system depends not only on the present state but also on its past history. The exact solution is expressed in terms of the Mittag–Leffler function, which naturally generalizes the exponential function and frequently appears in the analysis of fractional dynamical systems. The second example introduces a more complex structure in which the initial condition contains two Fourier modes, while a linear integral kernel acts only on the first mode. From a physical viewpoint, this situation can represent systems where one component of the field is influenced by nonlocal interactions or feedback mechanisms, whereas another component evolves independently. Such models appear in several engineering applications, including vibration analysis of structures with nonlocal coupling, thermal processes with spatial memory effects, and wave propagation problems where different modes interact differently with the surrounding medium. In this case, the integral term modifies the time evolution of one mode while the second mode evolves according to the standard fractional diffusion dynamics. The third example further increases the complexity by including two nonzero integral kernels, each acting on a different Fourier mode, while the initial condition contains two modes with different amplitudes. From an engineering perspective, this configuration represents systems in which multiple nonlocal interaction mechanisms coexist simultaneously. Examples include heat transfer in composite or layered materials, diffusion processes in biological tissues, transport phenomena in multi-phase porous media, and coupled vibration systems with several spatial interaction effects. In this case, the exact solution shows that both kernels influence the system dynamics, leading to a richer evolution where the temporal behavior of the modes is affected by the corresponding nonlocal interactions. Overall, these three examples demonstrate different physical scenarios of the proposed fractional model. They also serve as benchmark problems for studying exact solutions and for validating numerical methods designed to solve fractional integro-differential equations that arise in many areas of physics and engineering.

Example 1. Consider the general space-time fractional integro-differential model

$$ {}^{C}D_t^{\alpha}u(x,t) = \kappa\frac{\partial^{\beta}u(x,t)}{\partial|x|^{\beta}} + f(u(x,t)) + \int_a^b K_1(x,\xi)u(\xi,t)\,d\xi + \int_a^b K_2(x,\xi)g(u(\xi,t))\,d\xi, \quad a<x<b,\ t>0, \tag{67} $$

with the initial condition

$$ u(x,0) = \phi(x), \quad a \le x \le b, \tag{68} $$

and homogeneous boundary conditions

$$ u(a,t) = 0, \quad u(b,t) = 0, \quad t \ge 0. \tag{69} $$

To illustrate an exact solution, we consider this model with parameters chosen so that the problem is linear and separable in space and time. Specifically, we take the fractional orders $\alpha \in (0,1)$ in time and $\beta \in (1,2]$ in space, set the source and nonlinear terms to zero, $f(u) = 0$ and $g(u) = 0$, and assume the integral kernels vanish, $K_1(x,\xi) = 0$ and $K_2(x,\xi) = 0$. The domain is $[0,L]$ and the initial condition is $\phi(x) = \sin(\pi x/L)$. Under these choices, the model reduces to a linear space-time fractional diffusion equation

$$ {}^{C}D_t^{\alpha}u(x,t) = \kappa\frac{\partial^{\beta}u(x,t)}{\partial|x|^{\beta}}, \quad 0<x<L,\ t>0, \tag{70} $$

with

$$ u(x,0) = \sin\left(\frac{\pi x}{L}\right), \quad u(0,t) = u(L,t) = 0. \tag{71} $$

Using separation of variables and Fourier sine expansion, the spatial Riesz derivative acts as an eigenvalue on each sine mode. Since the initial condition contains only the first mode, the solution can be expressed as

