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Research article

Application of Fractional Quantum Calculus on Lane-Emden Type Problem Involving Two Fractional q-Derivatives

Mohamed Houas[1], Mohammad Esmael Samei[2,*], Mohammed K. A. Kaabar[3,4], Manochehr Kazemi[5,*]

[1] Laboratory, FIMA, UDBKM, Khemis Miliana University, Algeria
[2] Department of Mathematics, Faculty of Science, Bu-Ali Sina University, Hamedan, Iran
[3] Gofa Camp, Near Gofa Industrial College and German Adebabay, Nifas Silk-Lafto, 26649 Addis Ababa, Ethiopia
[4] Institute of Mathematical Sciences, Faculty of Science, University of Malaya, Kuala Lumpur 50603, Malaysia
[5] Department of Mathematics, Ashtian Branch, Islamic Azad University, Ashtian, Iran

* **Corresponding author(s):** mesamei@basu.ac.ir; manochehr.kazemi@iau.ac.ir

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Abstract

In this manuscript, we investigate the nonlinear singular $q-$differential equation of Lane-Emden type, by using the general Riemann-Liouville integral and Caputo derivative of $q-$fractional order operators. First, our approach to prove existence and uniqueness is Banach's contraction principle. Then, in the next step, to confirm the existence of at least one solution, we take help from fixed point theorem of Schaefer. Moreover, the stabilities in the sense of Ulam-Hyers and Ulam-Hyers-Rassias are also defined and examined. Finally, we present a comprehensive example to show the applicability of the outcomes.

Keywords: Lane-Emden equation, Fractional $q-$difference equations, Existence, Stability

Mathematics Subject Classification (2020): 39B72, 39A13,30C45

1 Introduction

The differential equations ($\mathbb{DE}$s), in which fractional quantum calculus are used, play an effective role in the analysis of physical mathematical problems, dynamical system and quantum models [1, 2]. There has been a very important progress in the study of the theory of fractional $q-\mathbb{DE}$s [3–6]. Considerable attention has been given to the study of the existence, uniqueness and stability of solutions of $\mathbb{DE}$s involving fractional $q-$calculus [7–9]. The singular $\mathbb{DE}$s and fractional $q-\mathbb{DE}$s are also very important in applied sciences [10–13].

The generalization to fractional Lane-Emden equations (L-$\mathbb{EE}$s) [14–16] significantly enhances their capability to KdV type and Chazy-type second-degree difference equations, integral equations and hereditary properties [17, 18]. Among fractional operators, the Caputo derivative is particularly advantageous for physical modeling due to its tractable initial conditions involving integer-order derivative. In 1908, Jackson provided experimental problem to develop $\mathbb{DE}$s namely as q-differential equations, in under discrete cases [19]. This stands making the Riemann-Liouville (RL) integral and Caputo derivative operators more suitable for boundary value problems with discrete initial conditions.

The classical form of L-E$\mathbb{E}$, can be displayed by [20],

$$\mathbb{D}^2\mathfrak{s}(\tilde{\tau})+\frac{\eta}{\tilde{\tau}}\mathbb{D}^1\mathfrak{s}(\tilde{\tau})+\varphi(\tilde{\tau},\mathfrak{s}(\tilde{\tau}))=\mathfrak{m}(\tilde{\tau}),\qquad \eta\in\mathbb{R}^{\geq 0},\ \tilde{\tau}\in J_0:=]0,1],$$

with continuous real functions φ, $\mathfrak{m}$, under conditions $\mathfrak{s}(0)=\ell_1$, $\mathbb{D}^1\mathfrak{s}(0)=\ell_2$ where $\ell_j\in\mathbb{R}$, $j=1,2$. The L-E type problem involving the fractional calculus have recently been studied by several scholars, see [21–23]. In [23], Mechee and Senu studied the $\mathbb{F}$L-E$\mathbb{P}$ in sense of Caputo fractional derivative,

$$_{C}\mathbb{D}^{\gamma}\mathfrak{s}(\tilde{\tau})+\frac{\eta}{\tilde{\tau}^{\gamma-\delta}}\,{}_{C}\mathbb{D}^{\delta}\mathfrak{s}(\tilde{\tau})+\varphi(\tau,\mathfrak{s}(\tilde{\tau}))=\mathfrak{m}(\tilde{\tau}),\qquad \tau\in J_0,\ \eta\geq 0,$$

under conditions $\mathfrak{s}(0)=\ell_1$, $\mathbb{D}^1\mathfrak{s}(0)=\ell_2$, $\ell_j\in\mathbb{R}$, $j=1,2$, base on the collocation method, where $0<\gamma\leq 2$, $\delta\in J_0$, see [14,21–23]. Also, Ibrahim studied an $\mathbb{F}$L-E type problem involving two Caputo fractional derivatives, as form

$$_{C}\mathbb{D}^{\gamma}\left[{}_{C}\mathbb{D}^{\delta}+\frac{\eta}{\tilde{\tau}}\right]\mathfrak{s}(\tilde{\tau})+\varphi(\tilde{\tau},\mathfrak{s}(\tilde{\tau}))=\mathfrak{m}(\tilde{\tau}),\qquad \tilde{\tau}\in J_0,\ \eta\geq 0,$$

with continuous function φ, under conditions $\mathfrak{s}(0)=\ell_1$, $\mathfrak{s}(1)=\ell_2$, $\ell_j\in\mathbb{R}$, $j=1,2$, where $\mathfrak{m}\in C(\Omega)$, $\Omega:=[0,1]$ [24]. For more information on $\mathbb{F}$L-E type problem, we refer the reader to the works [25,26]. Halder *et al.* in [27], studied the nonlinear fractional functional integral equation ($\mathbb{IE}$) involving RL operator which is defined within the Banach algebra $C([0,a])$, $a>0$ and analyzed it on the Petryshyn's fixed point theorem and the notion of measure of non-compactness, see for more details and related works [28–32].

The L-E$\mathbb{E}$ can analyze some mathematical physics subjects such as astrophysics and stellar structure [33–35]. We consider a class of singular $q-$fractional Lane-Emden problem ($q-\mathbb{F}$L-E$\mathbb{P}$) and examine the existence, uniqueness and stabilities of Ulam-Hyers (UH), Ulam-Hyers-Rassias (UHR) of solutions results of solutions by generalization simultaneously both $q-$fractional derivatives of RL and Caputo type of the form,

$$\begin{cases} {}_{R.L}\mathbb{D}_q^{\gamma}\left[{}_{C}\mathbb{D}_q^{\delta}+\frac{\eta}{\tilde{\tau}^{\varpi}}\right]\mathfrak{s}(\tilde{\tau})=\mathfrak{m}(\tilde{\tau})-\theta\varphi\left(\tilde{\tau},\mathfrak{s}(\tilde{\tau}),{}_{C}\mathbb{D}_q^{\alpha}\mathfrak{s}(\tilde{\tau})\right)-\lambda\psi\left(\tilde{\tau},\mathfrak{s}(\tilde{\tau}),\mathbb{I}_q^{\beta}\mathfrak{s}(\tilde{\tau})\right), & \tilde{\tau}\in J_0,\ 0<q<1,\\ \left[{}_{C}\mathbb{D}_q^{\delta}+\eta\right]\mathfrak{s}(1)=0,\ \vartheta_1\mathfrak{s}(\omega)=\vartheta_2\mathfrak{s}(1), & \mathfrak{s}(0)=0,\end{cases}\tag{1}$$

η, θ, $\lambda>0$, $\varpi\in J_0$, $0<\omega<1$, $\vartheta_j\in\mathbb{R}$, $j=1,2$, for $1<\gamma<2$, $0<\delta<1$, $\alpha<\delta$, where $\mathfrak{m}\in C(\Omega)$ and $\varphi,\psi\in C(\Omega\times\mathbb{R}^2)$ which will be introduced later.

In the next sections, first, some auxiliary notions of fractional quantum calculus, is reminded. The main existence and uniqueness results are proved in Section 3. Followed by, in Section 4, the proposed stability will be checked for $q-\mathbb{F}$L-E$\mathbb{P}$ (1). An illustrative example presented in Section 5, well confirms the details of the obtained results. Finally, in Section 6, we close the paper with the conclusion, to set goals for future works.

