

Fibonacci Polynomial Reproducing Kernel Collocation Method for 2D Time-Fractional Diffusion Equations

Mahdi Emamjomeh*

Basic Sciences Group, Golpayegan College of Engineering, Isfahan University of Technology, Isfahan 84156-83111, Iran

* Corresponding author(s): m.emamjomeh@iut.ac.ir

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Abstract

In this paper, a collocation approach based on reproducing kernels is presented for the numerical solution of the 2D time-fractional diffusion equation. Some finite-dimensional positive definite reproducing kernel spaces are constructed using the bases of the Fibonacci polynomials. The spatial discretization in the proposed method is based on the Fibonacci polynomial reproducing kernel method, which is combined with a finite difference scheme for temporal discretization. Handling the boundary conditions in the numerical solution of partial differential equations is a challenging issue in numerical methods. To deal with the boundary conditions in the proposed method, the reproducing kernels are constructed in a way that exactly satisfy the boundary conditions. Some numerical simulations are conducted to prove the efficiency and ability of the Fibonacci kernel approach combined with the time-stepping scheme. The numerical results show the effectiveness and accuracy of the proposed method.

Keywords: Time-fractional diffusion equations, Fibonacci polynomial, Reproducing kernel collocation method, Semi-discrete, Polynomial reproducing kernel

Mathematics Subject Classification (2020): 35K57, 35R11, 65Mxx

1 Introduction

In recent decades, fractional calculus and its applications have exhibited various spatiotemporal patterns to provide more realistic models for physical and chemical processes as well as biological systems [1–3]. For example, they can describe many areas of science such as electromagnetism, electrochemistry, diffusion, and general transport theory [4, 5]. Du et al. [6] showed that we cannot obtain the analytical solutions for most fractional differential equations; therefore, they investigated practical methods to give numerical solutions to these equations. In recent research works, various numerical techniques have been introduced for dealing with problems containing derivatives of fractional orders [7–16].

In contrast to integer-order derivatives, fractional-order derivatives, from both the mathematical and engineering perspectives, focus on the nonlocality property of the derivative operators. As it is known, nonlocality means that the next behavior of a system depends not only on its current behavior, but also on its past behavior and state. The principle of the direct methods consists of various non-locality fractional operators to describe the real-world phenomena that depend on different factors such as time, space, etc. This

approach evaluates variable-order fractional operators with a non-stationary power-law kernel [17–19]. The direct methods have been widely applied to a variable-order fractional calculus in a variety of sciences and engineering phenomena, ranging from anomalous diffusion [20–22] and viscoelastic mechanics to control systems and petroleum engineering, among many other applications [23–28]. An important generalization of classical partial differential equations widely has been widely studied in recent years is fractional partial differential equations. The diffusion and the diffusion-wave equations with derivatives of fractional orders were introduced and intensely discussed in different applications for investigating anomalous diffusion in transport processes through complex systems, as well as in the mathematical literature [29]. The time-fractional diffusion equation (TFDE) has been successfully used to study superdiffusion and subdiffusion phenomena [30–32]. It has received wide attention in several research areas such as chemistry, biochemistry, finance, medicine, and systems biology [33–37]. Consequently, a time-fractional diffusion has been used to solve some practical problems as diverse as a part of the boundary data, initial data, and diffusion coefficients. Interestingly, some fractional diffusion inverse problems constitute an interesting challenge and an active research area [38–40]. In addition, since obtaining analytical solutions for fractional-order differential equations is often difficult or impossible, studying efficient numerical methods for solving this class of equations is of particular importance.

Zaremba considered the reproducing kernel Hilbert spaces (RKHS) in the early 20th century. After a few decades, some mathematicians such as Zigo, Bergman, and Bacchner have investigated the idea of reproducing kernel methods. Aronszajn and Bergman in 1950 provided the theory of the reproducing kernel Hilbert spaces method. Javan et al. [41] have used the RKHS method for obtaining the solution of nonlinear integral equations. In [42], an iterative reproducing kernel Hilbert space method is introduced for the fundamental solution of the Riccati differential equation.

The reproducing kernel method has already been used by many researchers for several scientific problems. Toutian Isfahani et al., [43] applied the reproducing kernel Hilbert space technique to obtain approximate solution of some initial optimal control problems. The convergence order of the reproducing kernel method is used in [44] for investigating boundary value problems. Lin et al [45] showed that a numerical solution for an anomalous subdiffusion equation by using the reproducing kernel Hilbert space technique. Sahihi et al. [46] work on the numerical treatment system of second-order BVPs using reproducing kernel Hilbert space.

It is noted that the interesting application of the reproducing kernel method for some differential equations such as non-local initial-boundary value problems for parabolic and hyperbolic integro-differential equations [47], numerical solution of integral equations [48], the generalized Black-Scholes equation [49], multiple solutions of nonlinear boundary value problems [50] were developed by many authors. However, for some practical problems, fractional derivative problems in two and three dimensions [51], multi-term time-fractional diffusion and diffusion-wave equations [52], fractional partial differential equation with non-classical conditions [56] and weakly singular Volterra integral equation on graded mesh [57], have been solved successfully by reproducing kernel method and numerical approximate solutions obtained. Reproducing kernel methods for numerical approximation have already been used by researchers such as Jiang and Lin [58], Beyrami et al. [57], Geng and Cui [59], and Foroutan et al. [60]. On the other hand, in recent years, the polynomial reproducing kernel-based methods have attracted the attention of many researchers for the numerical solution of the differential and integral problems [61–65].

