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Research article

Fibonacci Polynomial Reproducing Kernel Collocation Method for 2D Time-Fractional Diffusion Equations

Mahdi Emamjomeh*

Basic Sciences Group, Golpayegan College of Engineering, Isfahan University of Technology, Isfahan 84156–83111, Iran

* **Corresponding author(s):** m.emamjomeh@iut.ac.ir

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Abstract

In this paper, a collocation approach based on reproducing kernels is presented for the numerical solution of the 2D time-fractional diffusion equation. Some finite-dimensional positive definite reproducing kernel spaces are constructed using the bases of the Fibonacci polynomials. The spatial discretization in the proposed method is based on the Fibonacci polynomial reproducing kernel method, which is combined with a finite difference scheme for temporal discretization. Handling the boundary conditions in the numerical solution of partial differential equations is a challenging issue in numerical methods. To deal with the boundary conditions in the proposed method, the reproducing kernels are constructed in a way that exactly satisfy the boundary conditions. Some numerical simulations are conducted to prove the efficiency and ability of the Fibonacci kernel approach combined with the time-stepping scheme. The numerical results show the effectiveness and accuracy of the proposed method.

Keywords: Time-fractional diffusion equations, Fibonacci polynomial, Reproducing kernel collocation method, Semi-discrete, Polynomial reproducing kernel

Mathematics Subject Classification (2020): 35K57, 35R11, 65Mxx

1 Introduction

In recent decades, fractional calculus and its applications have exhibited various spatiotemporal patterns to provide more realistic models for physical and chemical processes as well as biological systems [1–3]. For example, they can describe many areas of science such as electromagnetism, electrochemistry, diffusion, and general transport theory [4, 5]. Du et al. [6] showed that we cannot obtain the analytical solutions for most fractional differential equations; therefore, they investigated practical methods to give numerical solutions to these equations. In recent research works, various numerical techniques have been introduced for dealing with problems containing derivatives of fractional orders [7–16].

In contrast to integer-order derivatives, fractional-order derivatives, from both the mathematical and engineering perspectives, focus on the nonlocality property of the derivative operators. As it is known, nonlocality means that the next behavior of a system depends not only on its current behavior, but also on its past behavior and state. The principle of the direct methods consists of various non-locality fractional operators to describe the real-world phenomena that depend on different factors such as time, space, etc. This approach evaluates variable-order fractional operators with a non-stationary power-law kernel [17–19]. The direct methods have been widely applied to a variable-order fractional calculus in a variety of sciences and engineering phenomena, ranging from anomalous diffusion [20–22] and viscoelastic mechanics to control systems and petroleum engineering, among many other applications [23–28]. An important generalization of classical partial differential equations widely has been widely studied in recent years is fractional partial differential equations. The diffusion and the diffusion-wave equations with derivaticves of fractional orders were introduced and intensely discussed in different applications for investigating anomalous diffusion in transport processes through complex systems, as well as in the mathematical literature [29]. The time-fractional diffusion equation (TFDE) has been successfully used to study superdiffusion and subdiffusion phenomena [30–32]. It has received wide attention in several research areas such as chemistry, biochemistry, finance, medicine, and systems biology [33–37]. Consequently, a time-fractional diffusion has been used to solve some practical problems as diverse as a part of the boundary data, initial data, and diffusion coefficients. Interestingly, some fractional diffusion inverse problems constitute an interesting challenge and an active research area [38–40]. In addition, since obtaining analytical solutions for fractional-order differential equations is often difficult or impossible, studying efficient numerical methods for solving this class of equations is of particular importance.

Zaremba considered the reproducing kernel Hilbert spaces (RKHS) in the early 20th century. After a few decades, some mathematicians such as Zigo, Bergman, and Bacchner have investigated the idea of reproducing kernel methods. Aronszajn and Bergman in 1950 provided the theory of the reproducing kernel Hilbert spaces method. Javan et al. [41] have used the RKHS method for obtaining the solution of nonlinear integral equations. In [42], an iterative reproducing kernel Hilbert space method is introduced for the fundamental solution of the Riccati differential equation.

The reproducing kernel method has already been used by many researchers for several scientific problems. Toutian Isfahani et al., [43] applied the reproducing kernel Hilbert space technique to obtain approximate solution of some initial optimal control problems. The convergence order of the reproducing kernel method is used in [44] for investigating boundary value problems. Lin et al [45] showed that a numerical solution for an anomalous subdiffusion equation by using the reproducing kernel Hilbert space technique. Sahihi et al. [46] work on the numerical treatment system of second-order BVPs using reproducing kernel Hilbert space.

It is noted that the interesting application of the reproducing kernel method for some differential equations such as non-local initial-boundary value problems for parabolic and hyperbolic integro-differential equations [47], numerical solution of integral equations [48], the generalized Black-Scholes equation [49], multiple solutions of nonlinear boundary value problems [50] were developed by many authors. However, for some practical problems, fractional derivative problems in two and three dimensions [51], multi-term time-fractional diffusion and diffusion-wave equations [52], fractional partial differential equation with non-classical conditions [56] and weakly singular Volterra integral equation on graded mesh [57], have been solved successfully by reproducing kernel method and numerical approximate solutions obtained. Reproducing kernel methods for numerical approximation have already been used by researchers such as Jiang and Lin [58], Beyrami et al. [57], Geng and Cui [59], and Foroutan et al. [60]. On the other hand, in recent years, the polynomial reproducing kernel-based methods have attracted the attention of many researchers for the numerical solution of the differential and integral problems [61–65].

The purpose of this paper is to provide a simple framework for the following problem

$$\begin{cases} {}_0^C D_t^{\alpha} u(X,t) + \mathfrak{L}u(X,t) = f(X,t), & for\ (X,t) \in \Omega \times J, \\ u(X,t) = 0, & for\ (X,t) \in \partial\Omega \times J, \\ u(X,0) = u_0(X), & for\ X \in \overline{\Omega}, \end{cases} \tag{1}$$

where $(X,t) = (x_1,x_2,t) \in \Omega \times J$, $\Omega = (a,b) \times (c,d) \subset \mathbb{R}^2$, $J = (0,T]$, u is a adequately smooth function in $\overline{\Omega} \times [0,T]$.
The time-fractional derivative ${}_0^C D_t^{\alpha} u$, for some $\alpha \in (0,1)$, is the Caputo fractional derivative defined by

$${}_0^C D_t^{\alpha} u(X,t) := ({}_0I_t^{1-\alpha} \frac{\partial u}{\partial t})(X,t), \qquad 0 < \alpha < 1,$$

in which ${}_0I_t$ is the Riemann-Liouville integral of fractional order

$$({}_0I_t^{\alpha} u)(X,t) = \int_0^t u(X,s) \nu_{\alpha}(t-s) ds,$$

with $\nu_{\alpha}(t) := \frac{t^{\alpha-1}}{\Gamma(\alpha)}$.