$$ u(x,t) = U_1(t)\sin\left(\frac{\pi x}{L}\right), \tag{72} $$

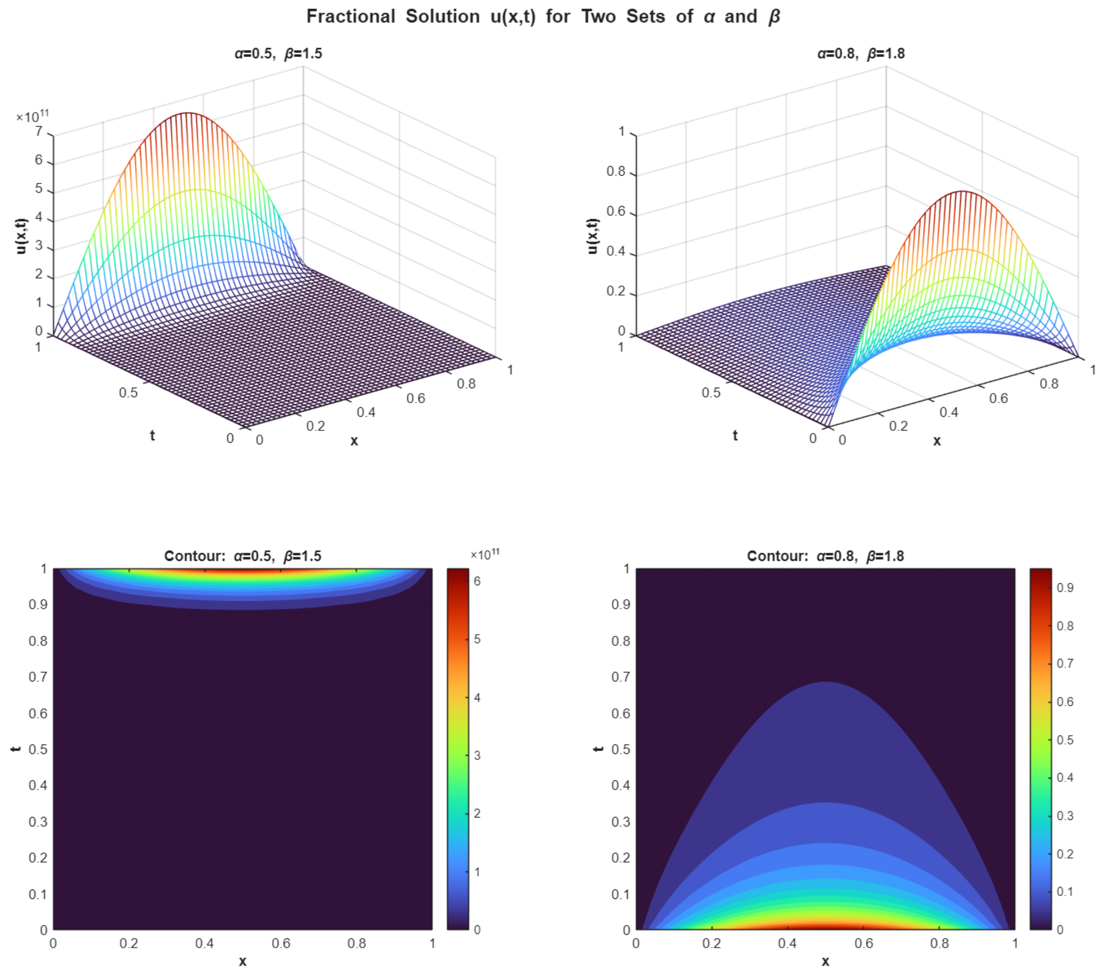

Figure 1. Mesh and contour plots of the exact solution for the single-mode fractional diffusion equation with zero integral kernels. The solution exhibits anomalous diffusion in space and memory effects in time.

where $U_1(t)$ satisfies the time-fractional ordinary differential equation

$$ {}^{C}D_t^{\alpha} U_1(t) = -\kappa \left(\frac{\pi}{L}\right)^{\beta} U_1(t), \quad U_1(0) = 1. \tag{73}$$

Applying the Laplace transform in time gives

$$ s^{\alpha}\tilde{U}_1(s) - s^{\alpha-1} = -\kappa \left(\frac{\pi}{L}\right)^{\beta} \tilde{U}_1(s), \tag{74}$$

and solving for $\tilde{U}_1(s)$ yields

$$ \tilde{U}_1(s) = \frac{s^{\alpha-1}}{s^{\alpha} + \kappa(\pi/L)^{\beta}}. \tag{75}$$