The purpose of this paper is to provide a simple framework for the following problem

$$\begin{cases} {}_0^C D_t^\alpha u(X, t) + \mathfrak{L}u(X, t) = f(X, t), & \text{for } (X, t) \in \Omega \times J, \\ u(X, t) = 0, & \text{for } (X, t) \in \partial\Omega \times J, \\ u(X, 0) = u_0(X), & \text{for } X \in \bar{\Omega}, \end{cases} \quad (1)$$

where $(X, t) = (x_1, x_2, t) \in \Omega \times J$, $\Omega = (a, b) \times (c, d) \subset \mathbb{R}^2$, $J = (0, T]$, u is a adequately smooth function in $\bar{\Omega} \times [0, T]$.

The time-fractional derivative ${}_0^C D_t^\alpha u$, for some $\alpha \in (0, 1)$, is the Caputo fractional derivative defined by

$${}_0^C D_t^\alpha u(X, t) := ({}_0 I_t^{1-\alpha} \frac{\partial u}{\partial t})(X, t), \quad 0 < \alpha < 1,$$

in which ${}_0 I_t$ is the Riemann-Liouville integral of fractional order

$$({}_0 I_t^\alpha u)(X, t) = \int_0^t u(X, s) v_\alpha(t-s) ds,$$

with $v_\alpha(t) := \frac{t^{\alpha-1}}{\Gamma(\alpha)}$.

The spatial operator \mathfrak{L} is a second-order elliptic operator and is defined by

$$\mathfrak{L}u := - \sum_{i=1}^2 \partial_{x_i}^2 u + c(X)u, \quad c(X) \text{ for } X \in \overline{\Omega}.$$

The main objective of this paper is to present a novel hybrid method for solving time-fractional diffusion equations (TFDEs) based on the combination of the finite difference method and the Fibonacci polynomial reproducing kernel collocation method. In this approach, spatial discretization is performed using Fibonacci polynomial reproducing kernels, which are combined with temporal discretization through a finite difference scheme. We employ a finite difference scheme to discretize the time derivatives and consider an approximation based on the discretization of the reproducing kernel in semi-discrete space to solve the problem.

In this paper, a collocation approach based on reproducing kernels is introduced for the numerical solution of two-dimensional time-fractional partial differential equations. The positive definite reproducing kernel spaces are constructed using the Fibonacci polynomial basis. One of the main challenges in numerical methods is the management of boundary conditions in solving partial differential equations. In the proposed method, the reproducing kernels are designed to satisfy the boundary conditions exactly. Also, several numerical simulations have been carried out to demonstrate the efficiency and performance of the Fibonacci kernel approach along with the time scheme. The numerical results demonstrate the effectiveness and accuracy of the proposed method.

2 Implementation of Method

2.1 Temporal Discretization

Initially, we derive the semi-discrete scheme for the main equation in (1). Set $t_n = T(\frac{n}{N})$, $n = 0 : N$, where $N \in \mathcal{N}^*$, and also $\tau = t_n - t_{n-1}$, for $n = 0 : N$. Letting $u^n = u(x, t_n)$, for a discrete function $\{u^n\}_{n=0}^N$, we focus on the discretization of the time-fractional derivative ${}_0^C D_t^\alpha u(X, t)$ for some $0 < \alpha < 1$. A Caputo derivative approximation formula for ${}_0^C D_t^\alpha u(X, t_n) = {}_0^C D_t^\alpha u^n(X)$ ($n \geq 0$) with $0 < \alpha < 1$ can be defined as following [52]

$$\begin{aligned} {}_0^C D_t^\alpha u^n &\simeq D_N^\alpha u^n = \frac{1}{\Gamma(1-\alpha)} \sum_{i=0}^{n-1} \frac{u^{i+1} - u^i}{\tau} \int_{t_i}^{t_{i+1}} (t_n - s)^{-\alpha} ds \\ &= \frac{1}{\Gamma(1-\alpha)} \sum_{i=0}^{n-1} \frac{u^{i+1} - u^i}{\tau} [(t_n - t_i)^{1-\alpha} - (t_n - t_{i+1})^{1-\alpha}] \\ &= \frac{b_1}{\Gamma(2-\alpha)} u^n - \frac{b_n}{\Gamma(2-\alpha)} u^0 + \frac{1}{\Gamma(2-\alpha)} \sum_{i=1}^{n-1} u^{n-i} (b_{i+1} - b_i), \end{aligned} \quad (2)$$

where

$$b_1 = \tau^{-\alpha},$$

and

$$b_{i+1} = \frac{(i+1)^{1-\alpha} - (i)^{1-\alpha}}{\tau^\alpha}, \quad i \geq 1.$$