The spatial operator $\mathfrak{L}$ is a second-order elliptic operator and is defined by

$$\mathfrak{L}u := -\sum_{i=1}^{2} \partial^2_{x_k^2} u + c(X)u,\ c(X)\ for\ X \in \overline{\Omega}.$$

The main objective of this paper is to present a novel hybrid method for solving time-fractional diffusion equations (TFDEs) based on the combination of the finite difference method and the Fibonacci polynomial reproducing kernel collocation method. In this approach, spatial discretization is performed using Fibonacci polynomial reproducing kernels, which are combined with temporal discretization through a finite difference scheme. We employ a finite difference scheme to discretize the time derivatives and consider an approximation based on the discretization of the reproducing kernel in semi-discrete space to solve the problem.

In this paper, a collocation approach based on reproducing kernels is introduced for the numerical solution of two-dimensional time-fractional partial differential equations. The positive definite reproducing kernel spaces are constructed using the Fibonacci polynomial basis. One of the main challenges in numerical methods is the management of boundary conditions in solving partial differential equations. In the proposed method, the reproducing kernels are designed to satisfy the boundary conditions exactly. Also, several numerical simulations have been carried out to demonstrate the efficiency and performance of the Fibonacci kernel approach along with the time scheme. The numerical results demonstrate the effectiveness and accuracy of the proposed method.

2 Implementation of Method

2.1 Temporal Discretization

Initially, we derive the semi-discrete scheme for the main equation in (1). Set $t_n = T\left(\frac{n}{N}\right), n = 0:N$, where $N \in \mathcal{N}^*$, and also $\tau = t_n - t_{n-1}$, for $n = 0:N$. Letting $u^n = u(x,t_n)$, for a discrete function $\{u^n\}_{n=0}^{N}$, we focus on the discretization of the time-fractional derivative ${}_0^C D_t^\alpha u(X,t)$ for some $0 < \alpha < 1$. A Caputo derivative approximation formula for ${}_0^C D_t^\alpha u(X,t_n) = {}_0^C D_t^\alpha u^n(X)\ (n \geq 0)$ with $0 < \alpha < 1$ can be defined as following [52]

$$\begin{aligned} {}_0^C D_t^\alpha u^n &\simeq D_N^\alpha u^n = \frac{1}{\Gamma(1-\alpha)} \sum_{i=0}^{n-1} \frac{u^{i+1}-u^i}{\tau} \int_{t_i}^{t_{i+1}} (t_n - s)^{-\alpha} ds \\ &= \frac{1}{\Gamma(1-\alpha)} \sum_{i=0}^{n-1} \frac{u^{i+1}-u^i}{\tau} \left[(t_n - t_i)^{1-\alpha} - (t_n - t_{i+1})^{1-\alpha} \right] \\ &= \frac{b_1}{\Gamma(2-\alpha)} u^n - \frac{b_n}{\Gamma(2-\alpha)} u^0 + \frac{1}{\Gamma(2-\alpha)} \sum_{i=1}^{n-1} u^{n-i} (b_{i+1} - b_i), \end{aligned} \tag{2}$$

where

$$b_1 = \tau^{-\alpha},$$

and

$$b_{i+1} = \frac{(i+1)^{1-\alpha} - (i)^{1-\alpha}}{\tau^\alpha}, \qquad i \geq 1.$$

The local truncation error $R^n(u) = {}_0^C D_t^\alpha u^n - D_N^\alpha u^n$ is given in [52] as:

$$|R^n(u)| \leq C_u \tau^{2-\alpha}.$$

Substituting $D_N^\alpha u^n$ into (1), we get

$$D_N^\alpha u^n + \mathfrak{L}u^n = f^n - R^n(u). \tag{3}$$

Replacing u^n by the approximate solution $\mathbf{u}^n$, we can obtain the semi-discrete problem as following:
Semi-discrete scheme: Set $\mathbf{u}^0 = u_0$ and find $\mathbf{u}^{n+1}$ $(n = 0:N-1)$, such that

$$\begin{cases} D_N^\alpha \mathbf{u}^n + \mathfrak{L}\mathbf{u}^n = f^n, \\ \mathbf{u}^n|_{X \in \partial\Omega} = 0, \qquad 0 \leq n \leq N. \end{cases} \tag{4}$$

2.2 Spatial Discretization

Here, we will discuss the spatial discretization of the problem using rproducing kernels Hilbert space method. Firstly, we present the definitions and theorems related to RKHS. For more details, one can refer to the references [56, 67, 68].

2.2.1 Kernel Space

Definition 1. *([67]) Let $\mathscr{F}(\Omega)$ be the set of all real-valued functions on the domain Ω for an arbitrary non-empty set $\Omega \subseteq \mathbb{R}^d (d \geq 1)$. A reproducing kernel over the set Ω is a Hilbert space $H \subset \mathscr{F}(\Omega)$ equipped with a function $K(X,Y) : \Omega \times \Omega \to H$, known as the reproducing kernel (RK). The kernel satisfies the following properties:*

1) $K(X,.) \in H, \ \forall X \in \Omega,$

2) $u(X) = \langle u(.), K(X,.)\rangle_H, \ \forall u \in H, \forall X \in \Omega.$

It is known that the RK of a Hilbert space is unique and that the existence of an RK is due to the Riesz representation Theorem.

Theorem 1. *([67]) Let H be a reproducing kernel Hilbert space with reproducing kernel $K : \Omega \times \Omega \to \mathbb{R}$. Then $K(X,\cdot)$ defines a positive definite function.*

Theorem 2. *([67]) The reproducing kernel (RK) associated with a RKHS H satisfies the following properties:*

- *For every $u \in H$ and $X \in \Omega$, the point-evaluation functional is bounded and*

$$|I_X(u)| \leq \sqrt{K(X,X)} \times \|u\|_H.$$

- *For all $X,Y \in \Omega$, the reproducing property yields*

$$K(X,Y) = \langle K(X,\cdot), K(Y,\cdot)\rangle_H = \langle I_X, I_Y\rangle_{H^*}.$$

- *For any $\lambda, \mu \in H^*$, their inner product in the dual space is given by*

$$\langle \lambda, \mu\rangle_{H^*} = \lambda^X \times \mu^Y \times K(X,Y),$$

where λ^X denotes the action of the functional λ on the variable X.