2 Preliminaries of $q-$fractional Calculus

The fractional $q-$integral of the RL type [4,36] is defined by

$$\mathbb{I}_q^{\varsigma}\left[\mathfrak{s}(\tilde{\tau})\right]=\begin{cases}\displaystyle\int_0^{\tilde{\tau}}\frac{(\tilde{\tau}-qr)^{(\varsigma-1)}}{\Gamma_q(\varsigma)}\mathfrak{s}(r)\,\mathrm{d}_qr, & \varsigma>0,\\ \mathfrak{s}(\tilde{\tau}), & \varsigma=0,\end{cases}$$

where the $q-$gamma function is defined by $\Gamma_q(\varsigma)=\frac{(1-q)^{(\varsigma-1)}}{(1-q)^{\varsigma-1}}$ and satisfies $\Gamma_q(\varsigma+1)=[\varsigma]_q\Gamma_q(\varsigma)$, such that,

$$[\varsigma]_q=\frac{1-q^{\varsigma}}{1-q},\ (1-q)^{(0)}=1\quad\&\quad(1-q)^{(n)}=\prod_{l=0}^{n-1}\left(1-q^{l+1}\right),n\in\mathbb{N}.$$

Also, $(1-q)^{(\varsigma)}=\prod_{l=0}^{\infty}\frac{1-q^{l+1}}{1-q^{\varsigma+l+1}}$, for $\varsigma\in\mathbb{R}$ [37]. The fractional $q-$derivatives of the RL and Caputo type, are given by

$${}_{R.L}\mathbb{D}_q^{\gamma}\left[\mathfrak{s}(\tilde{\tau})\right]=\begin{cases}\mathbb{D}_q^{[\gamma]}\left[\mathbb{I}_q^{[\gamma]-\gamma}\left[\mathfrak{s}(\tilde{\tau})\right]\right], & \gamma>0,\\ \mathfrak{s}(\tilde{\tau}), & \gamma=0,\end{cases}$$

and

$$ {}_{C}\mathbb{D}_q^{\upsilon}\left[\mathfrak{s}(\tilde{\tau})\right] = \begin{cases} \mathbb{I}_q^{[\upsilon]-\upsilon}\left[\mathbb{D}_q^{[\upsilon]}[\mathfrak{s}(\tilde{\tau})]\right], & \upsilon > 0, \\ \mathfrak{s}(\tilde{\tau}), & \upsilon = 0, \end{cases} $$

respectively, where $0 < q < 1$, $[\gamma]$ is the smallest integer greater than or equal to γ [4, 36].

Lemma 1 ([4, 36]). *Let $\xi, \varsigma \geq 0$ and $\mathfrak{s}$ be a function defined in Ω. Then $\mathbb{I}_q^{\xi}\left[\mathbb{I}_q^{\varsigma}\left[\mathfrak{s}(\tilde{\tau})\right]\right] = \mathbb{I}_q^{\xi+\varsigma}\left[\mathfrak{s}(\tilde{\tau})\right]$ and $\mathbb{D}_q^{\xi}\mathbb{I}_q^{\xi}\left[\mathfrak{s}(\tilde{\tau})\right] = \mathfrak{s}(\tilde{\tau})$.*

Lemma 2 ([4, 36]). *Let $\xi \in \mathbb{R}^{+}\backslash\mathbb{N}$. Then, the following equality is valid*

$$ \mathbb{I}_q^{\xi}\left[{}_{C}\mathbb{D}_q^{\xi}\left[\mathfrak{s}(\tilde{\tau})\right]\right] = \mathfrak{s}(\tilde{\tau}) - \sum_{l=0}^{n-1}\frac{\tilde{\tau}^{l}}{\Gamma_q(l+1)}\mathbb{D}_q^{l}\mathfrak{s}(0), \qquad n = [\xi]. $$

Lemma 3 ([36]). *Suppose that $\xi > 0$ and μ be a positive integer. Then, we have*

$$ \mathbb{I}_q^{\xi}\left[{}_{R.L}\mathbb{D}_q^{\mu}\left[\mathfrak{s}(\tilde{\tau})\right]\right] = {}_{R.L}\mathbb{D}_q^{\mu}\left[\mathbb{I}_q^{\xi}\left[\mathfrak{s}(\tilde{\tau})\right]\right] - \sum_{l=0}^{\mu-1}\frac{\tilde{\tau}^{\xi-\mu+l}}{\Gamma_q(\xi+l-\mu+1)}\mathbb{D}_q^{l}\mathfrak{s}(0). $$

Lemma 4 ([36]). *For $\xi \in \mathbb{R}_{+}$ and $\nu > -1$, we have $\mathbb{I}_q^{\xi}\left[\tilde{\tau}^{(\nu)}\right] = \frac{\Gamma_q(\nu+1)}{\Gamma_q(\xi+\nu+1)}\tilde{\tau}^{(\xi+\nu)}$. If $\nu = 0$, we can obtain $\mathbb{I}_q^{\xi}[1] = \frac{1}{\Gamma_q(\xi+1)}\tilde{\tau}^{(\xi)}$.*

Lemma 5 ([38]). *Let F be a Banach space and $Q : F \to F$ a completely continuous operator. If*

$$ G = \left\{\mathfrak{s} \in F \,:\, \mathfrak{s} = \mu Q\mathfrak{s},\, 0 < \mu < 1\right\}, $$

is bounded, then there exists $\mathfrak{s}^{*} \in G$ such that $Q\mathfrak{s}^{*} = \mathfrak{s}^{*}$.

We consider the Banach space $W = \left\{\mathfrak{s} \,:\, \mathfrak{s} \in C(\Omega)\ \&\ {}_{C}\mathbb{D}_q^{\alpha}\mathfrak{s} \in C(\Omega)\right\}$, to study the $q-\mathbb{F}$L-E$\mathbb{P}$ (1), endowed with the norm

$$ \|\mathfrak{s}\|_W = \|\mathfrak{s}\| + \left\|{}_{C}\mathbb{D}_q^{\alpha}\mathfrak{s}\right\| = \sup_{\tilde{\tau}\in\Omega}|\mathfrak{s}(\tilde{\tau})| + \sup_{\tilde{\tau}\in\Omega}\left|{}_{C}\mathbb{D}_q^{\alpha}\mathfrak{s}(\tilde{\tau})\right|. $$

Now, we consider the Ulam-stability for the $q-\mathbb{F}$L-E$\mathbb{P}$ (1). For $a > 0$ and $g : \Omega \to \mathbb{R}_{+}$, we give the following inequalities

$$ \left|{}_{R.L}\mathbb{D}_q^{\gamma}\left[{}_{C}\mathbb{D}_q^{\delta} + \frac{\eta}{\tilde{\tau}^{\varpi}}\right]\mathfrak{s}(\tilde{\tau}) - \left[\mathfrak{m}(\tilde{\tau}) - \theta\varphi_{\mathfrak{s}}^{\diamond}(\tilde{\tau}) - \lambda\psi_{\mathfrak{s}}^{\diamond}(\tilde{\tau})\right]\right| \leq a, \tag{2} $$

and

$$ \left|{}_{R.L}\mathbb{D}_q^{\gamma}\left[{}_{C}\mathbb{D}_q^{\delta} + \frac{\eta}{\tilde{\tau}^{\varpi}}\right]\mathfrak{s}(\tilde{\tau}) - \left[\mathfrak{m}(\tilde{\tau}) - \theta\varphi_{\mathfrak{s}}^{\diamond}(\tilde{\tau}) - \lambda\psi_{\mathfrak{s}}^{\diamond}(\tilde{\tau})\right]\right| \leq ag(\tilde{\tau}), \tag{3} $$

for $\tilde{\tau} \in \Omega$, where $\varphi_{\mathfrak{s}}^{\diamond}(\tilde{\tau}) = \varphi\left(\tilde{\tau}, \mathfrak{s}(\tilde{\tau}), {}_{C}\mathbb{D}_q^{\alpha}\mathfrak{s}(\tilde{\tau})\right)$ and $\psi_{\mathfrak{s}}^{\diamond}(\tilde{\tau}) = \psi\left(\tilde{\tau}, \mathfrak{s}(\tilde{\tau}), \mathbb{I}_q^{\beta}\mathfrak{s}(\tilde{\tau})\right)$.

Definition 1. *The $q-\mathbb{F}$L-E$\mathbb{P}$ in (1) is stable in*

- UH *sense if there exists a real number $A_{\varphi^{\diamond},\psi^{\diamond},m} > 0$ such that for each $a > 0$ and for each solution t of the inequality (2) there exists a solution $\mathfrak{s}$ of the $q-\mathbb{F}$L-E$\mathbb{P}$ (1) with $\|t-\mathfrak{s}\|_W \leq A_{\varphi^{\diamond},\psi^{\diamond},\mathfrak{m}}a$.*

- UHR *sense with respect to $g \in C(\Omega, \mathbb{R}_{+})$ if there exists a real number $A_{\varphi^{\diamond},\psi^{\diamond},\mathfrak{m},g} > 0$ such that for each $a > 0$ and for each solution t of the inequality (2) there exists a solution $\mathfrak{s}$ of the $q-\mathbb{F}$L-E$\mathbb{P}$ (1) with $\|t-\mathfrak{s}\|_E \leq A_{\varphi^{\diamond},\psi^{\diamond},\mathfrak{m},g}ag(\tilde{\tau})$.*

Remark 1. A function $\mathfrak{s} \in W$ is a solution of the inequality (2) if and only if there exists a function $f : \Omega \to \mathbb{R}$ (which depend on $\mathfrak{s}$) such that $|f(\tilde{\tau})| \leq a$, for each $\tilde{\tau} \in \Omega$ and

$$ {}_{R.L}\mathbb{D}_q^{\gamma}\left[{}_{C}\mathbb{D}_q^{\delta} + \frac{\eta}{\tilde{\tau}^{\varpi}}\right]\mathfrak{s}(\tilde{\tau}) = \mathfrak{m}(r) - \theta\varphi_{\mathfrak{s}}^{\diamond}(\tilde{\tau}) - \lambda\psi_{\mathfrak{s}}^{\diamond}(\tilde{\tau}) + f(\tilde{\tau}), \qquad \tilde{\tau} \in \Omega. $$