The local truncation error $R^n(u) = {}_0^C D_t^\alpha u^n - D_N^\alpha u^n$ is given in [52] as:

$$|R^n(u)| \leq C_u \tau^{2-\alpha}.$$

Substituting $D_N^\alpha u^n$ into (1), we get

$$D_N^\alpha u^n + \mathfrak{L}u^n = f^n - R^n(u). \quad (3)$$

Replacing u^n by the approximate solution \mathbf{u}^n , we can obtain the semi-discrete problem as following:

Semi-discrete scheme: Set $\mathbf{u}^0 = u_0$ and find \mathbf{u}^{n+1} ($n = 0 : N - 1$), such that

$$\begin{cases} D_N^\alpha \mathbf{u}^n + \mathfrak{L}\mathbf{u}^n = f^n, \\ \mathbf{u}^n|_{X \in \partial\Omega} = 0, \quad 0 \leq n \leq N. \end{cases} \quad (4)$$

2.2 Spatial Discretization

Here, we will discuss the spatial discretization of the problem using reproducing kernels Hilbert space method. Firstly, we present the definitions and theorems related to RKHS. For more details, one can refer to the references [56, 67, 68].

2.2.1 Kernel Space

Definition 1. ([67]) Let $\mathcal{F}(\Omega)$ be the set of all real-valued functions on the domain Ω for an arbitrary non-empty set $\Omega \subseteq \mathbb{R}^d$ ($d \geq 1$). A reproducing kernel over the set Ω is a Hilbert space $H \subset \mathcal{F}(\Omega)$ equipped with a function $K(X, Y) : \Omega \times \Omega \rightarrow H$, known as the reproducing kernel (RK). The kernel satisfies the following properties:

- 1) $K(X, \cdot) \in H, \forall X \in \Omega$,
- 2) $u(X) = \langle u(\cdot), K(X, \cdot) \rangle_H, \forall u \in H, \forall X \in \Omega$.

It is known that the RK of a Hilbert space is unique and that the existence of an RK is due to the Riesz representation Theorem.

Theorem 1. ([67]) Let H be a reproducing kernel Hilbert space with reproducing kernel $K : \Omega \times \Omega \rightarrow \mathbb{R}$. Then $K(X, \cdot)$ defines a positive definite function.

Theorem 2. ([67]) The reproducing kernel (RK) associated with a RKHS H satisfies the following properties:

- For every $u \in H$ and $X \in \Omega$, the point-evaluation functional is bounded and

$$|I_X(u)| \leq \sqrt{K(X, X)} \times \|u\|_H.$$

- For all $X, Y \in \Omega$, the reproducing property yields

$$K(X, Y) = \langle K(X, \cdot), K(Y, \cdot) \rangle_H = \langle I_X, I_Y \rangle_{H^*}.$$

- For any $\lambda, \mu \in H^*$, their inner product in the dual space is given by

$$\langle \lambda, \mu \rangle_{H^*} = \lambda^X \times \mu^Y \times K(X, Y),$$

where λ^X denotes the action of the functional λ on the variable X .

Remark 1. ([67]) The RKHS contains all functions of the form $u = \sum_{j=1}^n c_j K(\cdot, X_j)$ provided $\{X_j\}_{j=1}^n \subset \Omega$. Using the properties of RKHS, we have

$$\|u\|_H = \langle u, u \rangle_H = \sum_{i=1}^n \sum_{j=1}^n c_i c_j K(X_i, X_j). \quad (5)$$

Theorem 3. ([67]) Suppose that the given symmetric RK $K : \Omega \times \Omega \rightarrow \mathbb{R}$ is positive definite. Then there is a unique RKHS associated with it.

2.3 Fibonacci Polynomials

Next, we first introduce the Fibonacci polynomials and then use them to construct the reproducing kernels of a Hilbert space. Fibonacci polynomials are a sequence of polynomials that can be obtained using the following recursive equation [53]:

$$F_n(x) = xF_{n-1}(x) + F_{n-2}(x), \quad n \geq 3, \quad (6)$$

where $F_1(x) = 1$ and $F_2(x) = x$. These polynomials also satisfy the following recurrence relation [53]:

$$F'_{n+1}(x) = \frac{n+1}{2} F_n(x) + \frac{x}{2} F'_n(x). \quad (7)$$

The first five Fibonacci polynomials are as follows:

$$F_1(x) = 1, F_2(x) = x, F_3(x) = x^2 + 1, \quad (8)$$

$$F_4(x) = x^3 + 2x, F_5(x) = x^4 + 3x^2 + 1, \quad (9)$$

It is also shown in [54] that the standard bases of the polynomial functions, x^{n-1} for $n \geq 1$, can be uniquely written in terms of $F_0, F_1, F_2, \dots, F_n$ as follows:

$$x^{n-1} = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^j \left(\binom{n}{j} - \binom{n}{j-1} \right) F_{n-2j}, \quad (10)$$

where $F_0(x) = 0$ is assumed.

2.4 Finite-Dimensional Reproducing Kernel Hilbert Space

Here, we construct a Fibonacci polynomial RKHS of finite dimension and derive its reproducing kernel.