Remark 1. ([67]) The RKHS contains all functions of the form $u = \sum_{j=1}^{n} c_j K(.,X_j)$ provided $\{X_j\}_{j=1}^n \subset \Omega$. Using the properties of RKHS, we have

$$\|u\|_H = \langle u,u\rangle_H = \sum_{i=1}^{n}\sum_{j=1}^{n} c_i c_j K(X_i,X_j). \tag{5}$$

Theorem 3. *([67]) Suppose that the given symmetric RK $K : \Omega \times \Omega \to \mathbb{R}$ is positive definite. Then there is a unique RKHS associated with it.*

2.3 Fibonacci Polynomials

Next, we first introduce the Fibonacci polynomials and then use them to construct the reproducing kernels of a Hilbert space. Fibonacci polynomials are a sequence of polynomials that can be obtained using the following recursive equation [53]:

$$F_n(x) = xF_{n-1}(x) + F_{n-2}(x), \qquad n \geq 3, \tag{6}$$

where $F_1(x) = 1$ and $F_2(x) = x$. These polynomials also satisfy the following recurrence relation [53]:

$$F'_{n+1}(x) = \frac{n+1}{2}F_n(x) + \frac{x}{2}F'_n(x). \tag{7}$$

The first five Fibonacci polynomials are as follows:

$$F_1(x) = 1, F_2(x) = x, F_3(x) = x^2 + 1, \tag{8}$$

$$F_4(x) = x^3 + 2x, F_5(x) = x^4 + 3x^2 + 1, \tag{9}$$

It is also shown in [54] that the standard bases of the polynomial functions, x^{n-1} for $n \geq 1$, can be uniquely written in terms of $F_0, F_1, F_2, \ldots, F_n$ as follows:

$$x^{n-1} = \Sigma_{j=0}^{\left[\frac{n}{2}\right]} (-1)^j \left(\begin{pmatrix} n \\ j \end{pmatrix} - \begin{pmatrix} n \\ j-1 \end{pmatrix} \right) F_{n-2j}, \tag{10}$$

where $F_0(x) = 0$ is assumed.

2.4 Finite-Dimensional Reproducing Kernel Hilbert Space

Here, we construct a Fibonacci polynomial RKHS of finite dimension and derive its reproducing kernel.

Definition 2. *Let $H_N[0,1] = \text{span}\{F_1(x), \ldots, F_N(x)\}$. Every function $f \in H_N[0,1]$ admits a unique representation of the form*

$$f(x) = \sum_{j=1}^{N} \omega_j F_j(x), \tag{11}$$

for some coefficients $\omega_1, \ldots, \omega_N \in \mathbb{R}$. The inner product and the induced norm on H_N are defined by

$$(f,h)_{H_N} = \sum_{j=1}^{N} \omega_j \mu_j, \tag{12}$$

$$\|f\|_{H_N} = \sqrt{(f,f)_{H_N}} = \sqrt{\sum_{j=1}^{N} \omega_j^2}, \tag{13}$$

where

$$f(x) = \sum_{j=1}^{N} \omega_j F_j(x), \qquad h(x) = \sum_{j=1}^{N} \mu_j F_j(x).$$

Theorem 4. *The space $H_N[0,1]$ is a finite-dimensional Hilbert space and possesses a reproducing kernel. Its reproducing kernel is given by*

$$K(x,y) = \sum_{j=1}^{N} F_j(x) F_j(y),$$

and it satisfies the following properties:

$$1) \quad K(\cdot,y) \in H_N, \qquad \forall\, y \in [0,1], \tag{14}$$

$$2) \quad (K(\cdot,y), f(\cdot))_{H_N} = f(y), \qquad \forall\, f \in H_N, \qquad \forall\, y \in [0,1], \tag{15}$$

where property (2) is referred to as the reproducing property of the kernel.

Proof. It can be easily seen that for any fixed y in $[0,1]$, we have:

$$K(x,y) = \sum_{j=1}^{N} F_j(x) F_j(y) \in H_N[0,1]. \tag{16}$$

Furthermore, for any function $f(x) = \sum_{j=1}^{N} \omega_j F_j(x)$ belonging to $H_N[0,1]$, the reproducing property of the kernel is obtained using the definition of the inner product in (12) as follows:

$$\begin{aligned}(K(\cdot,y), f(\cdot))_{H_N} &= \Big(\sum_{j=1}^{N} F_j(\cdot)F_j(y),\, f(\cdot)\Big)_{H_N} \\ &= \Big(\sum_{j=1}^{N} F_j(\cdot)F_j(y),\, \sum_{j=1}^{N} \omega_j F_j(\cdot)\Big)_{H_N} \\ &= \sum_{j=1}^{N} \omega_j F_j(y) \\ &= f(y).\end{aligned} \tag{17}$$

□

Without loss of generality, let $L_1 u = 0$ and $L_2 u = 0$ denote the boundary condition of the given PDE, where L_1 and L_2 are linear operators. Now, we reconstruct the reproducing kernel $K(x,y)$ such that it satisfies the boundary conditions of the given problem. To this end, using the Fibonacci reproducing kernel $K(x,y)$, the kernels $K_1(x,y)$ and $K_2(x,y)$ are constructed as follows:

$$K_1(x,y) = K(x,y) - \frac{L_{1,x}K(x,y)L_{1,y}K(x,y)}{L_{1,x}L_{1,y}K(x,y)}, \tag{18}$$

$$K_2(x,y) = K_1(x,y) - \frac{L_{2,x}K_1(x,y)L_{2,y}K_1(x,y)}{L_{2,x}L_{2,y}K_1(x,y)}, \tag{19}$$

where the subscript x in $L_{1,x}$ indicates that the operator L_1 acts on the single-variable function in terms of x in the kernel, considering y as a constant. In the next theorem, we see that $K_2(x,y)$ will be a reproducing kernel under certain conditions and will precisely satisfy the boundary conditions $L_1 u = 0$ and $L_2 u = 0$.

Theorem 5. *[55] If $L_{1,x}L_{1,y}K(x,y) \neq 0$ and $L_{2,x}L_{2,y}K_1(x,y) \neq 0$, then $K_2(x,y)$ is a reproducing kernel and precisely satisfies the boundary conditions $L_1 u = 0$ and $L_2 u = 0$.*

Theorem 6. *([68]) (Two-dimensional RKHS) Suppose that $H_1[a,b]$ and $H_2[c,d]$ are the reproducing kernel Hilbert spaces, then the direct product $H(\overline{\Omega}) = H_1[a,b] \otimes H_2[c,d]$ is a two-dimensional reproducing kernel space with the following inner product*

$$\langle u_1 u_2, v_1 v_2 \rangle_H = \langle u_1, v_1 \rangle_{H_1} \langle u_2, v_2 \rangle_{H_2},$$

and we can obtain the reproducing kernel of $H(\overline{\Omega})$ as $K(X,Y) = K(x_1,x_2,y_1,y_2) = R_1(x_1,y_1)R_2(x_2,y_2)$, where $R_1(x_1,y_1)$ and $R_2(x_2,y_2)$ are the reproducing kernels of $H_1[a,b]$ and $H_2[c,d]$, respectively.