In fact, if $h_{\acute{\mathfrak{s}}}(\tilde{\tau}) = \mathfrak{m}(\tilde{\tau}) - \theta\varphi_{\mathfrak{s}}^{\diamond}(\tilde{\tau}) - \lambda\psi_{\mathfrak{s}}^{\diamond}(\tilde{\tau})$ then, from

$$ \acute{\mathfrak{s}}(\tilde{\tau}) = \mathbb{I}_q^{\gamma+\delta}\left[h_{\acute{\mathfrak{s}}}(\tilde{\tau})\right] - \mathbb{I}_q^{\delta}\left[\frac{\eta}{\tilde{\tau}^{\varpi}}\acute{\mathfrak{s}}(\tilde{\tau})\right] + \frac{c_0\tilde{\tau}^{\delta}}{\Gamma_q(\delta+1)} + \frac{c_1\tilde{\tau}^{\delta+1}}{\Gamma_q(\delta+2)} + c_2\tilde{\tau}^{\delta-1} + \mathbb{I}_q^{\gamma+\delta}\left[f(\tilde{\tau})\right], $$

for $c_j \in \mathbb{R}$, $j = 0, 1, 2$, for $\tilde{\tau} \in \Omega$, we will show that

$$ |\acute{\mathfrak{s}}(\tilde{\tau}) - Q\acute{\mathfrak{s}}(\tilde{\tau})| = \left|\mathbb{I}_q^{\gamma+\delta}[f(\tilde{\tau})]\right| \leq a\mathbb{I}_q^{\gamma+\delta}[1], $$

and help us Lemma 4 we have

$$ |\acute{\mathfrak{s}}(\tilde{\tau}) - Q\acute{\mathfrak{s}}(\tilde{\tau})| \leq \frac{1}{\Gamma_q(\gamma+\delta+1)}a\tilde{\tau}^{(\gamma+\delta)}, $$

where $Q : W \to W$ is a completely continuous operator.

3 Main Results

In this section, by utilizing the fixed point theory, we will prove the existence and uniqueness of the solution of $q-\mathbb{FL}$-$\mathbb{EP}$ (1). First, we need the following key lemma.

Lemma 6. *Let $h \in W$ and assume that*

$$\Delta := \vartheta_1 \left[(\delta+1)\,\omega^{\delta} - \omega^{\delta+1}\right] - \vartheta_2 \delta \neq 0. \tag{4}$$

Then the unique solution of $q-\mathbb{FL}$-$\mathbb{EP}$

$$\begin{cases} {}_{R.L}\mathbb{D}_q^{\gamma}\left[{}_{C}\mathbb{D}_q^{\delta} + \frac{\eta}{\tilde{\tau}^{\varpi}}\right]\mathfrak{s}(\tilde{\tau}) = h(\tilde{\tau}), & \tilde{\tau} \in J_0, q<1, \\ \left[{}_{C}\mathbb{D}_q^{\delta} + \eta\right]\mathfrak{s}(1) = 0, & \eta > 0, \\ \vartheta_1 \mathfrak{s}(\omega) = \vartheta_2 \mathfrak{s}(1), \mathfrak{s}(0) = 0, & \vartheta_i \in \mathbb{R}, i = 1,2, \end{cases} \tag{5}$$

$1<\gamma<2, 0<\delta<1, 0<\varpi\leq 1, 0<\omega<1$, *is given by*

$$\begin{aligned} \mathfrak{s}(\tilde{\tau}) = & \int_0^{\tilde{\tau}} \frac{(\tilde{\tau}-qr)^{(\gamma+\delta-1)}}{\Gamma_q(\gamma+\delta)} h(r)\,\mathrm{d}_q r - \int_0^{\tilde{\tau}} \frac{\eta(\tilde{\tau}-qr)^{(\delta-1)}}{\Gamma_q(\delta) r^{\varpi}} \mathfrak{s}(r)\,\mathrm{d}_q r \\ & + \frac{(\delta+1)\tilde{\tau}^{\delta} - \tilde{\tau}^{\delta+1}}{\Delta}\left[\int_0^1 \frac{\vartheta_2(1-qr)^{(\gamma+\delta-1)}}{\Gamma_q(\gamma+\delta)} h(r)\,\mathrm{d}_q r - \int_0^{\omega} \frac{\vartheta_1(\omega-qr)^{(\gamma+\delta-1)}}{\Gamma_q(\gamma+\delta)} h(r)\,\mathrm{d}_q r - \int_0^1 \frac{\vartheta_2\eta(1-qr)^{(\delta-1)}}{\Gamma_q(\delta)} \mathfrak{s}(r)\,\mathrm{d}_q r \right. \\ & \left. + \int_0^{\omega} \frac{\vartheta_1\eta(\omega-qr)^{(\delta-1)}}{\Gamma_q(\delta) r^{\varpi}} \mathfrak{s}(r)\,\mathrm{d}_q r + \frac{2\left[\left(\vartheta_1\omega^{\delta+1}-\vartheta_2\right)+\Delta\right]}{\Gamma_q(\delta+2)} \int_0^1 \frac{(1-qr)^{(\gamma-1)}}{\Gamma_q(\gamma)} h(r)\,\mathrm{d}_q r\right]. \end{aligned} \tag{6}$$

Proof. First, by employing Lemma 2, taking RL-integral fractional operator in the both sides of $q-\mathbb{FL}$-$\mathbb{EE}$ (5), we get

$$\left[{}_{R.L}\mathbb{D}_q^{\delta} + \frac{\eta}{\tilde{\tau}^{\varpi}}\right]\mathfrak{s}(\tilde{\tau}) = \mathbb{I}_q^{\gamma}[h(\tilde{\tau})] + e_0 + e_1\tilde{\tau}, \tag{7}$$

and utilizing Lemma 3,

$$\mathfrak{s}(\tilde{\tau}) = \mathbb{I}_q^{\gamma+\delta}[h(\tilde{\tau})] - \mathbb{I}_q^{\delta}\left[\frac{\eta}{\tilde{\tau}^{\varpi}}\mathfrak{s}(\tilde{\tau})\right] + \frac{e_0\tilde{\tau}^{\delta}}{\Gamma_q(\delta+1)} + \frac{e_1\tilde{\tau}^{\delta+1}}{\Gamma_q(\delta+2)} + e_2\tilde{\tau}^{\delta-1}, \tag{8}$$

where $e_j \in \mathbb{R}$, $j=0,1,2$. The condition $\mathfrak{s}(0)=0$, implies that $e_2=0$. Now, using the conditions $\left[{}_{C}\mathbb{D}_q^{\delta}+\eta\right]\mathfrak{s}(1)=0$ and $\vartheta_1\mathfrak{s}(\omega)=\vartheta_2\mathfrak{s}(1)$, we obtain

$$\begin{aligned} e_0 &= \frac{\Gamma_q(\delta+2)}{\Delta}\left(\vartheta_2\mathbb{I}_q^{\gamma+\delta}[h(1)] - \vartheta_1\mathbb{I}_q^{\gamma+\delta}[h(\omega)] - \vartheta_2\mathbb{I}_q^{\delta}[\eta\mathfrak{s}(1)] + \vartheta_1\mathbb{I}_q^{\delta}\left[\frac{\eta}{\omega^{\varpi}}\mathfrak{s}(\omega)\right] + \frac{\vartheta_1\omega^{\delta+1}-\vartheta_2}{\Gamma_q(\delta+2)}\mathbb{I}_q^{\gamma}[h(1)]\right), \\ e_1 &= \frac{\Gamma_q(\delta+2)}{\Delta}\left(\vartheta_1\mathbb{I}_q^{\gamma+\delta}[h(\omega)] - \vartheta_2\mathbb{I}_q^{\gamma+\delta}[h(1)] + \vartheta_2\mathbb{I}_q^{\delta}[\eta\mathfrak{s}(1)] - \vartheta_1\mathbb{I}_q^{\delta}\left[\frac{\eta}{\omega^{\varpi}}\mathfrak{s}(\omega)\right] - \frac{\vartheta_1\omega^{\delta+1}-\vartheta_2+\Delta}{\Gamma_q(\delta+2)}\mathbb{I}_q^{\gamma}[h(1)]\right). \end{aligned}$$