Definition 2. Let $H_N[0, 1] = \text{span}\{F_1(x), \dots, F_N(x)\}$. Every function $f \in H_N[0, 1]$ admits a unique representation of the form

$$f(x) = \sum_{j=1}^N \omega_j F_j(x), \quad (11)$$

for some coefficients $\omega_1, \dots, \omega_N \in \mathbb{R}$. The inner product and the induced norm on H_N are defined by

$$(f, h)_{H_N} = \sum_{j=1}^N \omega_j \mu_j, \quad (12)$$

$$\|f\|_{H_N} = \sqrt{(f, f)_{H_N}} = \sqrt{\sum_{j=1}^N \omega_j^2}, \quad (13)$$

where

$$f(x) = \sum_{j=1}^N \omega_j F_j(x), \quad h(x) = \sum_{j=1}^N \mu_j F_j(x).$$

Theorem 4. The space $H_N[0, 1]$ is a finite-dimensional Hilbert space and possesses a reproducing kernel. Its reproducing kernel is given by

$$K(x, y) = \sum_{j=1}^N F_j(x) F_j(y),$$

and it satisfies the following properties:

$$1) \quad K(\cdot, y) \in H_N, \quad \forall y \in [0, 1], \quad (14)$$

$$2) \quad (K(\cdot, y), f(\cdot))_{H_N} = f(y), \quad \forall f \in H_N, \quad \forall y \in [0, 1], \quad (15)$$

where property (2) is referred to as the reproducing property of the kernel.

Proof. It can be easily seen that for any fixed y in $[0, 1]$, we have:

$$K(x, y) = \sum_{j=1}^N F_j(x) F_j(y) \in H_N[0, 1]. \quad (16)$$

Furthermore, for any function $f(x) = \sum_{j=1}^N \omega_j F_j(x)$ belonging to $H_N[0, 1]$, the reproducing property of the kernel is obtained using the definition of the inner product in (12) as follows:

$$\begin{aligned} (K(\cdot, y), f(\cdot))_{H_N} &= \left(\sum_{j=1}^N F_j(\cdot) F_j(y), f(\cdot) \right)_{H_N} \\ &= \left(\sum_{j=1}^N F_j(\cdot) F_j(y), \sum_{j=1}^N \omega_j F_j(\cdot) \right)_{H_N} \\ &= \sum_{j=1}^N \omega_j F_j(y) \\ &= f(y). \end{aligned} \tag{17}$$

□

Without loss of generality, let $L_1 u = 0$ and $L_2 u = 0$ denote the boundary condition of the given PDE, where L_1 and L_2 are linear operators. Now, we reconstruct the reproducing kernel $K(x, y)$ such that it satisfies the boundary conditions of the given problem. To this end, using the Fibonacci reproducing kernel $K(x, y)$, the kernels $K_1(x, y)$ and $K_2(x, y)$ are constructed as follows:

$$K_1(x, y) = K(x, y) - \frac{L_{1,x}K(x, y)L_{1,y}K(x, y)}{L_{1,x}L_{1,y}K(x, y)}, \tag{18}$$

$$K_2(x, y) = K_1(x, y) - \frac{L_{2,x}K_1(x, y)L_{2,y}K_1(x, y)}{L_{2,x}L_{2,y}K_1(x, y)}, \tag{19}$$

where the subscript x in $L_{1,x}$ indicates that the operator L_1 acts on the single-variable function in terms of x in the kernel, considering y as a constant. In the next theorem, we see that $K_2(x, y)$ will be a reproducing kernel under certain conditions and will precisely satisfy the boundary conditions $L_1 u = 0$ and $L_2 u = 0$.

Theorem 5. [55] If $L_{1,x}L_{1,y}K(x, y) \neq 0$ and $L_{2,x}L_{2,y}K_1(x, y) \neq 0$, then $K_2(x, y)$ is a reproducing kernel and precisely satisfies the boundary conditions $L_1 u = 0$ and $L_2 u = 0$.

Theorem 6. ([68]) (Two-dimensional RKHS) Suppose that $H_1[a, b]$ and $H_2[c, d]$ are the reproducing kernel Hilbert spaces, then the direct product $H(\overline{\Omega}) = H_1[a, b] \otimes H_2[c, d]$ is a two-dimensional reproducing kernel space with the following inner product

$$\langle u_1 u_2, v_1 v_2 \rangle_H = \langle u_1, v_1 \rangle_{H_1} \langle u_2, v_2 \rangle_{H_2},$$

and we can obtain the reproducing kernel of $H(\overline{\Omega})$ as $K(X, Y) = K(x_1, x_2, y_1, y_2) = R_1(x_1, y_1)R_2(x_2, y_2)$, where $R_1(x_1, y_1)$ and $R_2(x_2, y_2)$ are the reproducing kernels of $H_1[a, b]$ and $H_2[c, d]$, respectively.