2.5 RKHS on Bounded Domains

Definition 3. *([56]) (One-dimensional RKHS) Suppose that $-\infty < a < b < \infty$, we define the space $H^m[a,b]$ as follows*

$$H^m[a,b] = \{u : [a,b] \to \mathbb{R} | u^{(i)}(x) \in AC[a,b], i = 0,1,\ldots,m-1,\ u^{(m)}(x) \in L^2[a,b]\}.$$

The inner product in the space $H^m[a,b]$ is given by

$$\langle u,v \rangle_{H^m} = \langle u^{(m)}, v^{(m)} \rangle + \sum_{j=0}^{m-1} u^{(j)}(a)v^{(j)}(a), \qquad u,v \in H^m, \tag{20}$$

where $\langle u,v \rangle = \int_a^b u(x)v(x)dx$.

Theorem 7. *([56]) The space $H^m[a,b]$ is a RKHS and reproducing kernel $K(x,y) : [a,b] \times [a,b] \to \mathbb{R}$ can be denoted by*

$$K(x,y) = \begin{cases} ((m-1)!)^{-2} \int_a^x (x-z)^{m-1}(y-z)^{m-1}dz + \sum_{i=0}^{m-1} (i!)^{-2}(x-a)^i(y-a)^i, & x < y, \\ ((m-1)!)^{-2} \int_a^y (x-z)^{m-1}(y-z)^{m-1}dz + \sum_{i=0}^{m-1} (i!)^{-2}(x-a)^i(y-a)^i, & y \leq x. \end{cases}$$

We now consider the subspace of $H^m[a,b]$ by applying condition $B(u) = 0$ on it. Since the norm and inner product of this subspace are defined exactly as $H^m[a,b]$, it is also a RKHS.

Definition 4. *(Subspace of $H^m[a,b]$) The inner product space $\widetilde{H}^m[a,b]$ for function u is defined as $\widetilde{H}^m[a,b] = \{u|u(x) \in H^m[a,b],\ B(u) = 0\}$.*

It is shown in [68] that $\widetilde{H}^m[a,b]$ is a closed subspace of $\mathsf{H}^m[a,b]$. The following theorem describes how to find the reproducing kernel of a RKHS $\widetilde{H}^m[a,b]$ associated with $B(u) = 0$.

Theorem 8. *([56]) The space $\widetilde{H}^m[a,b]$ is a RKHS and its RK is given by*

$$R(x,y) = K(x,y) - \frac{B_x K(x,y) B_y K(x,y)}{B_x B_y K(x,y)}, \qquad B_x B_y K(x,y) \neq 0, \qquad K(x,y) \in H^m[a,b],$$

and the subscript x on the operators B_x and B_y indicate that the operator B applies to the function of x and y, respectively.

2.5.1 A Reproducing Kernel Collocation Method

We proceed by expressing the numerical approximation of the semi-discrete formulation within the finite-dimensional reproducing kernel Hilbert space H_N. To simplify the presentation, we introduce the following linear operator

$$\mathbb{L} : H(\overline{\Omega}) \longrightarrow C(\overline{\Omega}),$$

as follows:

$$\mathrm{L}\mathbf{u}^n = \frac{b_1}{\Gamma(2-\alpha)}\mathbf{u}^n + \mathfrak{L}\mathbf{u}^n, \qquad n \geq 1. \tag{21}$$

Accordingly, the semi-discrete formulation is transformed into an equivalent operator equation with domain $H(\overline{\Omega})$ and range $C(\overline{\Omega})$:

$$\mathrm{L}\mathbf{u}^n = \mathbf{f}^n, \qquad n \geq 1, \tag{22}$$

where

$$\mathbf{f}^n = f^n + \frac{b_n}{\Gamma(2-\alpha)}\mathbf{u}^0 - \frac{1}{\Gamma(2-\alpha)}\sum_{i=1}^{n-1}\mathbf{u}^{n-i}(b_{i+1} - b_i), \qquad k \geq 1. \tag{23}$$

Assume that $\mathscr{Q}_M$ denotes a collection of M pairwise distinct points located in the closure of the domain Ω, that is,

$$\mathscr{Q}_M := \{X_1, X_2, \ldots, X_M\} \subset \overline{\Omega}.$$

Based on this set of nodes, we introduce a finite-dimensional function space defined as

$$\mathscr{U}_M = \mathbf{span}\{\mathbf{u}_j(x) = K(X, X_j),\ X_j \in \mathscr{Q}_M\} \subset H(\overline{\Omega}), \tag{24}$$

where $K(X,Y) \in H(\overline{\Omega})$. An approximant $\mathbf{u}_M^n$ to $\mathbf{u}^n$ can be obtained by calculating a truncated series based on trial functions as follows

$$\mathbf{u}^n(X) \approx \mathbf{u}_M^n(X) := \sum_{j=1}^{M}\alpha_j^n \mathbf{u}_j(X) = [\mathbf{u}_1(X)\ \mathbf{u}_2(X)\ \cdots\ \mathbf{u}_M(X)]\begin{pmatrix}\alpha_1^n \\ \alpha_2^n \\ \vdots \\ \alpha_M^n\end{pmatrix}. \tag{25}$$

To compute the interpolation coefficients $\{\alpha_j^n\}_{j=1}^M$, we enforce the collocation conditions on the points in $\mathscr{Q}_M$ according to (22). This leads to the relations

$$\mathrm{L}\mathbf{u}_M^n(X_i) = \sum_{j=1}^{M}\alpha_j^n \mathrm{L}\mathbf{u}_j(X_i) = \mathbf{f}^n(X_i), \quad i = 1, 2, \ldots, M. \tag{26}$$

Here, λ_i denotes the functional corresponding to the application of the differential operator followed by evaluation at $X_i \in \mathscr{Q}_M$. Collectively, the set of functionals is written as

$$\Lambda_M = \{\lambda_1, \lambda_2, \dots, \lambda_M\},$$

which may include different differential operators. The associated collocation matrix $\mathbf{K}_{\Lambda_M,\mathscr{Q}_M}$ is in general non-symmetric, with entries

$$(\mathbf{K}_{\Lambda_M,\mathscr{Q}_M})_{ij} = \lambda_i[\mathbf{u}_j] = \lambda_i^X \widetilde{K}(X, X_j), \quad 1 \le i, j \le M, \tag{27}$$

where the superscript X emphasizes that the functional λ_i acts on the variable X. Consequently, the unknown coefficients $\{\alpha_j^n\}_{j=1}^M$ can be obtained by solving this system of linear equations:

$$\mathbf{K}^n_{\Lambda_M,\mathscr{Q}_M}[\alpha]^n = \mathbf{f}^n_{\mathscr{Q}_M},$$

where $[\alpha]^n = [\alpha_1^n \ \alpha_2^k \ \cdots \ \alpha_M^n]^T$, $\mathbf{f}^n_{\mathscr{Q}_M} = [\mathbf{f}^n(X_1) \ \mathbf{f}^n(X_2) \ \cdots \ \mathbf{f}^n(X_M)]^T$ and

$$\mathbf{K}_{\Lambda_M,\mathscr{Q}_M} = \begin{pmatrix} \lambda_1^X \widetilde{K}(X,X_1) & \lambda_2^X \widetilde{K}(X,X_1) & \cdots & \lambda_n^x \widetilde{K}(X,X_1) \\ \lambda_1^X \widetilde{K}(X,X_2) & \lambda_2^X \widetilde{K}(X,X_2) & \cdots & \lambda_n^x \widetilde{K}(X,X_2) \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^X \widetilde{K}(X,X_M) & \lambda_2^X \widetilde{K}(X,X_M) & \cdots & \lambda_M^X \widetilde{K}(X,X_M) \end{pmatrix}. \tag{28}$$