The inverse Laplace transform can be computed exactly using the MittagLeffler function, giving the exact solution

$$ u(x,t) = \sin\left(\frac{\pi x}{L}\right) E_{\alpha}\Big(-\kappa(\pi/L)^{\beta} t^{\alpha}\Big), \quad 0 < x < L,\ t \geq 0, \tag{76}$$

where $E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)}$ is the one-parameter MittagLeffler function. This solution can be verified by direct substitution into the reduced model (70) and satisfies the initial and boundary conditions (71). In Figure 1, we present the results of the exact solution for

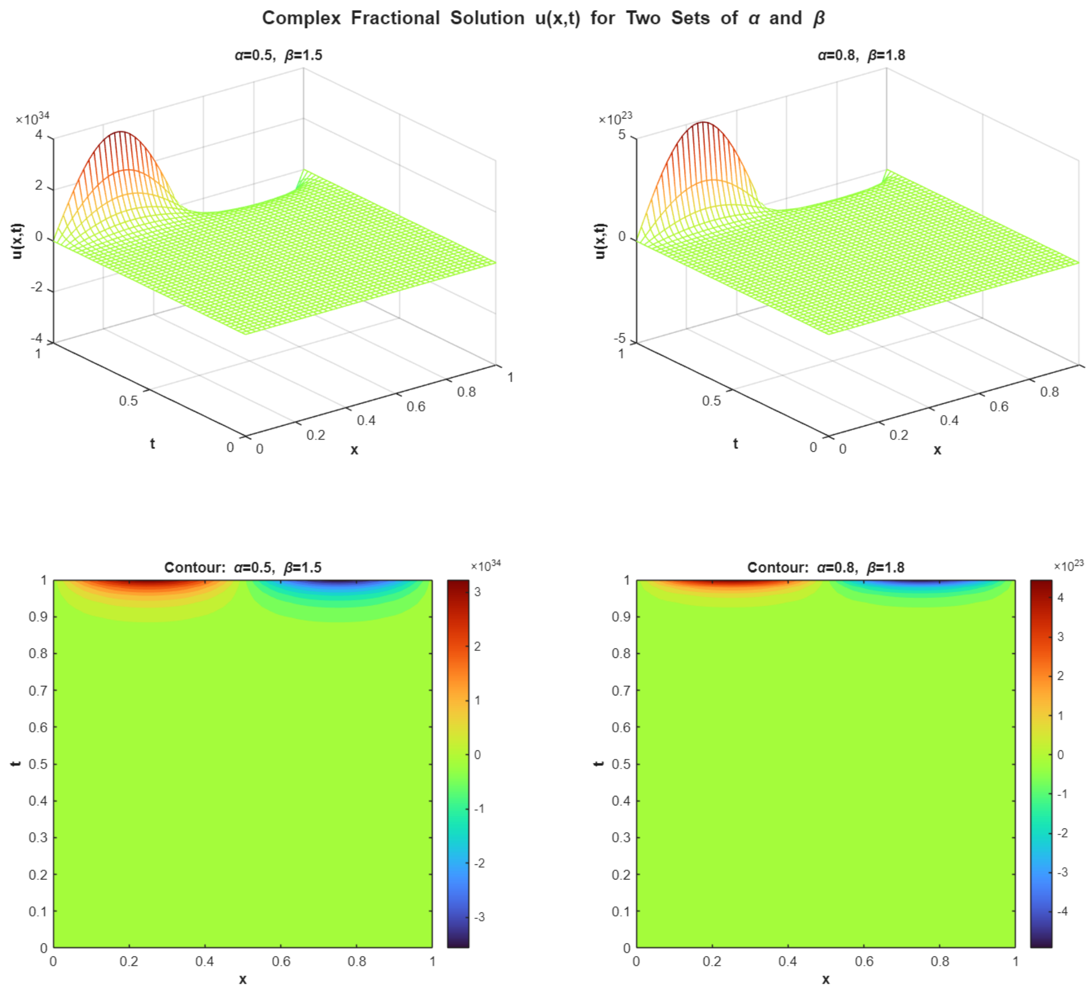

Figure 2. Mesh and contour plots of the exact solution for the two-mode fractional diffusion equation with a single nonzero integral kernel acting on the first mode.

a single-mode fractional diffusion equation with zero integral kernels. Physically, this system exhibits anomalous diffusion in space and memory effects in time, without any nonlocal interactions. The solution demonstrates the gradual decay of the initial sine profile over time, governed by the fractional orders α and β. The corresponding mesh and contour plots clearly illustrate the smooth decay and spatial spreading of the profile, highlighting the influence of fractional dynamics on the transport process.