2.5 RKHS on Bounded Domains

Definition 3. ([56]) (One-dimensional RKHS) Suppose that $-\infty < a < b < \infty$, we define the space $H^m[a, b]$ as follows

$$H^m[a, b] = \{u : [a, b] \rightarrow \mathbb{R} \mid u^{(i)}(x) \in AC[a, b], i = 0, 1, \dots, m - 1, u^{(m)}(x) \in L^2[a, b]\}.$$

The inner product in the space $H^m[a, b]$ is given by

$$\langle u, v \rangle_{H^m} = \langle u^{(m)}, v^{(m)} \rangle + \sum_{j=0}^{m-1} u^{(j)}(a)v^{(j)}(a), \quad u, v \in H^m, \tag{20}$$

where $\langle u, v \rangle = \int_a^b u(x)v(x)dx$.

Theorem 7. ([56]) The space $H^m[a, b]$ is a RKHS and reproducing kernel $K(x, y) : [a, b] \times [a, b] \rightarrow \mathbb{R}$ can be denoted by

$$K(x, y) = \begin{cases} ((m - 1)!)^{-2} \int_a^x (x - z)^{m-1} (y - z)^{m-1} dz + \sum_{i=0}^{m-1} (i!)^{-2} (x - a)^i (y - a)^i, & x < y, \\ ((m - 1)!)^{-2} \int_a^y (x - z)^{m-1} (y - z)^{m-1} dz + \sum_{i=0}^{m-1} (i!)^{-2} (x - a)^i (y - a)^i, & y \leq x. \end{cases}$$

We now consider the subspace of $H^m[a, b]$ by applying condition $B(u) = 0$ on it. Since the norm and inner product of this subspace are defined exactly as $H^m[a, b]$, it is also a RKHS.

Definition 4. (Subspace of $H^m[a, b]$) The inner product space $\tilde{H}^m[a, b]$ for function u is defined as $\tilde{H}^m[a, b] = \{u | u(x) \in H^m[a, b], B(u) = 0\}$.

It is shown in [68] that $\tilde{H}^m[a, b]$ is a closed subspace of $H^m[a, b]$. The following theorem describes how to find the reproducing kernel of a RKHS $\tilde{H}^m[a, b]$ associated with $B(u) = 0$.

Theorem 8. ([56]) The space $\tilde{H}^m[a, b]$ is a RKHS and its RK is given by

$$R(x, y) = K(x, y) - \frac{B_x K(x, y) B_y K(x, y)}{B_x B_y K(x, y)}, \quad B_x B_y K(x, y) \neq 0, \quad K(x, y) \in H^m[a, b],$$

and the subscript x on the operators B_x and B_y indicate that the operator B applies to the function of x and y , respectively.

2.5.1 A Reproducing Kernel Collocation Method

We proceed by expressing the numerical approximation of the semi-discrete formulation within the finite-dimensional reproducing kernel Hilbert space H_N . To simplify the presentation, we introduce the following linear operator

$$\mathbb{L} : H(\bar{\Omega}) \rightarrow C(\bar{\Omega}),$$

as follows:

$$\mathbb{L} \mathbf{u}^n = \frac{b_1}{\Gamma(2-\alpha)} \mathbf{u}^n + \mathfrak{L} \mathbf{u}^n, \quad n \geq 1. \quad (21)$$

Accordingly, the semi-discrete formulation is transformed into an equivalent operator equation with domain $H(\bar{\Omega})$ and range $C(\bar{\Omega})$:

$$\mathbb{L} \mathbf{u}^n = \mathbf{f}^n, \quad n \geq 1, \quad (22)$$

where

$$\mathbf{f}^n = f^n + \frac{b_n}{\Gamma(2-\alpha)} \mathbf{u}^0 - \frac{1}{\Gamma(2-\alpha)} \sum_{i=1}^{n-1} \mathbf{u}^{n-i} (b_{i+1} - b_i), \quad k \geq 1. \quad (23)$$

Assume that \mathcal{Q}_M denotes a collection of M pairwise distinct points located in the closure of the domain Ω , that is,

$$\mathcal{Q}_M := \{X_1, X_2, \dots, X_M\} \subset \bar{\Omega}.$$

Based on this set of nodes, we introduce a finite-dimensional function space defined as

$$\mathcal{U}_M = \text{span}\{\mathbf{u}_j(x) = K(X, X_j), X_j \in \mathcal{Q}_M\} \subset H(\bar{\Omega}), \quad (24)$$

where $K(X, Y) \in H(\bar{\Omega})$. An approximant \mathbf{u}_M^n to \mathbf{u}^n can be obtained by calculating a truncated series based on trial functions as follows

$$\mathbf{u}^n(X) \approx \mathbf{u}_M^n(X) := \sum_{j=1}^M \alpha_j^n \mathbf{u}_j(X) = [\mathbf{u}_1(X) \ \mathbf{u}_2(X) \ \cdots \ \mathbf{u}_M(X)] \begin{pmatrix} \alpha_1^n \\ \alpha_2^n \\ \vdots \\ \alpha_M^n \end{pmatrix}. \quad (25)$$