Suppose that $\mathscr{Q}_n = \{X_i\}_{i=1}^M$ and $\mathscr{U}_M = Span\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_M\}$, applying Gram-Schmidt orthogonalization process to $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_M\}$, we can obtain

$$\bar{\mathbf{u}}_i(X) = \sum_{k=1}^{i} \beta_{ik} \mathbf{u}_k(X), \ (\beta_{ii} > 0, i = 1, 2, \dots, M). \tag{29}$$

Therefore $\{\bar{\mathbf{u}}_1, \bar{\mathbf{u}}_2, \dots, \bar{\mathbf{u}}_M\}$ is a orthonormal basis for $\mathscr{U}_M$. Now, we can write the interpolant $\mathbf{u}_M^n(x)$ to $\mathbf{u}^n$ at $\mathscr{Q}_M$ in the following form

$$\mathbf{u}^n(X) \approx \mathbf{u}_M^n(X) = \sum_{i=1}^{M} \mathbf{u}^n(X_i) \bar{\mathbf{u}}_i(X). \tag{30}$$

3 Numerical Results

Several numerical simulations are carried out to evaluate the efficiency of the proposed technique. The outcomes demonstrate its stable performance and high level of accuracy across different configurations of the discretization parameters M and N.

Example 1. Consider the time-fractional diffusion problem with

$$\mathfrak{L}u := -\sum_{i=1}^{2} \partial^2_{x_k^2} u, \tag{31}$$

and

$$f(X,t) = -\frac{\sin(\pi x_1)(t\pi^2 x_2^2 \Gamma(2-\alpha) - t\pi^2 x_2 \Gamma(2-\alpha) - 2t\Gamma(2-\alpha) + t^{1-\alpha} x_2^2 - t^{1-\alpha} x_2)}{\Gamma(2-\alpha)}. \tag{32}$$

Let $X = (x_1, x_2)$ represent the spatial variables and let $\Omega = (0,1) \times (0,1)$ denote the computational domain. The problem admits the exact solution

$$u(X,t) = t \, \sin(\pi x_1)(x_2^2 - x_2).$$

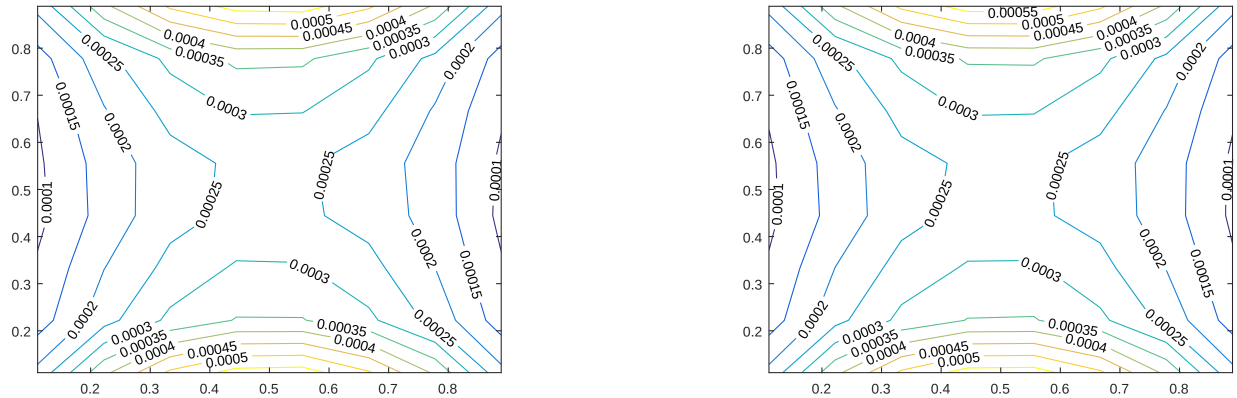
To carry out the numerical experiments, the proposed algorithm is applied on a uniform tensor-product discretization of the domain, given by

$$\{(x_i, y_i)\}_{i=1}^M = \left\{ \frac{i}{M_1 + 1} \right\}_{i=1}^{M_1} \times \left\{ \frac{i}{M_1 + 1} \right\}_{i=1}^{M_1}, \quad M = M_1 \times M_1.$$

The numerical results are presented in Figures 1a and 1b, which display the contour plots of the error

$$|u^N(X_i) - u_M^N(X_i)|, \quad i = 1, 2, \dots, M,$$

for $M = 64$ and $N = 15$.



(a) $\alpha = \frac{9}{10}$ **(b)** $\alpha = \frac{3}{5}$

Figure 1. The error $|u^N(X_i) - u_M^N(X_i)|,\ i = 1,2,\ldots,M$, with $M = 64$, $N = 15$ for Example 1.

Example 2. Consider the time-fractional diffusion problem with

$$\mathfrak{L}u := -\sum_{i=1}^{2} \partial^2_{x_k^2} u, \tag{33}$$

and

$$f(X,t) = -2tx_2^2 + 2tx_2 - 2tx_1^2 + 2x_1 t + \frac{t^{1-\alpha} x_1 x_2 (x_1 x_2 - x_1 - x_2 + 1)}{\Gamma(2-\alpha)}. \tag{34}$$

In this setting, $X = (x_1, x_2)$ refers to the spatial variable and $\Omega = (0,1)\times(0,1)$ denotes the solution domain. The problem admits the exact solution

$$u(X,t) = t\,(x_1^2 - x_1)(x_2^2 - x_2).$$

For the numerical investigation, the proposed method is implemented on a uniform tensor-product mesh, which is constructed as

$$\{(x_i, y_j) : 1 \le i, j \le M_1\} = \left\{\frac{i}{M_1+1}\right\}_{i=1}^{M_1} \times \left\{\frac{j}{M_1+1}\right\}_{j=1}^{M_1}, \qquad M = M_1^2.$$

Figures 2a and 2b present the numerical results, showing contour plots of the error

$$|u^N(X_i) - u_M^N(X_i)|, \quad i = 1,2,\ldots,M,$$

for $M = 64$ and $N = 15$.