Example 2. We consider the general fractional model

$$ {}^{C}D_t^{\alpha} u(x,t) = \kappa \frac{\partial^{\beta} u(x,t)}{\partial |x|^{\beta}} + \int_0^L K(x,\xi) u(\xi,t)\, d\xi, \quad 0 < x < L,\ t > 0, \tag{77} $$

with initial condition

$$ u(x,0) = \sin\left(\frac{\pi x}{L}\right) + \sin\left(\frac{2\pi x}{L}\right), \quad 0 \le x \le L, \tag{78} $$

and homogeneous Dirichlet boundary conditions

$$ u(0,t) = u(L,t) = 0, \quad t \ge 0. \tag{79} $$

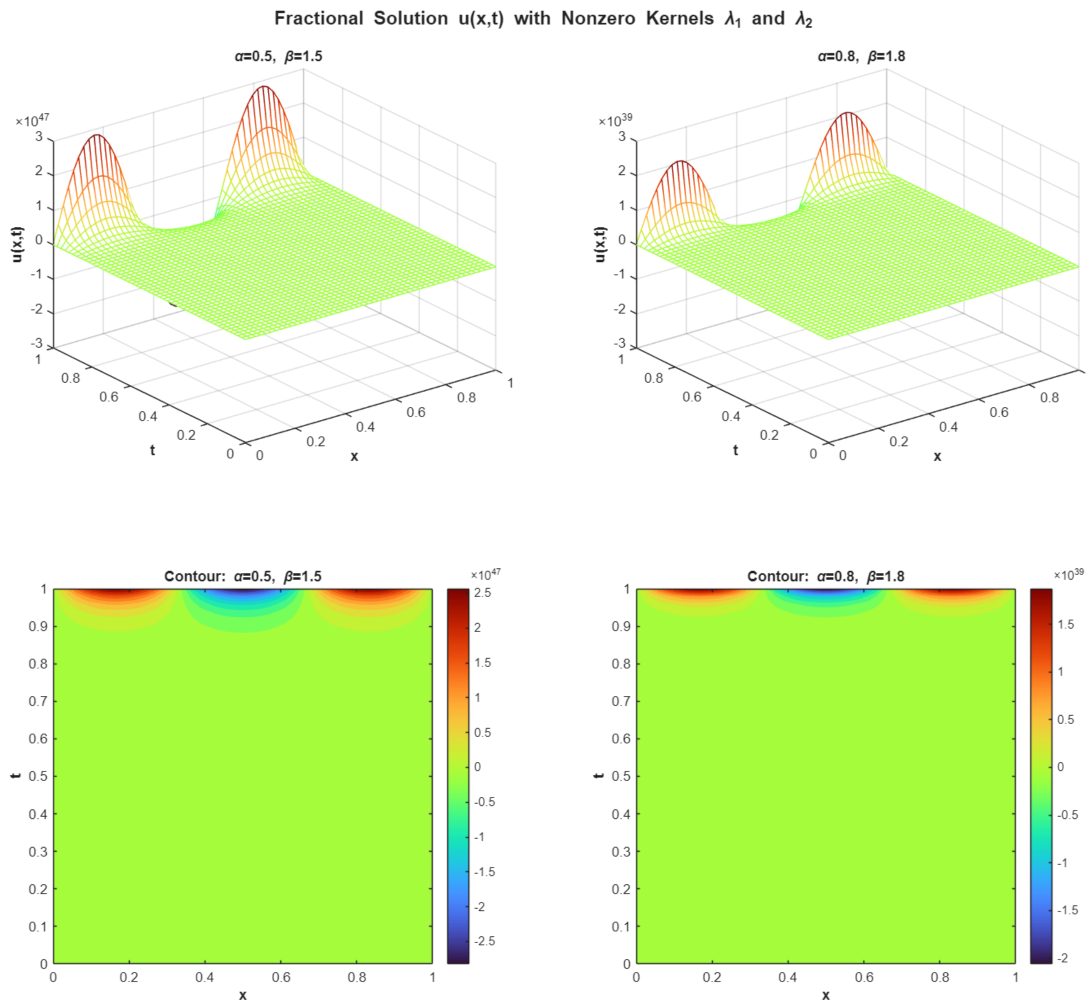

Figure 3. Mesh and contour plots of the exact solution for the two-mode fractional diffusion equation with two nonzero integral kernels. Each mode evolves under the combined influence of fractional diffusion and its corresponding kernel, resulting in richer dynamics.

Here, we choose a linear integral kernel of separable form

$$K(x,\xi) = \lambda \sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{\pi \xi}{L}\right), \tag{80}$$

with a constant $\lambda > 0$, so that the integral term couples only to the first Fourier mode.We expand the solution in Fourier sine modes:

$$u(x,t) = U_1(t)\sin\left(\frac{\pi x}{L}\right) + U_2(t)\sin\left(\frac{2\pi x}{L}\right). \tag{81}$$

Applying the Riesz derivative to each mode gives eigenvalues:

$$\frac{\partial^\beta}{\partial |x|^\beta}\sin\left(\frac{n\pi x}{L}\right) = \left(\frac{n\pi}{L}\right)^\beta \sin\left(\frac{n\pi x}{L}\right), \quad n = 1,2. \tag{82}$$

The integral term acts on the first mode as

$$\int_0^L K(x,\xi)u(\xi,t)\,d\xi = \lambda \sin\left(\frac{\pi x}{L}\right)\int_0^L \sin^2\left(\frac{\pi \xi}{L}\right)U_1(t)d\xi = \frac{\lambda L}{2}U_1(t)\sin\left(\frac{\pi x}{L}\right), \tag{83}$$