Thus, inserting the values e_j, $j=0,1,2$ in (8), we get Eq. (6). □

Now, thanks to Lemma 6, we define operator $Q: W \to W$ as form

$$\begin{aligned} Q\mathfrak{s}(\tilde{\tau}) = & \int_0^{\tilde{\tau}} \frac{(\tilde{\tau}-qr)^{(\gamma+\delta-1)}}{\Gamma_q(\gamma+\delta)}\left[\mathfrak{m}(r) - \theta\varphi_{\mathfrak{s}}^{\diamond}(\tilde{\tau}) - \lambda\psi_{\mathfrak{s}}^{\diamond}(\tilde{\tau})\right]\mathrm{d}_q r \\ & - \int_0^{\tilde{\tau}} \frac{\eta(\tilde{\tau}-qr)^{(\delta-1)}}{\Gamma_q(\delta)r^{\varpi}}\mathfrak{s}(r)\,\mathrm{d}_q r + \frac{(\delta+1)\tilde{\tau}^{\delta}-\tilde{\tau}^{\delta+1}}{\Delta}\left[\int_0^1 \frac{\vartheta_2(1-qr)^{(\gamma+\delta-1)}}{\Gamma_q(\gamma+\delta)}\left[\mathfrak{m}(r) - \theta\varphi_{\mathfrak{s}}^{\diamond}(\tilde{\tau}) - \lambda\psi_{\mathfrak{s}}^{\diamond}(\tilde{\tau})\right]\mathrm{d}_q r\right. \\ & - \int_0^{\omega} \frac{\vartheta_1(\omega-qr)^{(\gamma+\delta-1)}}{\Gamma_q(\gamma+\delta)}\left[\mathfrak{m}(r) - \theta\varphi_{\mathfrak{s}}^{\diamond}(\tilde{\tau}) - \lambda\psi_{\mathfrak{s}}^{\diamond}(\tilde{\tau})\right]\mathrm{d}_q r - \int_0^1 \frac{\vartheta_2\eta(1-qr)^{(\delta-1)}}{\Gamma_q(\delta)}\mathfrak{s}(r)\,\mathrm{d}_q r \\ & \left. + \int_0^{\omega} \frac{\vartheta_1\eta(\omega-qr)^{(\delta-1)}}{\Gamma_q(\delta)r^{\varpi}}\mathfrak{s}(r)\,\mathrm{d}_q r + \frac{2\left[\left(\vartheta_1\omega^{\delta+1}-\vartheta_2\right)+\Delta\right]}{\Gamma_q(\delta+2)}\int_0^1 \frac{(1-qr)^{(\gamma-1)}}{\Gamma_q(\gamma)}\left[\mathfrak{m}(r) - \theta\varphi_{\mathfrak{s}}^{\diamond}(\tilde{\tau}) - \lambda\psi_{\mathfrak{s}}^{\diamond}(\tilde{\tau})\right]\mathrm{d}_q r\right], \end{aligned} \tag{9}$$

also, we have

$$\mathbb{D}_q^{\alpha}[Q\mathfrak{s}(\tilde{\tau})] = \int_0^{\tilde{\tau}} \frac{(\tilde{\tau}-qr)^{(-\alpha)}}{\Gamma_q(1-\alpha)}\left(\mathbb{D}_q Q\mathfrak{s}\right)(r)\,\mathrm{d}r, \tag{10}$$

where

$$\begin{aligned}\mathbb{D}_q[Q\mathfrak{s}(\tilde{\tau})] &= \int_0^{\tilde{\tau}} \frac{(\tilde{\tau}-qr)^{(\gamma+\delta-2)}}{\Gamma_q(\gamma+\delta-1)}[\mathfrak{m}(r)-\theta\varphi_{\mathfrak{s}}^{\diamond}(\tilde{\tau})-\lambda\psi_s^{\diamond}(\tilde{\tau})]\,\mathrm{d}_q r - \frac{\eta}{\Gamma_q(\delta-1)}\int_0^{\tilde{\tau}}(\tilde{\tau}-qr)^{(\delta-2)}\frac{s(r)}{r^{\varpi}}\,\mathrm{d}_q r\\ &+ \frac{(\delta+1)[\delta]_q\tilde{\tau}^{\delta-1}-[\delta+1]_q\tilde{\tau}^{\delta}}{\Delta}\Bigg[\int_0^1 \frac{\vartheta_2(1-qr)^{(\gamma+\delta-1)}}{\Gamma_q(\gamma+\delta)}[\mathfrak{m}(r)-\theta\varphi_{\mathfrak{s}}^{\diamond}(\tilde{\tau})-\lambda\psi_s^{\diamond}(\tilde{\tau})]\,\mathrm{d}_q r\\ &-\int_0^{\omega}\frac{\vartheta_1(\omega-qr)^{(\gamma+\delta-1)}}{\Gamma_q(\gamma+\delta)}[\mathfrak{m}(r)-\theta\varphi_{\mathfrak{s}}^{\diamond}(\tilde{\tau})-\lambda\psi_{\mathfrak{s}}^{\diamond}(\tilde{\tau})]\,\mathrm{d}_q r\\ &+\int_0^1\frac{\vartheta_2\eta(1-qr)^{(\delta-1)}}{\Gamma_q(\delta)}s(r)\,\mathrm{d}_q r+\int_0^{\omega}\frac{\vartheta_1\eta(\omega-qr)^{(\delta-1)}}{\Gamma_q(\delta)r^{\varpi}}\mathfrak{s}(r)\,\mathrm{d}_q r\\ &+\frac{2\left[\left(\vartheta_1\omega^{\delta+1}-\vartheta_2\right)+\Delta\right]}{\Gamma_q(\delta+2)}\int_0^1\frac{(1-qr)^{(\gamma-1)}}{\Gamma_q(\gamma)}[m(r)-\theta\varphi_{\mathfrak{s}}^{\diamond}(\tilde{\tau})-\lambda\psi_{\mathfrak{s}}^{\diamond}(\tilde{\tau})]\,\mathrm{d}_q r\Bigg],\end{aligned}$$

such that Δ is defined by (4) and $\varpi \le 1$.

The first result is concerned with the existence and uniqueness of the solution for $q-\mathbb{F}$L-E$\mathbb{P}$ (1) and is based on Banach's fixed point theorem.

Theorem 1. *Let $\varphi,\psi \in C(\Omega\times\mathbb{R}^2)$ and $\mathfrak{m}\in C(\Omega)$ and assume that*

H$_1$) *there exist $k_1, k_2 > 0$ such that for each $\tilde{\tau}\in\Omega$ and $\mathfrak{s}_j, \acute{\mathfrak{s}}_j \in \mathbb{R}$, $j=1,2$, we have*

$$\begin{aligned}|\varphi(\tilde{\tau},\mathfrak{s}_1,\acute{\mathfrak{s}}_1)-\varphi(\tilde{\tau},\mathfrak{s}_2,\acute{\mathfrak{s}}_2)| &\le k_1\left(|\mathfrak{s}_1-\mathfrak{s}_2|+|\acute{\mathfrak{s}}_1-\acute{\mathfrak{s}}_2|\right),\\ |\psi(\tilde{\tau},\mathfrak{s}_1,\acute{\mathfrak{s}}_1)-\psi(\tilde{\tau},\mathfrak{s}_2,\acute{\mathfrak{s}}_2)| &\le k_2\left(|\mathfrak{s}_1-\mathfrak{s}_2|+|\acute{\mathfrak{s}}_1-\acute{\mathfrak{s}}_2|\right).\end{aligned}$$

Then $q-\mathbb{F}$L-E$\mathbb{P}$ (1) has a unique solution if

$$\theta k_1+\lambda\frac{(\Gamma_q(\beta+1)+1)k_2}{\Gamma_q(\beta+1)} < [1-\eta\Phi_2]\,\Phi_1^{-1}, \tag{11}$$

where

$$\Phi_i := \Pi_i + \frac{\Theta_i}{\Gamma_q(2-\alpha)}, \qquad i=1,2, \tag{12}$$

and

$$\Pi_1 := \frac{1}{\Gamma_q(\gamma+\delta+1)}+\frac{(\delta+2)|\vartheta_2|}{|\Delta|\Gamma_q(\gamma+\delta+1)}+\frac{(\delta+2)|\vartheta_1|\omega^{(\gamma+\delta)}}{|\Delta|\Gamma_q(\gamma+\delta+1)}+\frac{2(\delta+2)\left[\left(|\vartheta_1|\omega^{(\delta+1)}+|\vartheta_2|\right)+|\Delta|\right]}{|\Delta|\Gamma_q(\delta+2)\Gamma_q(\gamma+1)},$$

$$\Pi_2 := \frac{\Gamma_q(1-\varpi)}{\Gamma_q(\delta-\varpi+1)}+\frac{(\delta+2)|\vartheta_2|}{|\Delta|\Gamma_q(\delta+1)}+\frac{(\delta+2)|\vartheta_1|\Gamma_q(1-\varpi)\omega^{(\delta-\varpi)}}{|\Delta|\Gamma_q(\delta-\varpi+1)},$$

$$\Theta_1 := \frac{1}{\Gamma_q(\gamma+\delta)}+\frac{\left((\delta+1)[\delta]_q+[\delta+1]_q\right)|\vartheta_2|}{|\Delta|\Gamma_q(\gamma+\delta+1)}+\frac{\left((\delta+1)[\delta]_q+[\delta+1]_q\right)|\vartheta_1|\omega^{\gamma+\delta}}{|\Delta|\Gamma_q(\gamma+\delta+1)}+\frac{2\left((\delta+1)[\delta]_q+[\delta+1]_q\right)\left[\left(|\vartheta_1|\omega^{\delta+1}+|\vartheta_2|\right)+|\Delta|\right]}{|\Delta|\Gamma_q(\delta+2)\Gamma_q(\gamma+1)},$$

$$\Theta_2 := \frac{\Gamma_q(1-\varpi)}{\Gamma_q(\delta-\varpi)}+\frac{\left((\delta+1)[\delta]_q+[\delta+1]_q\right)|\vartheta_2|}{|\Delta|\Gamma_q(\delta+1)}+\frac{\left((\delta+1)[\delta]_q+[\delta+1]_q\right)|\vartheta_1|\Gamma_q(1-\varpi)\omega^{\delta-\varpi}}{|\Delta|\Gamma_q(\delta+1)\Gamma_q(\delta-\varpi+1)}. \tag{13}$$