Example 3. Consider the time-fractional diffusion problem with

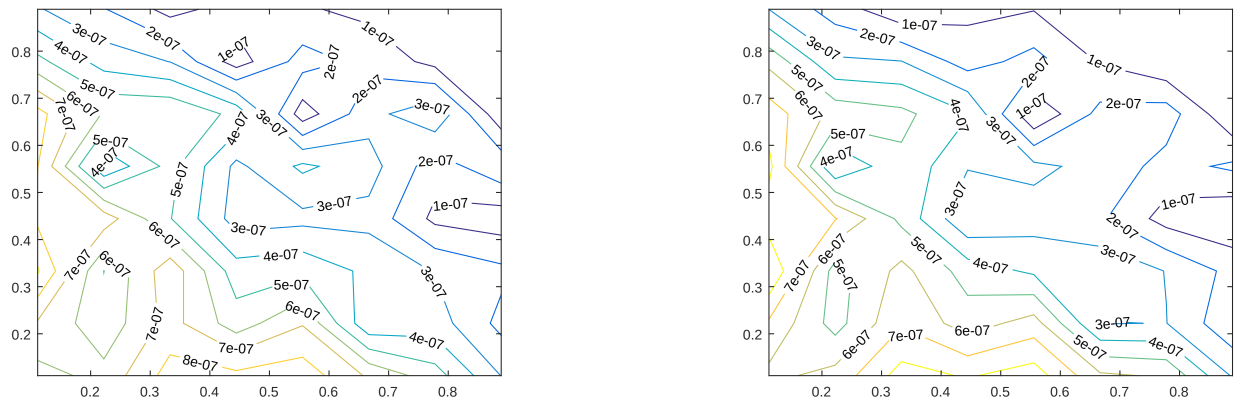
$$\mathfrak{L}u := -\sum_{i=1}^{2} \partial^2_{x_k^2} u, \tag{35}$$

and

$$\begin{aligned} f(X,t) &= -t^2(2x_2e^{x_2} - 2x_2 - 2e^{x_2} + 2 + x_1^2x_2e^{x_2} + x_1^2e^{x_2} - x_2e^{x_2}x_1 - x_1e^{x_2}) \\ &+ \frac{2t^{-\alpha+2}x_1(x_2e^{x_2}x_1 - x_2e^{x_2} - x_2x_1 + x_2 - e^{x_2}x_1 + e^{x_2} + x_1 - 1)}{\Gamma(-\alpha+3)}, \end{aligned} \tag{36}$$

where $X = (x_1, x_2)$, $\Omega = (0,1)\times(0,1)$. Then $u(X,t) = t^2(x_1^2 - x_1)(e^{x_2} - 1)(-1 + x_2)$ is the exact solution. To evaluate the proposed method, we use a uniform tensor-product grid defined by

$$\{(x_i, y_i)\}_{i=1}^{M} = \left\{\frac{i}{M_1+1}\right\}_{i=1}^{M_1} \times \left\{\frac{i}{M_1+1}\right\}_{i=1}^{M_1}, \quad M = M_1 \times M_1.$$



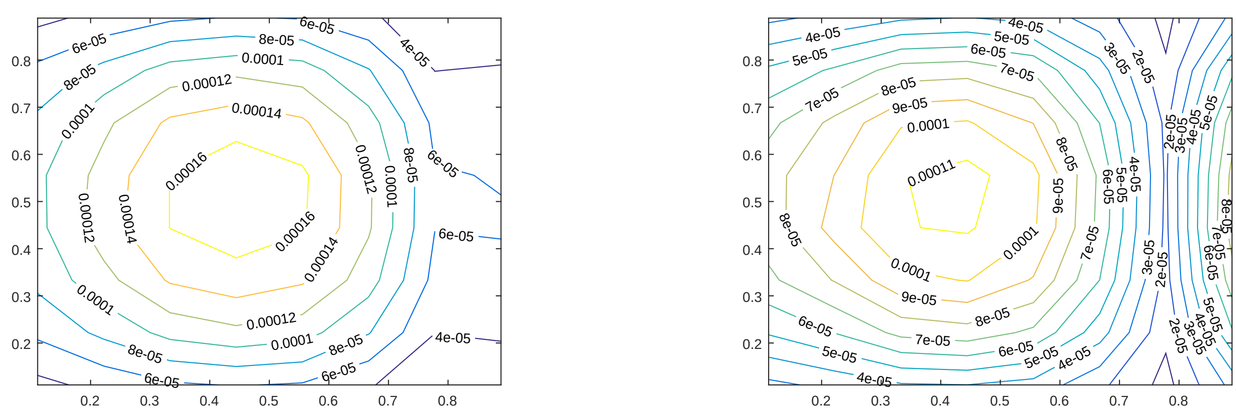
(a) $\alpha = \frac{9}{10}$ **(b)** $\alpha = \frac{7}{10}$

Figure 2. The accuracy of the proposed method is assessed by evaluating the pointwise errors $E_i = \left|u^N(X_i) - u_M^N(X_i)\right|$, $i = 1,2,\ldots,M$, for $M = 64$ and $N = 15$ in Example 2.

The results are illustrated in Figures 3a and 3b, where the contour plots depict the error

$$|u^N(X_i) - u_M^N(X_i)|, \quad i = 1,2,\ldots,M,$$

with $M = 64$ and $N = 15$.



(a) $\alpha = \frac{4}{5}$ **(b)** $\alpha = \frac{1}{2}$

Figure 3. The pointwise error, defined as $\left|u^N(X_i) - u_M^N(X_i)\right|$, $i = 1,2,\ldots,M$, is computed for Example 3 with $M = 64$ and $N = 15$.

4 Conclusion

In this paper, a reproducing kernel collocation method is developed to solve two-dimensional time-fractional diffusion equations. The proposed approach combines spatial discretization using a Fibonacci polynomial reproducing kernel method with temporal discretization via a finite difference scheme, making it computationally efficient due to its simple implementation and reasonable accuracy. The effectiveness of the method is demonstrated through the obtained numerical solutions and error norms, showcasing its ability to achieve accurate numerical solutions for time-fractional diffusion equations.

Data Availability

All data in the paper are available from the corresponding authors upon reasonable request.

Conflicts of Interest

The author declares that there is no conflict of interest.

Ethical Considerations

The author has diligently addressed ethical concerns, such as informed consent, plagiarism, data fabrication, misconduct, falsification, double publication, redundancy, submission, and other related matters.

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References

[1] I. Podlubny, Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some in Science and Engineering, Academic Press, (1998).