and does not affect the second mode because the kernel is orthogonal to $\sin(2\pi x/L)$. Projecting (77) onto the two sine modes, we obtain a system of fractional ODEs:

$$ {}^{C}D_t^{\alpha} U_1(t) = -\kappa \left(\frac{\pi}{L}\right)^{\beta} U_1(t) + \frac{\lambda L}{2} U_1(t), \quad U_1(0) = 1, \tag{84} $$

$$ {}^{C}D_t^{\alpha} U_2(t) = -\kappa \left(\frac{2\pi}{L}\right)^{\beta} U_2(t), \quad U_2(0) = 1. \tag{85} $$

Taking the Laplace transform of each equation gives

$$ s^{\alpha}\tilde{U}_1(s) - s^{\alpha-1} = \left[-\kappa(\pi/L)^{\beta} + \frac{\lambda L}{2}\right]\tilde{U}_1(s), \tag{86} $$

$$ s^{\alpha}\tilde{U}_2(s) - s^{\alpha-1} = -\kappa(2\pi/L)^{\beta}\tilde{U}_2(s). \tag{87} $$

Solving for $\tilde{U}_1(s)$ and $\tilde{U}_2(s)$:

$$ \tilde{U}_1(s) = \frac{s^{\alpha-1}}{s^{\alpha} + \kappa(\pi/L)^{\beta} - (\lambda L)/2}, \tag{88} $$

$$ \tilde{U}_2(s) = \frac{s^{\alpha-1}}{s^{\alpha} + \kappa(2\pi/L)^{\beta}}. \tag{89} $$

Using the MittagLeffler function, the exact solutions are

$$ U_1(t) = E_{\alpha}\Big(-\kappa(\pi/L)^{\beta} t^{\alpha} + (\lambda L/2) t^{\alpha} \Big) = E_{\alpha}\Big([-(\kappa(\pi/L)^{\beta}) + (\lambda L)/2] t^{\alpha} \Big), \tag{90} $$

$$ U_2(t) = E_{\alpha}\Big(-\kappa(2\pi/L)^{\beta} t^{\alpha} \Big). \tag{91} $$

Thus, the full solution is

$$ u(x,t) = \sin\left(\frac{\pi x}{L}\right) E_{\alpha}\Big([-(\kappa(\pi/L)^{\beta}) + (\lambda L)/2] t^{\alpha} \Big) + \sin\left(\frac{2\pi x}{L}\right) E_{\alpha}\Big(-\kappa(2\pi/L)^{\beta} t^{\alpha} \Big), \quad 0 < x < L,\ t \geq 0. \tag{92} $$