To compute the interpolation coefficients $\{\alpha_j^n\}_{j=1}^M$, we enforce the collocation conditions on the points in \mathcal{Q}_M according to (22). This leads to the relations

$$\mathbb{L} \mathbf{u}_M^n(X_i) = \sum_{j=1}^M \alpha_j^n \mathbb{L} \mathbf{u}_j(X_i) = \mathbf{f}^n(X_i), \quad i = 1, 2, \dots, M. \quad (26)$$

Here, λ_i denotes the functional corresponding to the application of the differential operator followed by evaluation at $X_i \in \mathcal{Q}_M$. Collectively, the set of functionals is written as

$$\Lambda_M = \{\lambda_1, \lambda_2, \dots, \lambda_M\},$$

which may include different differential operators. The associated collocation matrix $\mathbf{K}_{\Lambda_M, \mathcal{Q}_M}$ is in general non-symmetric, with entries

$$(\mathbf{K}_{\Lambda_M, \mathcal{Q}_M})_{ij} = \lambda_i[\mathbf{u}_j] = \lambda_i^X \tilde{K}(X, X_j), \quad 1 \leq i, j \leq M, \quad (27)$$

where the superscript X emphasizes that the functional λ_i acts on the variable X . Consequently, the unknown coefficients $\{\alpha_j^n\}_{j=1}^M$ can be obtained by solving this system of linear equations:

$$\mathbf{K}_{\Lambda_M, \mathcal{Q}_M}^n [\alpha]^n = \mathbf{f}_{\mathcal{Q}_M}^n,$$

where $[\alpha]^n = [\alpha_1^n \ \alpha_2^n \ \dots \ \alpha_M^n]^T$, $\mathbf{f}_{\mathcal{Q}_M}^n = [\mathbf{f}^n(X_1) \ \mathbf{f}^n(X_2) \ \dots \ \mathbf{f}^n(X_M)]^T$ and

$$\mathbf{K}_{\Lambda_M, \mathcal{Q}_M} = \begin{pmatrix} \lambda_1^X \tilde{K}(X, X_1) & \lambda_2^X \tilde{K}(X, X_1) & \dots & \lambda_n^X \tilde{K}(X, X_1) \\ \lambda_1^X \tilde{K}(X, X_2) & \lambda_2^X \tilde{K}(X, X_2) & \dots & \lambda_n^X \tilde{K}(X, X_2) \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^X \tilde{K}(X, X_M) & \lambda_2^X \tilde{K}(X, X_M) & \dots & \lambda_M^X \tilde{K}(X, X_M) \end{pmatrix}. \quad (28)$$

Suppose that $\mathcal{Q}_n = \{X_{ij}\}_{i=1}^M$ and $\mathcal{U}_M = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_M\}$, applying Gram-Schmidt orthogonalization process to $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_M\}$, we can obtain

$$\bar{\mathbf{u}}_i(X) = \sum_{k=1}^i \beta_{ik} \mathbf{u}_k(X), \quad (\beta_{ii} > 0, i = 1, 2, \dots, M). \quad (29)$$

Therefore $\{\bar{\mathbf{u}}_1, \bar{\mathbf{u}}_2, \dots, \bar{\mathbf{u}}_M\}$ is a orthonormal basis for \mathcal{U}_M . Now, we can write the interpolant $\mathbf{u}_M^n(x)$ to \mathbf{u}^n at \mathcal{Q}_M in the following form

$$\mathbf{u}^n(X) \approx \mathbf{u}_M^n(X) = \sum_{i=1}^M \mathbf{u}^n(X_i) \bar{\mathbf{u}}_i(X). \quad (30)$$

3 Numerical Results

Several numerical simulations are carried out to evaluate the efficiency of the proposed technique. The outcomes demonstrate its stable performance and high level of accuracy across different configurations of the discretization parameters M and N .

Example 1. Consider the time-fractional diffusion problem with

$$\mathfrak{L}u := - \sum_{i=1}^2 \partial_{x_i}^2 u, \quad (31)$$

and

$$f(X, t) = - \frac{\sin(\pi x_1)(t\pi^2 x_2^2 \Gamma(2-\alpha) - t\pi^2 x_2 \Gamma(2-\alpha) - 2t\Gamma(2-\alpha) + t^{1-\alpha} x_2^2 - t^{1-\alpha} x_2)}{\Gamma(2-\alpha)}. \quad (32)$$

Let $X = (x_1, x_2)$ represent the spatial variables and let $\Omega = (0, 1) \times (0, 1)$ denote the computational domain. The problem admits the exact solution

$$u(X, t) = t \sin(\pi x_1)(x_2^2 - x_2).$$

To carry out the numerical experiments, the proposed algorithm is applied on a uniform tensor-product discretization of the domain, given by

$$\{(x_i, y_i)\}_{i=1}^M = \left\{ \frac{i}{M_1 + 1} \right\}_{i=1}^{M_1} \times \left\{ \frac{i}{M_1 + 1} \right\}_{i=1}^{M_1}, \quad M = M_1 \times M_1.$$

The numerical results are presented in Figures 1a and 1b, which display the contour plots of the error

$$|u^N(X_i) - u_M^N(X_i)|, \quad i = 1, 2, \dots, M,$$

for $M = 64$ and $N = 15$.