Proof. First, we show that $QB_\sigma \subset B_\sigma$, where $B_\sigma = \left\{\mathfrak{s}\in W : \|\mathfrak{s}\|_W \le \sigma\right\}$, with

$$\sigma \ge \Phi_1(\theta+\lambda+1)\Lambda\left[1-\left(\Phi_1\left(\theta k_1+\frac{\lambda(\Gamma_q(\beta+1)+1)k_2}{\Gamma_q(\beta+1)}\right)+\eta\Phi_2\right)\right]^{-1},$$

such that $\Lambda=\max\left\{\Lambda_j,\ j=1,2,3\right\}$ and

$$\Lambda_1=\sup_{\tilde{\tau}\in\Omega}|\varphi(\tilde{\tau},0,0)|,\quad \Lambda_2=\sup_{\tilde{\tau}\in\Omega}|\psi(\tilde{\tau},0,0)|,\quad \Lambda_3=\sup_{\tilde{\tau}\in\Omega}|m(\tilde{\tau})|.$$

By (H_1), we have

$$\begin{aligned}
|\psi_{\mathfrak{s}}^{\diamond}(\tilde{\tau})| &= \left|\psi\left(\tilde{\tau},\mathfrak{s}(\tilde{\tau}),\mathbb{I}_q^{\beta}\mathfrak{s}(\tilde{\tau})\right)\right| \\
&\leq \left|\psi\left(\tilde{\tau},\mathfrak{s}(\tilde{\tau}),\mathbb{I}_q^{\beta}\mathfrak{s}(\tilde{\tau})\right)-\psi(\tilde{\tau},0,0)\right|+|\psi(\tilde{\tau},0,0)| \\
&\leq k_2\left(\|\mathfrak{s}\|+\left|\mathbb{I}_q^{\beta}[\|\mathfrak{s}\|]\right|\right)+\Lambda_2 \leq k_2\left(\|\mathfrak{s}\|_S+\frac{\|\mathfrak{s}\|_{\mathfrak{s}}}{\Gamma_q(\beta+1)}\right)+\Lambda_2 \\
&\leq k_2\frac{(\Gamma_q(\beta+1)+1)}{\Gamma_q(\beta+1)}\sigma+\Lambda, \\
|\varphi_{\mathfrak{s}}^{\diamond}(\tilde{\tau})| &= \left|\varphi\left(\tilde{\tau},s(\tilde{\tau}),{}_C\mathbb{D}_q^{\alpha}\mathfrak{s}(\tilde{\tau})\right)\right| \\
&\leq \left|\varphi\left(\tilde{\tau},\mathfrak{s}(\tilde{\tau}),{}_C\mathbb{D}_q^{\alpha}\mathfrak{s}(\tilde{\tau})\right)-\varphi(\tilde{\tau},0,0)\right|+|\varphi(\tilde{\tau},0,0)| \\
&\leq k_1\left(\|\mathfrak{s}\|+{}_C\mathbb{D}_q^{\alpha}\|\mathfrak{s}\|\right)+\Lambda_1 \leq k_1\|\mathfrak{s}\|_{\mathfrak{s}}+\Lambda_1 \leq k_1\sigma+\Lambda.
\end{aligned}$$

Thanks to the above estimates, we get

$$\begin{aligned}
\|Q(s)\| \leq& \left[\frac{1}{\Gamma_q(\gamma+\delta+1)}+\frac{(\delta+2)|\vartheta_2|}{|\Delta|\Gamma_q(\gamma+\delta+1)}+\frac{(\delta+2)|\vartheta_1|\omega^{(\gamma+\delta)}}{|\Delta|\Gamma_q(\gamma+\delta+1)}\right. \\
&\left.+\frac{2(\delta+2)\left[\left(|\vartheta_1|\omega^{(\delta+1)}+|\vartheta_2|\right)+|\Delta|\right]}{|\Delta|\Gamma_q(\delta+2)\Gamma_q(\gamma+1)}\right]\left(\left[\theta k_1+\frac{\lambda(\Gamma_q(\beta+1)+1)k_2}{\Gamma_q(\beta+1)}\right]\sigma+(\theta+\lambda+1)\Lambda\right) \\
&+\eta\left[\frac{\Gamma_q(1-\varpi)}{\Gamma_q(\delta-\varpi+1)}+\frac{(\delta+2)|\vartheta_2|}{|\Delta|\Gamma_q(\delta+1)}+\frac{(\delta+2)|\vartheta_1|\Gamma_q(1-\varpi)\omega^{(\delta-\varpi)}}{|\Delta|\Gamma_q(\delta-\varpi+1)}\right]\sigma \\
=&\,\Pi_1\left(\left(\theta k_1+\frac{\lambda(\Gamma_q(\beta+1)+1)k_2}{\Gamma_q(\beta+1)}\right)\sigma+(\theta+\lambda+1)\Lambda\right)+\eta\Pi_2\sigma \\
=&\left(\Pi_1\left(\theta k_1+\frac{\lambda(\Gamma_q(\beta+1)+1)k_2}{\Gamma_q(\beta+1)}\right)+\eta\Pi_2\right)\sigma+\Pi_1(\theta+\lambda+1)\Lambda.
\end{aligned}$$

This implies that

$$\|Q(\mathfrak{s})\| \leq \left(\Pi_1\left(\theta k_1+\frac{\lambda(\Gamma_q(\beta+1)+1)k_2}{\Gamma_q(\beta+1)}\right)+\eta\Pi_2\right)\sigma+\Pi_1(\theta+\lambda+1)\Lambda.$$

Also, we have

$$\begin{aligned}
\left\|\mathbb{D}_q[Q(\mathfrak{s})]\right\| \leq& \left[\frac{1}{\Gamma_q(\gamma+\delta)}+\frac{\left((\delta+1)[\delta]_q+[\delta+1]_q\right)|\vartheta_2|}{|\Delta|\Gamma_q(\gamma+\delta+1)}+\frac{\left((\delta+1)[\delta]_q+[\delta+1]_q\right)|\vartheta_1|\omega^{\gamma+\delta}}{|\Delta|\Gamma_q(\gamma+\delta+1)}\right. \\
&\left.+\frac{2\left((\delta+1)[\delta]_q+[\delta+1]_q\right)\left[\left(|\vartheta_1|\omega^{\delta+1}+|\vartheta_2|\right)+|\Delta|\right]}{|\Delta|\Gamma_q(\delta+2)\Gamma_q(\gamma+1)}\right]\left(\left(\theta k_1+\frac{\lambda(\Gamma_q(\beta+1)+1)k_2}{\Gamma_q(\beta+1)}\right)\sigma+(\theta+\lambda+1)\Lambda\right) \\
&+\eta\left[\frac{\Gamma_q(1-\varpi)}{\Gamma_q(\delta-\varpi)}+\frac{\left((\delta+1)[\delta]_q+[\delta+1]_q\right)|\vartheta_2|}{|\Delta|\Gamma_q(\delta+1)}+\frac{\left((\delta+1)[\delta]_q+[\delta+1]_q\right)|\vartheta_1|\Gamma_q(1-\varpi)\omega^{\delta-\varpi}}{|\Delta|\Gamma_q(\delta+1)\Gamma_q(\delta-\varpi+1)}\right]\sigma \\
=&\left(\Theta_1\left(\theta k_1+\frac{\lambda(\Gamma_q(\beta+1)+1)k_2}{\Gamma_q(\beta+1)}\right)+\eta\Theta_2\right)\sigma+\Theta_1(\theta+\lambda+1)\Lambda.
\end{aligned}$$

Now, (10) implies that

$$\left\|\mathbb{D}_q^{\alpha}[Q(s)]\right\| \leq \left(\frac{\Theta_1}{\Gamma_q(2-\alpha)}\left(\theta k_1+\frac{\lambda(\Gamma_q(\beta+1)+1)k_2}{\Gamma_q(\beta+1)}\right)+\frac{\eta\Theta_2}{\Gamma_q(2-\alpha)}\right)\sigma+\frac{\Theta_1}{\Gamma_q(2-\alpha)}(\theta+\lambda+1)\Lambda.$$