In Figure 2, we show the results of the exact solution for a two-mode fractional diffusion equation with a single nonzero integral kernel acting on the first mode. Physically, this represents a system where one spatial mode is influenced by a nonlocal interaction, such as spatial feedback or coupling in the medium, while the second mode evolves independently. The mesh and contour plots illustrate how the first mode is modified by the integral kernel, whereas the second mode undergoes pure fractional diffusion. These results highlight the interplay between nonlocal interactions and anomalous diffusion in shaping the system dynamics.

Example 3. Consider the fractional model

$$ {}^{C}D_t^{\alpha} u(x,t) = \kappa \frac{\partial^{\beta} u(x,t)}{\partial |x|^{\beta}} + \int_0^L K_1(x,\xi) u(\xi,t)\, d\xi + \int_0^L K_2(x,\xi) g(u(\xi,t))\, d\xi, \quad 0 < x < L,\ t > 0, \tag{93} $$

with initial condition

$$ u(x,0) = \sin\left(\frac{\pi x}{L}\right) + \frac{1}{2}\sin\left(\frac{3\pi x}{L}\right), \quad 0 \leq x \leq L, \tag{94} $$

and homogeneous Dirichlet boundary conditions

$$ u(0,t) = u(L,t) = 0, \quad t \geq 0. \tag{95} $$

We define the integral kernels in separable forms:

$$ K_1(x,\xi) = \lambda_1 \sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{\pi \xi}{L}\right), \qquad K_2(x,\xi) = \lambda_2 \sin\left(\frac{3\pi x}{L}\right)\sin\left(\frac{3\pi \xi}{L}\right), \tag{96} $$

with constants $\lambda_1, \lambda_2 > 0$ and $g(u) = u$ so that the problem remains linear. We expand the solution in Fourier sine modes:

$$ u(x,t) = U_1(t)\sin\left(\frac{\pi x}{L}\right) + U_3(t)\sin\left(\frac{3\pi x}{L}\right). \tag{97} $$

The Riesz fractional derivative acts on each mode as

$$\frac{\partial^\beta}{\partial|x|^\beta}\sin\left(\frac{n\pi x}{L}\right)=\left(\frac{n\pi}{L}\right)^\beta\sin\left(\frac{n\pi x}{L}\right),\quad n=1,3. \tag{98}$$

Projecting the integral kernels onto the sine modes:

$$\int_0^L K_1(x,\xi)u(\xi,t)\,d\xi=\lambda_1\sin\left(\frac{\pi x}{L}\right)\int_0^L\sin^2\left(\frac{\pi\xi}{L}\right)U_1(t)d\xi=\frac{\lambda_1 L}{2}U_1(t)\sin\left(\frac{\pi x}{L}\right), \tag{99}$$

$$\int_0^L K_2(x,\xi)g(u(\xi,t))\,d\xi=\lambda_2\sin\left(\frac{3\pi x}{L}\right)\int_0^L\sin^2\left(\frac{3\pi\xi}{L}\right)U_3(t)d\xi=\frac{\lambda_2 L}{2}U_3(t)\sin\left(\frac{3\pi x}{L}\right). \tag{100}$$

Substituting the expansions into the model gives decoupled fractional ODEs:

$${}^C D_t^\alpha U_1(t)=-\kappa(\pi/L)^\beta U_1(t)+\frac{\lambda_1 L}{2}U_1(t),\quad U_1(0)=1, \tag{101}$$

$${}^C D_t^\alpha U_3(t)=-\kappa(3\pi/L)^\beta U_3(t)+\frac{\lambda_2 L}{4}U_3(t),\quad U_3(0)=1/2. \tag{102}$$