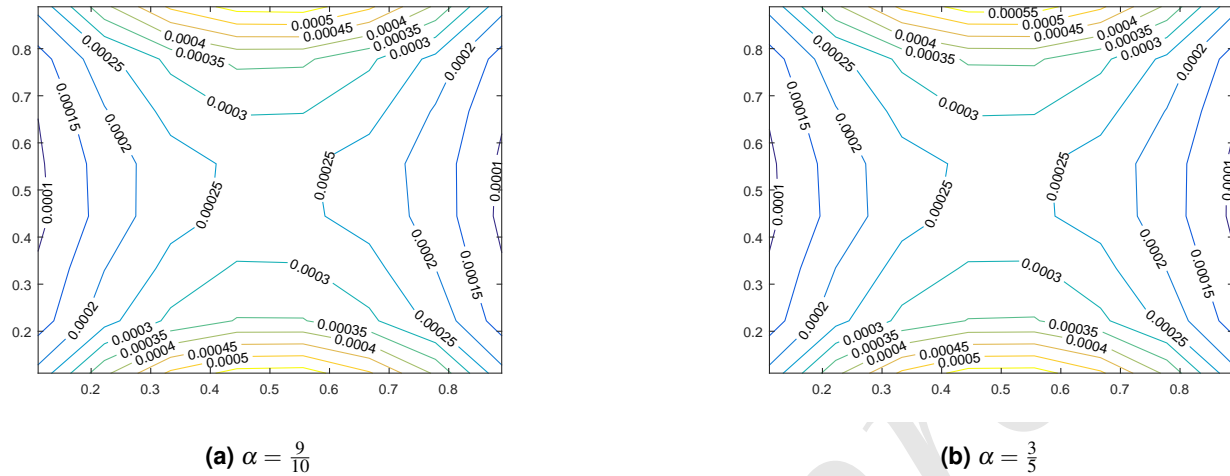


Figure 1. The error $|u^N(X_i) - u_M^N(X_i)|$, $i = 1, 2, \dots, M$, with $M = 64$, $N = 15$ for Example 1.

Example 2. Consider the time-fractional diffusion problem with

$$\mathfrak{L}u := - \sum_{k=1}^2 \partial_{x_k}^2 u, \tag{33}$$

and

$$f(X, t) = -2tx_2^2 + 2tx_2 - 2tx_1^2 + 2x_1t + \frac{t^{1-\alpha}x_1x_2(x_1x_2 - x_1 - x_2 + 1)}{\Gamma(2 - \alpha)}. \tag{34}$$

In this setting, $X = (x_1, x_2)$ refers to the spatial variable and $\Omega = (0, 1) \times (0, 1)$ denotes the solution domain. The problem admits the exact solution

$$u(X, t) = t(x_1^2 - x_1)(x_2^2 - x_2).$$

For the numerical investigation, the proposed method is implemented on a uniform tensor-product mesh, which is constructed as

$$\{(x_i, y_j) : 1 \leq i, j \leq M_1\} = \left\{ \frac{i}{M_1 + 1} \right\}_{i=1}^{M_1} \times \left\{ \frac{j}{M_1 + 1} \right\}_{j=1}^{M_1}, \quad M = M_1^2.$$

Figures 2a and 2b present the numerical results, showing contour plots of the error

$$|u^N(X_i) - u_M^N(X_i)|, \quad i = 1, 2, \dots, M,$$

for $M = 64$ and $N = 15$.

Example 3. Consider the time-fractional diffusion problem with

$$\mathfrak{L}u := - \sum_{k=1}^2 \partial_{x_k}^2 u, \tag{35}$$

and

$$f(X, t) = -t^2(2x_2e^{x_2} - 2x_2 - 2e^{x_2} + 2 + x_1^2x_2e^{x_2} + x_1^2e^{x_2} - x_2e^{x_2}x_1 - x_1e^{x_2}) + \frac{2t^{-\alpha+2}x_1(x_2e^{x_2}x_1 - x_2e^{x_2} - x_2x_1 + x_2 - e^{x_2}x_1 + e^{x_2} + x_1 - 1)}{\Gamma(-\alpha + 3)}, \tag{36}$$

where $X = (x_1, x_2)$, $\Omega = (0, 1) \times (0, 1)$. Then $u(X, t) = t^2(x_1^2 - x_1)(e^{x_2} - 1)(-1 + x_2)$ is the exact solution. To evaluate the proposed method, we use a uniform tensor-product grid defined by

$$\{(x_i, y_i)\}_{i=1}^M = \left\{ \frac{i}{M_1 + 1} \right\}_{i=1}^{M_1} \times \left\{ \frac{i}{M_1 + 1} \right\}_{i=1}^{M_1}, \quad M = M_1 \times M_1.$$