From the definition of $\|\cdot\|_W$ we have

$$\|Q(\mathfrak{s})\|_W \leq \left[\Phi_1\left(\theta k_1+\frac{\lambda(\Gamma_q(\beta+1)+1)k_2}{\Gamma_q(\beta+1)}\right)+\eta\Phi_2\right]\sigma+\Phi_1(\theta+\lambda+1)\Lambda \leq \sigma.$$

which means that $QB_\sigma \subset B_\sigma$. Now, for $\mathfrak{s}, \acute{\mathfrak{s}} \in B_\sigma$, we obtain

$$\begin{aligned}\left|Q\mathfrak{s}(\tilde{\tau}) - Q\acute{\mathfrak{s}}(\tilde{\tau})\right| \leq & \int_0^{\tilde{\tau}} \frac{(\tilde{\tau}-qr)^{(\gamma+\delta-1)}}{\Gamma_q(\gamma+\delta)} \left[\theta\left|\varphi_{\mathfrak{s}}^{\diamond}(\tilde{\tau}) - \varphi_{\acute{\mathfrak{s}}}^{\diamond}(\tilde{\tau})\right| + \lambda\left|\psi_{\mathfrak{s}}^{\diamond}(\tilde{\tau}) - \psi_{\acute{\mathfrak{s}}}^{\diamond}(\tilde{\tau})\right|\right] \mathrm{d}_q r \\ & + \eta \int_0^{\tilde{\tau}} \frac{(\tilde{\tau}-qr)^{(\delta-1)}}{\Gamma_q(\delta) r^{\varpi}} \left|\mathfrak{s}(r) - \acute{\mathfrak{s}}(r)\right| \mathrm{d}_q r + \frac{\left((\delta+1)\tilde{\tau}^{\delta}+\tilde{\tau}^{\delta+1}\right)|\vartheta_2|}{|\Delta|} \int_0^1 \frac{(1-qr)^{(\gamma+\delta-1)}}{\Gamma_q(\gamma+\delta)} \\ & \times \left[\theta\left|\varphi_{\mathfrak{s}}^{\diamond}(\tilde{\tau}) - \varphi_{\acute{\mathfrak{s}}}^{\diamond}(\tilde{\tau})\right| + \lambda\left|\psi_{\mathfrak{s}}^{\diamond}(\tilde{\tau}) - \psi_{\acute{\mathfrak{s}}}^{\diamond}(\tilde{\tau})\right|\right] \mathrm{d}_q r \\ & + \frac{\left((\delta+1)\tilde{\tau}^{\delta}+\tilde{\tau}^{\delta+1}\right)|\vartheta_1|}{|\Delta|} \int_0^{\omega} \frac{(\omega-qr)^{(\gamma+\delta-1)}}{\Gamma_q(\gamma+\delta)} \left[\theta\left|\varphi_{\mathfrak{s}}^{\diamond}(\tilde{\tau}) - \varphi_{\acute{\mathfrak{s}}}^{\diamond}(\tilde{\tau})\right| + \lambda\left|\psi_{\mathfrak{s}}^{\diamond}(\tilde{\tau}) - \psi_{\acute{\mathfrak{s}}}^{\diamond}(\tilde{\tau})\right|\right] \mathrm{d}_q r \\ & + \frac{\left((\delta+1)\tilde{\tau}^{\delta}+\tilde{\tau}^{\delta+1}\right)|\vartheta_2|\eta}{|\Delta|} \int_0^1 \frac{(1-qr)^{(\delta-1)}}{\Gamma_q(\delta)} \left|\mathfrak{s}(r) - \acute{\mathfrak{s}}(r)\right| \mathrm{d}_q r \\ & + \frac{\left((\delta+1)\tilde{\tau}^{\delta}+\tilde{\tau}^{\delta+1}\right)|\vartheta_1|\eta}{|\Delta|} \int_0^{\omega} \frac{(\omega-qr)^{(\delta-1)}}{\Gamma_q(\delta) r^{\varpi}} \left|\mathfrak{s}(r) - \acute{\mathfrak{s}}(r)\right| \mathrm{d}_q r \\ & + \frac{2\left((\delta+1)\tilde{\tau}^{\delta}+\tilde{\tau}^{\delta+1}\right)\left[\left(|\vartheta_1|\omega^{\delta+1}+|\vartheta_2|\right)+|\Delta|\right]}{|\Delta|\Gamma_q(\delta+2)} \\ & \times \int_0^1 \frac{(1-qr)^{(\gamma-1)}}{\Gamma_q(\gamma)} \left[\theta\left|\varphi_{\mathfrak{s}}^{\diamond}(\tilde{\tau}) - \varphi_{\acute{\mathfrak{s}}}^{\diamond}(\tilde{\tau})\right| + \lambda\left|\psi_{\mathfrak{s}}^{\diamond}(\tilde{\tau}) - \psi_{\acute{\mathfrak{s}}}^{\diamond}(\tilde{\tau})\right|\right] \mathrm{d}_q r, \qquad \tilde{\tau} \in \Omega.\end{aligned}$$

Thanks to (H_1), we get

$$\begin{aligned}\|Q(\mathfrak{s}) - Q(t)\| \leq & \left[\frac{1}{\Gamma_q(\gamma+\delta+1)} + \frac{(\delta+2)|\vartheta_2|}{|\Delta|\Gamma_q(\gamma+\delta+1)} + \frac{(\delta+2)|\vartheta_1|\omega^{(\gamma+\delta)}}{|\Delta|\Gamma_q(\gamma+\delta+1)}\right. \\ & \left. + \frac{2(\delta+2)\left[\left(|\vartheta_1|\omega^{\delta+1}+|\vartheta_2|\right)+|\Delta|\right]}{|\Delta|\Gamma_q(\delta+2)\Gamma_q(\gamma+1)}\right] \left(\theta k_1 + \frac{\lambda(\Gamma_q(\beta+1)+1)k_2}{\Gamma_q(\beta+1)}\right) \|\mathfrak{s} - \acute{\mathfrak{s}}\|_W \\ & + \eta\left(\frac{\Gamma_q(1-\varpi)}{\Gamma_q(\delta-\varpi+1)} + \frac{(\delta+2)|\vartheta_2|}{|\Delta|\Gamma_q(\delta+1)} + \frac{(\delta+2)|\vartheta_1|\Gamma_q(1-\varpi)\omega^{(\delta-\varpi)}}{|\Delta|\Gamma_q(\delta-\varpi+1)}\right) \|\mathfrak{s} - \acute{\mathfrak{s}}\|_W \\ = & \left[\Pi\left(\theta k_1 + \frac{\lambda(\Gamma_q(\beta+1)+1)k_2}{\Gamma_q(\beta+1)}\right) + \eta \Pi^*\right] \|\mathfrak{s} - \acute{\mathfrak{s}}\|_W .\end{aligned}$$

Hence,

$$\|Q(\mathfrak{s}) - Q(\acute{\mathfrak{s}})\| \leq \left[\Pi_1\left(\theta k_1 + \frac{\lambda(\Gamma_q(\beta+1)+1)k_2}{\Gamma_q(\beta+1)}\right) + \eta \Pi_2\right] \|\mathfrak{s} - \acute{\mathfrak{s}}\|_W .$$

Further, (H_1) implies that

$$\begin{aligned}\left\|\mathbb{D}_q[Q(\mathfrak{s})] - \mathbb{D}_q[Q(\acute{\mathfrak{s}})]\right\| \leq & \left[\frac{1}{\Gamma_q(\gamma+\delta)} + \frac{\left((\delta+1)[\delta]_q+[\delta+1]_q\right)|\vartheta_2|}{|\Delta|\Gamma_q(\gamma+\delta+1)} + \frac{\left((\delta+1)[\delta]_q+[\delta+1]_q\right)|\vartheta_1|\omega^{\gamma+\delta}}{|\Delta|\Gamma_q(\gamma+\delta+1)}\right. \\ & \left. + \frac{2\left((\delta+1)[\delta]_q+[\delta+1]_q\right)\left[\left(|\vartheta_1|\omega^{\delta+1}+|\vartheta_2|\right)+|\Delta|\right]}{|\Delta|\Gamma_q(\delta+2)\Gamma_q(\gamma+1)}\right] \left(\theta k_1 + \frac{\lambda(\Gamma_q(\beta+1)+1)k_2}{\Gamma_q(\beta+1)}\right) \|\mathfrak{s} - \acute{\mathfrak{s}}\|_W \\ & + \eta\left[\frac{\Gamma_q(1-\varpi)}{\Gamma_q(\delta-\varpi)} + \frac{\left((\delta+1)[\delta]_q+[\delta+1]_q\right)|\vartheta_2|}{|\Delta|\Gamma_q(\delta+1)} + \frac{\left((\delta+1)[\delta]_q+[\delta+1]_q\right)|\vartheta_1|\Gamma_q(1-\varpi)\omega^{\delta-\varpi}}{|\Delta|\Gamma_q(\delta+1)\Gamma_q(\delta-\varpi+1)}\right] \|\mathfrak{s} - \acute{\mathfrak{s}}\|_W \\ = & \left(\Theta_1\left(\theta k_1 + \frac{\lambda(\Gamma_q(\beta+1)+1)k_2}{\Gamma_q(\beta+1)}\right) + \eta \Theta_2\right) \|\mathfrak{s} - \acute{\mathfrak{s}}\|_W .\end{aligned}$$