[2] K. B. Oldham, J. Spanier, The Fractional Calculus: Theory and Applications of Differentiation and Integration to Arbitrary Order (Dover Books on Mathematics), Dover Publications, (2006).

[3] S. G. SSamko, O. I. Marichev, A. A. Kilbas, Fractional integrals and derivatives: Theory and applications, London: Taylor & Francis, (2002).

[4] R. Hilfer, Applications Of Fractional Calculus In Physics, Wspc, (2000).

[5] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, Theory and Applications of Fractional Differential Equations (Volume 204) (North-Holland Mathematics Studies, Volume 204) (1st ed.), Elsevier Science, (2006).

[6] R. Du, W. Cao, Z. Sun, A compact difference scheme for the fractional diffusion-wave equation, Applied Mathematical Modelling, 34(10), 2998–3007, (2010).

[7] F.Liu, P. Zhuang, V. Anh, I. Turner, K. Burrage, Stability and convergence of the difference methods for the space-time fractional advection-diffusion equation, Applied Mathematics and Computation, 191(1), 12–20, (2007).

[8] P. Zhuang, F. Liu, V. Anh, I. Turner, New Solution and Analytical Techniques of the Implicit Numerical Method for the Anomalous Subdiffusion Equation, SIAM Journal on Numerical Analysis, 46(2), 1079–1095, (2008).

[9] F. Liu, C. Yang, K. Burrage, Numerical method and analytical technique of the modified anomalous subdiffusion equation with a nonlinear source term, Journal of Computational and Applied Mathematics, 231(1), 160–176, (2009).

[10] R. Scherer, S. Kalla, L. Boyadjiev, B. Al-Saqabi, Numerical treatment of fractional heat equations, Applied Numerical Mathematics, 58(8), 1212–1223, (2008).

[11] M. Fardi, B., Raeisi, M. Ahmadi Darani, An RBF-based method with optimal point selection for solving two-dimensional multi-term time-fractional PIDEs with weakly, Mathematics and Computers in Simulation, 242, 96–120, (2026).

[12] S. Mohammadi, M. Fardi, M. Ghasemi, A numerical investigation with energy-preservation for nonlinear space-fractional Klein-Gordon-Schrodinger system, Computational and Applied Mathematics, 42, 356, (2023).

[13] M. Fardi, A kernel-based method for solving the time-fractional diffusion equation, Numerical Methods Partial Differential Equations, 1–15, (2023).

[14] M. Fardi, A kernel-based pseudo-spectral method for multi-term and distributed order time-fractional diffusion equations, Numerical Methods Partial Differential Equations, 1–22, (2022).

[15] M. Fardi, B. Raeisi, M. Ahmadi Darani, A stable numerical investigation based on geometric greedy points for 2D time-fractional partial integro-differential equations with singular kernels, Engineering Analysis with Boundary Elements, 172, 106129, (2025).

[16] M. Fardi, M.A. Zaky, B. Azarnavid, Fractional convection-diffusion systems in complex 2D and 3D geometries: A Bernoulli polynomial-based kernel method, Applied Numerical Mathematics, 218, 275–297, (2025).

[17] C. F. Lorenzo, T. T. Hartley, Variable order and distributed order fractional operators, Nonlinear Dynamic, 29(1-4), 57–98, (2002).

[18] H. Sun, W. Chen, H. Wei, Y. Chen, A comparative study of constant-order and variable-0rder fractional models in characterizing memory property of systems, Eur. Phys. J. Spec. Top., 193(1), 185–192, (2011).

[19] H. Sun, A. Chang, Y. Zhang, W. Chen, A review on variable-order fractional differential equations: mathematical foundations, physical models, numerical methods and applications, Fract. Calc. Appl. Anal., 22(1), 27–59, (2019).

[20] A. V. Chechkin, R. Gorenflo, I. M. Sokolov, Fractional diffusion in inhomogeneous media, J. Phys. A: Gen. Phys., 38 (42), 679–684, (2005).

[21] H. Sun, W. Chen, Y. Chen, Variable-order fractional differential operators in anomalous diffusion modeling, Physica A, 388(21), 4586–4592, (2009).

[22] P. Zhuang, F. Liu, V. Anh, I. Turner, Numerical methods for the variable-order fractional advection-diffusion equation with a nonlinear source term, SIAM J. Numer. Anal., 47(3), 1760–1781, (2009).

[23] C. F. M. Coimbra, Mechanics with variable-order differential operators, Ann. Phys., 12(11-12), 692–703, (2003).

[24] G. Diazand, C. F. M. Coimbra, Nonlinear dynamics and control of a variable order oscillator with application to the van der Pol equation, Nonlinear Dynamic, 56(1), 145–157, (2009).

[25] Y. Jia, M. Xu, Y. Lin, A numerical solution for variable order fractional functional differential equation, Applied Mathematics Letter, 64, 125–130, (2017).

[26] I. M. Sokolov, J. Klafter, From diffusion to anomalous diffusion: a century after Einstein's Brownian motion, Chaos, 15(2), 26103, (2005).

[27] P. Kumar, S. K. Chaudhary, Analysis of fractional order control system with performance and stability, Int. J. Eng. Sci. Tech., 9(5), 408–416, (2017).

[28] A. D. Obembe, M. E. Hossain, S. A. Abu-Khamsin, Variable-order derivative time fractional diffusion model for heterogeneous porous media, J. Petrol. Sci. Eng., 152, 391–405, (2017).

[29] S. Jordan, Metzler Lexikon der deutsch-judischen Literatur. Herausgegeben von Andreas B. Kilcher. Stuttgart: Metzler, 2000. xx + 664 Seiten. 39,90. Monatshefte, XCVII(2), 339–341, (2005).

[30] B. Berkowitz, H. Scher, S. E. Silliman, Anomalous transport in laboratory-scale, heterogeneous porous media, Water Resources Research, 36(1), 149–158, (2000).

[31] R. Metzler, J. Klafter, Subdiffusive transport close to thermal equilibrium: From the Langevin equation to fractional diffusion, Physical Review E, 61(6), 6308–6311, (2000).

[32] I. M. Sokolov, J. Klafter, From diffusion to anomalous diffusion: A century after Einstein's Brownian motion, Chaos: An Interdisciplinary Journal of Nonlinear Science, 15(2), 026103, (2005).

[33] S. B. Yuste, L. Acedo, K. Lindenberg, Reaction front in an $A+B \to C$ reaction-subdiffusion process. Physical Review E, 69(3), (2004).

[34] F. Mainardi, M. Raberto, R. Gorenflo, E. Scalas, Fractional calculus and continuous-time finance II: the waiting-time distribution. Physica A: Statistical Mechanics and Its Applications, 287(3-4), 468–481, (2000).

[35] L. Sabatelli, S. Keating, J. Dudley, P. Richmond, Waiting time distributions in financial markets. The European Physical Journal B - Condensed Matter, 27(2), 273–275, (2002).