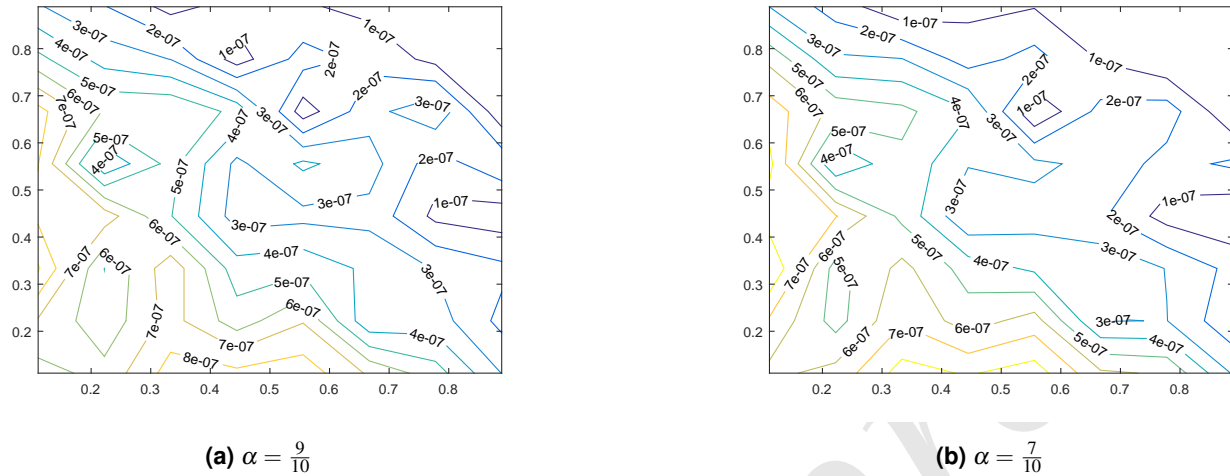


Figure 2. The accuracy of the proposed method is assessed by evaluating the pointwise errors $E_i = |u^N(X_i) - u_M^N(X_i)|, i = 1, 2, \dots, M$, for $M = 64$ and $N = 15$ in Example 2.

The results are illustrated in Figures 3a and 3b, where the contour plots depict the error

$$|u^N(X_i) - u_M^N(X_i)|, \quad i = 1, 2, \dots, M,$$

with $M = 64$ and $N = 15$.

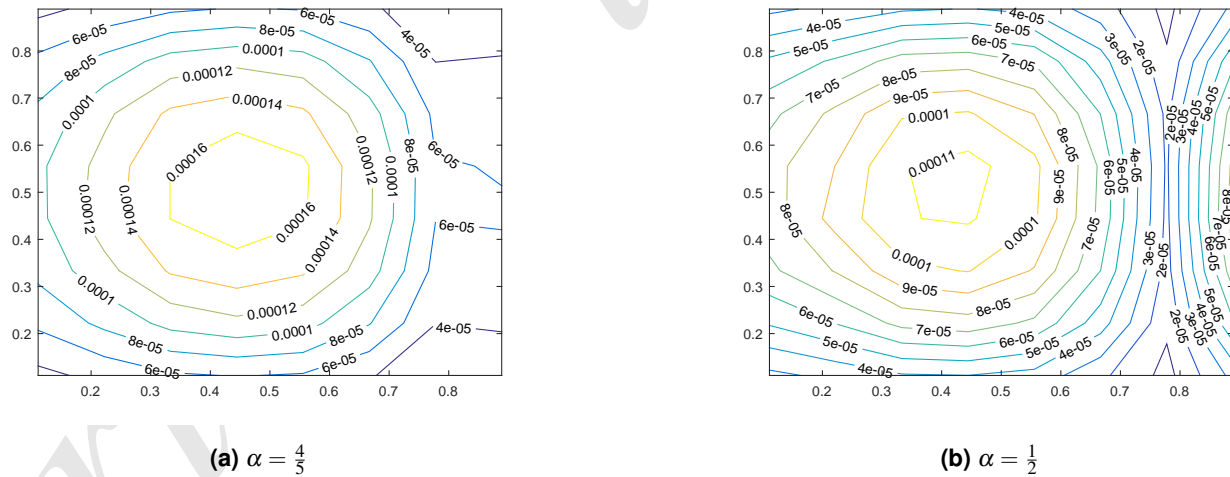


Figure 3. The pointwise error, defined as $|u^N(X_i) - u_M^N(X_i)|, i = 1, 2, \dots, M$, is computed for Example 3 with $M = 64$ and $N = 15$.

4 Conclusion

In this paper, a reproducing kernel collocation method is developed to solve two-dimensional time-fractional diffusion equations. The proposed approach combines spatial discretization using a Fibonacci polynomial reproducing kernel method with temporal discretization via a finite difference scheme, making it computationally efficient due to its simple implementation and reasonable accuracy. The effectiveness of the method is demonstrated through the obtained numerical solutions and error norms, showcasing its ability to achieve accurate numerical solutions for time-fractional diffusion equations.

Data Availability

All data in the paper are available from the corresponding authors upon reasonable request.

Conflicts of Interest

The author declares that there is no conflict of interest.

Ethical Considerations

The author has diligently addressed ethical concerns, such as informed consent, plagiarism, data fabrication, misconduct, falsification, double publication, redundancy, submission, and other related matters.

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