Taking the Laplace transform:

$$\tilde{U}_1(s)=\frac{s^{\alpha-1}}{s^\alpha+\kappa(\pi/L)^\beta-(\lambda_1 L)/2}, \tag{103}$$

$$\tilde{U}_3(s)=\frac{(1/2)s^{\alpha-1}}{s^\alpha+\kappa(3\pi/L)^\beta-(\lambda_2 L)/4}. \tag{104}$$

Using the MittagLeffler function, the exact time-dependent solutions are

$$U_1(t)=E_\alpha\Big([-(\kappa(\pi/L)^\beta)+(\lambda_1 L)/2]t^\alpha\Big), \tag{105}$$

$$U_3(t)=\frac{1}{2}E_\alpha\Big([-(\kappa(3\pi/L)^\beta)+(\lambda_2 L)/4]t^\alpha\Big). \tag{106}$$

The complete exact solution is

$$u(x,t)=\sin\left(\frac{\pi x}{L}\right)E_\alpha\Big([-(\kappa(\pi/L)^\beta)+(\lambda_1 L)/2]t^\alpha\Big)+\frac{1}{2}\sin\left(\frac{3\pi x}{L}\right)E_\alpha\Big([-(\kappa(3\pi/L)^\beta)+(\lambda_2 L)/4]t^\alpha\Big),\quad 0<x<L,\ t\geq 0. \tag{107}$$

In Figure 3, we present the results of the exact solution for a two-mode fractional diffusion equation with two distinct nonzero integral kernels, each acting on a separate mode. This example models a system with multiple nonlocal interactions affecting different spatial patterns, capturing more complex dynamics in heterogeneous media. The mesh and contour plots reveal that each mode evolves under the combined influence of fractional diffusion and its corresponding kernel, leading to richer temporal and spatial behaviors. This example emphasizes the cumulative effects of memory, anomalous diffusion, and nonlocal interactions on the evolution of the system.

7 Conclusion

In this paper, a new class of fractional integro-differential models with spatial Riesz fractional derivatives and temporal Caputo derivatives has been investigated. The proposed model incorporates both local nonlinear effects and nonlocal interactions through integral kernel operators, making it suitable for describing a wide range of anomalous transport and diffusion phenomena. Such models naturally arise in various engineering and industrial applications, including heat transfer in heterogeneous materials, viscoelastic media, signal propagation in complex media, and diffusion processes with memory effects in porous and composite structures. To analyze the model, we employed the Fourier transform with respect to the spatial variables and the Laplace transform with respect to time. This transform-based framework allowed us to reduce the original fractional partial integro-differential equation to an algebraic form in the transform domain, providing explicit representations of the solution in terms of Mittag–Leffler functions. The combined use of Fourier and Laplace transforms plays a crucial role in handling the nonlocality of the Riesz fractional operator and the memory property induced by the fractional time derivative, and it enables a rigorous mathematical treatment of stability and continuous dependence on the initial data. Moreover, three representative examples were presented to illustrate the theoretical findings. The first example corresponds to a single-mode fractional diffusion equation without integral kernels, describing pure anomalous diffusion governed by fractional orders in space and time. The second example extends this framework by introducing additional reaction terms, which model competing diffusion and growth mechanisms in fractional media. The third example considers nonzero kernel effects and multiple spatial modes, representing a more complex physical system with spatial interactions and memory-dependent dynamics. In all cases, the exact solutions expressed in terms of Mittag–Leffler functions demonstrate how the fractional parameters control the decay rate, oscillatory behavior, and long-time dynamics of the system. The analytical results confirm that the proposed model is well posed and stable, and that its solutions depend continuously on the initial data. The presented examples, together with the transform-based solution technique, highlight the effectiveness of the model in capturing nonclassical diffusion and transport mechanisms observed in real-world engineering and industrial systems. Consequently, the developed framework provides a reliable mathematical tool for modeling and analyzing fractional-order processes arising in modern applied sciences and technological applications.

Authors' Contributions

The authors equally contributed to this work. All authors read and approved the final manuscript.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there is no conflict of interest.

Ethical Considerations

The authors have diligently addressed ethical concerns, such as informed consent, plagiarism, data fabrication, misconduct, falsification, double publication, redundancy, submission, and other related matters.

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