[36] M. G. Hall, T. R. Barrick, From diffusion-weighted MRI to anomalous diffusion imaging. Magnetic Resonance in Medicine, 59(3), 447–455, (2008).

[37] S. B. Yuste, E. Abad, K. Lindenberg, Reaction-subdiffusion model of morphogen gradient formation. Physical Review E, 82(6), (2010).

[38] B. Jin, W. Rundell, An inverse problem for a one-dimensional time-fractional diffusion problem. Inverse Problems, 28(7), 075010, (2012).

[39] M. Krasnoschok, Time-fractional diffusion equation with dynamical boundary condition. Fractional Differential Calculus, 2, 151–178, (2016).

[40] G. H. Zheng, T. Wei, A new regularization method for a Cauchy problem of the time fractional diffusion equation. Advances in Computational Mathematics, 36(2), 377–398, (2011).

[41] S. Farzaneh Javan, S. Abbasbandy, M. A. Fariborzi Araghi, Application of Reproducing Kernel Hilbert Space Method for Solving a Class of Nonlinear Integral Equations. Mathematical Problems in Engineering, 2017, 1–10, (2017).

[42] M. G. Sakar, Iterative reproducing kernel Hilbert spaces method for Riccati differential equations. Journal of Computational and Applied Mathematics, 309, 163–174, (2017).

[43] F. Toutian Isfahani, R. Mokhtari, G. B. Loghmani, M. Mohammadi, Numerical solution of some initial optimal control problems using the reproducing kernel Hilbert space technique. International Journal of Control, 93(6), 1345–1352, (2018).

[44] Z. Zhao, Y. Lin, J. Niu, Convergence order of the reproducing kernel method for solving boundary value problems. Mathematical Modelling and Analysis, 21(4), 466–477, (2016).

[45] Y. Lin, W. Jiang, Numerical method for Stokes' first problem for a heated generalized second grade fluid with fractional derivative. Numerical Methods for Partial Differential Equations, 27(6), 1599–1609, (2010).

[46] H. Sahihi, T. Allahviranloo, S. Abbasbandy, Solving system of second-order BVPs using a new algorithm based on reproducing kernel Hilbert space. Applied Numerical Mathematics, 151, 27–39, (2020).

[47] M. Fardi, M. Ghasemi, Solving nonlocal initial-boundary value problems for parabolic and hyperbolic integro-differential equations in reproducing kernel Hilbert space. Numerical Methods for Partial Differential Equations, 33(1), 174–198, (2016).

[48] R. Ketabchi, R. Mokhtari, E. Babolian, Some error estimates for solving Volterra integral equations by using the reproducing kernel method. Journal of Computational and Applied Mathematics, 273, 245–250, (2015).

[49] S. Vahdati, M. Fardi, M. Ghasemi, Option pricing using a computational method based on reproducing kernel. Journal of Computational and Applied Mathematics, 328, 252–266, (2018).

[50] X. Lu, M. Cui, Existence and Numerical Method for Nonlinear Third-Order Boundary Value Problem in the Reproducing Kernel Space. Boundary Value Problems, 2010(1), 459754, (2010).

[51] N. Kopteva, Error analysis of the $L1$ method on graded and uniform meshes for a fractional-derivative problem in two and three dimensions. Mathematics of Computation, 88(319), 2135-2155, (2019).

[52] M. Li, C. Huang, W. Ming, Mixed finite-element method for multi-term time-fractional diffusion and diffusion-wave equations. Computational and Applied Mathematics, 37(2), 2309–2334, (2017).

[53] E. Erkus-Duman, H. Ciftci, Fibonacci and Lucas Differential Equations. Applications and Applied Mathematics: An International Journal (AAM), 13(2), 10, (2018).

[54] M. Esmaeili, M. Esmaeili, A Fibonacci-polynomial based coding method with error detection and correction. Computers & Mathematics with Applications, 60(10), 2738–2752, (2010).

[55] B. Azarnavid, M. Nabati, M. Emamjome, K. Parand, Imposing various boundary conditions on positive definite kernels. Applied Mathematics and Computation, 361, 453–465, (2019).

[56] Y. Wang, M. Du, F. Tan, Z. Li, T. Nie, Using reproducing kernel for solving a class of fractional partial differential equation with non-classical conditions. Applied Mathematics and Computation, 219(11), 5918–5925, (2013).

[57] H. Beyrami, T. Lotfi, K. Mahdiani, Stability and error analysis of the reproducing kernel Hilbert space method for the solution of weakly singular Volterra integral equation on graded mesh. Applied Numerical Mathematics, 120, 197–214, (2017).

[58] W. Jiang, Y. Lin, Approximate solution of the fractional advection-dispersion equation. Computer Physics Communications, 181(3), 557–561, (2010).

[59] F. Geng, M. Cui, Solving a nonlinear system of second order boundary value problems. Journal of Mathematical Analysis and Applications, 327(2), 1167–1181, (2007).

[60] M. Foroutan, A. Ebadian, R. Asadi, Reproducing kernel method in Hilbert spaces for solving the linear and nonlinear four-point boundary value problems. International Journal of Computer Mathematics, 95(10), 2128–2142, (2017).

[61] B. Azarnavid, The Bernoulli polynomials reproducing kernel method for nonlinear Volterra integro-differential equations of fractional order with convergence analysis. Computational and Applied Mathematics, 42(1), 8, (2023).

[62] M. Fardi, M. A. Zaky, B. Azarnavid, Fractional convection-diffusion systems in complex 2D and 3D geometries: A Bernoulli polynomial-based kernel method. Applied Numerical Mathematics, 218, 275–297, (2025).

[63] M. Khaleghi, M. T. Moghaddam, E. Babolian, S. Abbasbandy, Solving a class of singular two-point boundary value problems using new effective reproducing kernel technique. Applied Mathematics and Computation, 331, 264–273, (2018).

[64] B. Azarnavid, M. Fardi, H. Emami, Efficient polynomial kernel-based method for multi-term time-fractional diffusion systems in regular and irregular domains. Chaos, Solitons & Fractals, 199, 116666, (2025).

[65] B. Azarnavid, M. Nabati, M. Emamjomeh, The iterative reproducing kernel Hilbert space method based on the Fibonacci polynomials for the nonlinear fractional differential equations with fractional integral conditions. Journal of Advanced Mathematical Modeling, 13(3), 344–354, (2023).

[66] D. Henry, Geometric Theory of Semilinear Parabolic Equations (Lecture Notes in Mathematics, 840), Springer, (1981).

[67] N. Aronszajn, Theory of reproducing kernels. Cambridge, MA: Harvard University, (1951).

[68] M. Cui, Y. Lin, Nonlinear numerical analysis in reproducing kernel Hilbert space, Nova Science Publisher, New York, (2009).