By using (10), we have

$$\left\|\mathbb{D}_q^{\alpha}[Q(\mathfrak{s})] - \mathbb{D}_q^{\alpha}[Q(\acute{\mathfrak{s}})]\right\| \leq \left[\frac{\Theta_1}{\Gamma_q(2-\alpha)}\left(\theta k_1 + \frac{\lambda(\Gamma_q(\beta+1)+1)k_2}{\Gamma_q(\beta+1)}\right) + \frac{\eta\Theta_2}{\Gamma_q(2-\alpha)}\right] \|\mathfrak{s} - \acute{\mathfrak{s}}\|_W .$$

From the definition of $\|.\|_W$, we get

$$\begin{aligned}\|Q(\mathfrak{s}) - Q(\acute{\mathfrak{s}})\|_W = & \|Q(\mathfrak{s}) - Q(\acute{\mathfrak{s}})\| + \left\|\mathbb{D}_q^{\alpha}[Q(\mathfrak{s})] - \mathbb{D}_q^{\alpha}[Q(\acute{\mathfrak{s}})]\right\| \\ \leq & \left[\Phi_1\left(\theta k_1 + \frac{\lambda(\Gamma_q(\beta+1)+1)k_2}{\Gamma_q(\beta+1)}\right) + \eta \Phi_2\right] \|\mathfrak{s} - \acute{\mathfrak{s}}\|_W .\end{aligned}$$

By (11), Q is a contractive operator. Consequently, the fixed point theorem of Banach, Q has a fixed point which is a solution of (1). □

In the next theorem, we prove that there exists at least one solution, for the sequential $q-\mathbb{F}$L-E$\mathbb{P}$ (1), based on Lemma 5, and considered more assumption.

Theorem 2. *Suppose the functions φ, ψ and $\mathfrak{m}$ are with the same definitions as in Theorem 1, and let*

H$_2$) *there is $\nabla_j \in \mathbb{R}$, $j = 1,2,3$, such that for any $\mathfrak{s}, \acute{\mathfrak{s}} \in \mathbb{R}$, we have*

$$|\varphi(\tilde{\tau}, \mathfrak{s}, \acute{\mathfrak{s}})| \leq \nabla_1, \ |\psi(\tilde{\tau}, \mathfrak{s}, \acute{\mathfrak{s}})| \leq \nabla_2, \ |\mathfrak{m}(\tilde{\tau})| \leq \nabla_3, \qquad \forall \tilde{\tau} \in \Omega.$$

Then the $q-\mathbb{F}$L-E$\mathbb{P}$ (1) has at least one solution if $\Phi_2 < \eta^{-1}$, where Φ_2, is given by (13).

Proof. The continuity of functions φ, ψ and $\mathfrak{m}$ imply that Q is continuous. First, we demonstrate that Q maps bounded sets of W into bounded sets of W. Let us, put $B_{\ni} = \{\mathfrak{s} \in W : \|s\|_W \leq \ni, \ni > 0\}$. Then for all $\mathfrak{s} \in B_{\ni}$ and thanks to (H$_2$) and (10), we get

$$\|Q(\mathfrak{s})\| \leq \Pi_1 (\theta\nabla_1 + \lambda\nabla_2 + \nabla_3) + \eta\Pi_2 \ni < \infty,$$

and

$$\left\|\mathbb{D}_q^{\alpha}[Q(\mathfrak{s})]\right\| \leq \frac{\Theta_1}{\Gamma_q(2-\alpha)} (\theta\nabla_1 + \lambda\nabla_2 + \nabla_3) + \frac{\eta\Theta_2}{\Gamma_q(2-\alpha)} \ni < \infty.$$

From the above inequalities, it follows that $\|Q(\mathfrak{s})\|_W < \infty$. Next, we show that Q is equicontinuous. Now, let $\mathfrak{s} \in B_{\ni}$ and $\tilde{\tau}_1, \tilde{\tau}_2 \in \Omega$, with $\tilde{\tau}_2 < \tilde{\tau}_1$. In this case, we get

$$\begin{aligned}
\left|Q\mathfrak{s}(\tilde{\tau}_1) - Q\mathfrak{s}(\tilde{\tau}_2)\right| \leq{}& (\theta\nabla_1 + \lambda\nabla_2 + \nabla_3)\left(\frac{(\tilde{\tau}_1-\tilde{\tau}_2)^{\gamma+\delta} + \left|\tilde{\tau}_1^{\gamma+\delta} - \tilde{\tau}_2^{\gamma+\delta}\right|}{\Gamma_q(\gamma+\delta+1)}\right.\\
&+ \frac{\left((\delta+1)\left|\tilde{\tau}_1^{\delta}-\tilde{\tau}_2^{\delta}\right| + \left|\tilde{\tau}_1^{\delta+1}-\tilde{\tau}_2^{\delta+1}\right|\right)|\vartheta_1|\omega^{\gamma+\delta}}{|\Delta|\Gamma_q(\gamma+\delta+1)} + \frac{\left((\delta+1)\left|\tilde{\tau}_1^{\delta}-\tilde{\tau}_2^{\delta}\right| + \left|\tilde{\tau}_1^{\delta+1}-\tilde{\tau}_2^{\delta+1}\right|\right)|\vartheta_2|}{|\Delta|\Gamma_q(\gamma+\delta+1)}\\
&\left.+ \frac{2\left((\delta+1)\left|\tilde{\tau}_1^{\delta}-\tilde{\tau}_2^{\delta}\right| + \left|\tilde{\tau}_1^{\delta+1}-\tilde{\tau}_2^{\delta+1}\right|\right)\left[\left(|\vartheta_1|\omega^{\delta+1}+|\vartheta_2|\right)+|\Delta|\right]}{|\Delta|\Gamma_q(\delta+2)\Gamma_q(\gamma+1)}\right)\\
&+ \ni\eta\left(\frac{\Gamma_q(1-\varpi)\left[(\tilde{\tau}_1-\tilde{\tau}_2)^{\delta-\varpi} + \left|\tilde{\tau}_1^{\delta-\varpi}-\tilde{\tau}_2^{\delta-\varpi}\right|\right]}{\Gamma_q(\delta-\varpi+1)} + \frac{\left((\delta+1)\left|\tilde{\tau}_1^{\delta}-\tilde{\tau}_2^{\delta}\right| + \left|\tilde{\tau}_1^{\delta+1}-\tilde{\tau}_2^{\delta+1}\right|\right)|\vartheta_2|}{|\Delta|\Gamma_q(\delta+1)}\right.\\
&\left.+ \frac{\left((\delta+1)\left|\tilde{\tau}_1^{\delta}-\tilde{\tau}_2^{\delta}\right| + \left|\tilde{\tau}_1^{\delta+1}-\tilde{\tau}_2^{\delta+1}\right|\right)|\vartheta_1|\eta\Gamma_q(1-\varpi)\omega^{\delta-\varpi}}{|\Delta|\Gamma_q(\delta-\varpi+1)}\right).
\end{aligned} \tag{14}$$

Also, by (10) we have

$$\begin{aligned}
\left|\mathbb{D}_q^{\alpha}[Q\mathfrak{s}(\tilde{\tau}_1)] - \mathbb{D}_q^{\alpha}[Q\mathfrak{s}(\tilde{\tau}_2)]\right| \leq{}& \frac{(\theta\nabla_1+\lambda\nabla_2+\nabla_3)}{\Gamma_q(2-\alpha)}\left(\frac{\left[(\tilde{\tau}_1-\tilde{\tau}_2)^{\gamma+\delta-1} + \left|\tilde{\tau}_1^{\gamma+\delta-1}-\tilde{\tau}_2^{\gamma+\delta-1}\right|\right]}{\Gamma_q(\gamma+\delta)}\right.\\
&+ \frac{\left((\delta+1)[\delta]_q\left|\tilde{\tau}_1^{\delta-1}-\tilde{\tau}_2^{\delta-1}\right| + [\delta+1]_q\left|\tilde{\tau}_1^{\delta}-\tilde{\tau}_2^{\delta}\right|\right)|\vartheta_2|}{|\Delta|\Gamma_q(\gamma+\delta+1)}\\
&+ \frac{\left((\delta+1)[\delta]_q\left|\tilde{\tau}_1^{\delta-1}-\tilde{\tau}_2^{\delta-1}\right| + [\delta+1]_q\left|\tilde{\tau}_1^{\delta}-\tilde{\tau}_2^{\delta}\right|\right)|\vartheta_1|\omega^{\gamma+\delta}}{|\Delta|\Gamma_q(\gamma+\delta+1)}\\
&\left.+ \frac{2\left[\left(|\vartheta_1|\omega^{\delta+1}+|\vartheta_2|\right)+|\Delta|\right]}{|\Delta|\Gamma_q(\delta+2)\Gamma_q(\gamma+1)}\left[\left((\delta+1)[\delta]_q\left|\tilde{\tau}_1^{\delta-1}-\tilde{\tau}_2^{\delta-1}\right| + [\delta+1]_q\left|\tilde{\tau}_1^{\delta}-\tilde{\tau}_2^{\delta}\right|\right)\right]\right)\\
&+ \frac{\ni\eta}{\Gamma_q(2-\alpha)}\left\{\frac{\Gamma_q(1-\varpi)\left[(\tilde{\tau}_1-\tilde{\tau}_2)^{\gamma+\delta-1} + \left|\tilde{\tau}_1^{\gamma+\delta-1}-\tilde{\tau}_2^{\gamma+\delta-1}\right|\right]}{\Gamma_q(\delta-\varpi-1)}\right.\\
&+ \frac{\left((\delta+1)[\delta]_q\left|\tilde{\tau}_1^{\delta-1}-\tilde{\tau}_2^{\delta-1}\right| + [\delta+1]_q\left|\tilde{\tau}_1^{\delta}-\tilde{\tau}_2^{\delta}\right|\right)|\vartheta_2|\eta}{|\Delta|\Gamma_q(\delta+1)}\\
&\left.+ \frac{|\vartheta_1|\Gamma_q(1-\varpi)}{|\Delta|\Gamma_q(\delta-\varpi+1)}\left((\delta+1)[\delta]_q\left|\tilde{\tau}_1^{\delta-1}-\tilde{\tau}_2^{\delta-1}\right| + [\delta+1]_q\left|\tilde{\tau}_1^{\delta}-\tilde{\tau}_2^{\delta}\right|\right)\right\}.
\end{aligned} \tag{15}$$

Thanks to (14) and (15), we can state that $\|Q\mathfrak{s}(\tilde{\tau}_1) - Q\mathfrak{s}(\tilde{\tau}_2)\|_W \to 0$ as $\tilde{\tau}_1 \to \tilde{\tau}_2$. Hence, the theorem of Arzelà-Ascoli implies that Q is a completely continuous operator. Finally, we prove that the set Ψ, given by $\Psi = \{\mathfrak{s} \in W : \mathfrak{s} = \sigma Q(\mathfrak{s}), 0 < \sigma < 1\}$, is bounded. Let $\mathfrak{s} \in \Psi$, then for some $0 < \sigma < 1$, we have $\mathfrak{s} = \sigma Q(\mathfrak{s})$, and so $\mathfrak{s}(\tilde{\tau}) = \sigma Q\mathfrak{s}(\tilde{\tau})$, for $\tilde{\tau} \in \Omega$. Thanks to (H$_2$) and (10), we can write

$$\|\mathfrak{s}\| \leq \sigma\left(\Pi_1 (\theta\nabla_1 + \lambda\nabla_2 + \nabla_3) + \eta\Pi_2 \|\mathfrak{s}\|_W\right), \tag{16}$$

and

$$\left\|\mathbb{D}_q^{\alpha}\mathfrak{s}\right\| \leq \sigma\left(\frac{\Theta_1}{\Gamma_q(2-\alpha)} (\theta\nabla_1 + \lambda\nabla_2 + \nabla_3) + \frac{\eta\Theta_2}{\Gamma_q(2-\alpha)} \|\mathfrak{s}\|_W\right). \tag{17}$$

It follows from (16) and (17), that

$$\|\mathfrak{s}\|_W \leq \left(\Pi_1 + \frac{\Theta_1}{\Gamma_q(2-\alpha)}\right)(\theta\nabla_1 + \lambda\nabla_2 + \nabla_3) + \eta\left(\Pi_2 + \frac{\Theta_2}{\Gamma_q(2-\alpha)}\right)\|s\|_W,$$

which implies that

$$\|\mathfrak{s}\|_W \leq \left(\Pi_1 + \frac{\Theta_1}{\Gamma_q(2-\alpha)}\right)(\theta\nabla_1 + \lambda\nabla_2 + \nabla_3)\left[1 - \eta\left(\Pi_2 + \frac{\Theta_2}{\Gamma_q(2-\alpha)}\right)\right]^{-1}.$$

where Π_i and $\Theta_i, i = 1,2$, are given by (13). Indeed, Ψ is bounded. Eventually, Lemma 5 implies that Q has at least one fixed point, which is a solution of $q-\mathbb{F}$L-E$\mathbb{P}$ (1). □

4 Stability of Proposed Problem

Here, we discuss the stability of UH and UHR of the $q-\mathbb{F}$L-E$\mathbb{P}$ (1).

Theorem 3. *Consider the functions φ, ψ and $\mathfrak{m}$ with the same definitions as in Theorem 1, and assume that* (H_1) *and* (11) *are valid. Then, the $q-\mathbb{F}$L-E$\mathbb{P}$ (1) is stable in* UH *sense.*

Proof. Let $\mathfrak{s} \in W$ is the unique solution of problem

$$\begin{cases} {}_{R.L}\mathbb{D}_q^{\gamma}\left[{}_C\mathbb{D}_q^{\delta} + \frac{\eta}{\tilde{\tau}^{\varpi}}\right]\mathfrak{s}(\tilde{\tau}) = \mathfrak{m}(\tilde{\tau}) - \theta\varphi_{\mathfrak{s}}^{\diamond}(\tilde{\tau}) - \lambda\psi_{\mathfrak{s}}^{\diamond}(\tilde{\tau}), \quad \tilde{\tau} \in \Omega, 0 < q < 1, \\ \left[{}_C\mathbb{D}_q^{\delta} + \eta\right]\mathfrak{s}(1) = \left[{}_C\mathbb{D}_q^{\delta} + \eta\right]\acute{\mathfrak{s}}(1), \\ \mathfrak{s}(\omega) = \acute{\mathfrak{s}}(\omega), \mathfrak{s}(1) = \acute{\mathfrak{s}}(1), \mathfrak{s}(0) = \acute{\imath}(0), \end{cases} \tag{18}$$

for $1 < \gamma < 2$, $0 < \delta, \omega < 1$, $0 < \varpi \leq 1$, $\eta, \theta, \lambda > 0$, such that $\acute{\mathfrak{s}} \in W$ is a solution of the inequality (2). Remark 1, helps us that

$$\acute{\mathfrak{s}}(\tilde{\tau}) = \mathbb{I}_q^{\gamma+\delta}\left[h_{\acute{\mathfrak{s}}}(\tilde{\tau})\right] - \mathbb{I}_q^{\delta}\left[\frac{\eta}{\tilde{\tau}^{\varpi}}\acute{\mathfrak{s}}(\tilde{\tau})\right] + \frac{c_0\tilde{\tau}^{\delta}}{\Gamma_q(\delta+1)} + \frac{c_1\tilde{\tau}^{\delta+1}}{\Gamma_q(\delta+2)} + c_2\tilde{\tau}^{\delta-1} + \mathbb{I}_q^{\gamma+\delta}\left[f(\tilde{\tau})\right],$$

for $c_j \in \mathbb{R}$, $j = 0,1,2$, where $h_{\acute{\mathfrak{s}}}(\tilde{\tau}) = \mathfrak{m}(\tilde{\tau}) - \theta\varphi_{\mathfrak{s}}^{\diamond}(\tilde{\tau}) - \lambda\psi_{\mathfrak{s}}^{\diamond}(\tilde{\tau})$ and $|f(\tilde{\tau})| \leq a$, $\tilde{\tau} \in \Omega$. From Lemma 6, we have

$$|\acute{\mathfrak{s}}(\tilde{\tau}) - Q\acute{\mathfrak{s}}(\tilde{\tau})| = \left|\mathbb{I}_q^{\gamma+\delta}[f(\tilde{\tau})]\right| \leq a\mathbb{I}_q^{\gamma+\delta}[1],$$

and Lemma 4,

$$|\acute{\mathfrak{s}}(\tilde{\tau}) - Q\acute{\mathfrak{s}}(\tilde{\tau})| \leq \frac{1}{\Gamma_q(\gamma+\delta+1)}a\tilde{\tau}^{(\gamma+\delta)}.$$

Also, we have the following

$$\begin{aligned} |\acute{\mathfrak{s}}(\tilde{\tau}) - \mathfrak{s}(\tilde{\tau})| &= \left|\acute{\mathfrak{s}}(\tilde{\tau}) - \mathbb{I}_q^{\gamma+\delta}[h(\tilde{\tau})] - \mathbb{I}_q^{\delta}\left[\frac{\eta}{\tilde{\tau}^{\varpi}}\mathfrak{s}(\tilde{\tau})\right] + \frac{e_0\tilde{\tau}^{\delta}}{\Gamma_q(\delta+1)} + \frac{e_1\tilde{\tau}^{\delta+1}}{\Gamma_q(\delta+2)} + e_2\tilde{\tau}^{\delta-1}\right| \\ &= |\acute{\mathfrak{s}}(\tilde{\tau}) - Q\acute{\mathfrak{s}}(\tilde{\tau}) + Q\acute{\mathfrak{s}}(\tilde{\tau}) - Q\mathfrak{s}(\tilde{\tau})| \leq |\acute{\mathfrak{s}}(\tilde{\tau}) - Q\acute{\mathfrak{s}}(\tilde{\tau})| + |Q\acute{\mathfrak{s}}(\tilde{\tau}) - Q\mathfrak{s}(\tilde{\tau})|. \end{aligned}$$

Assumption (H_1), yields

$$\begin{aligned} \|\acute{\mathfrak{s}} - \mathfrak{s}\|_W &\leq \|\acute{\mathfrak{s}} - Q(\acute{\mathfrak{s}})\| + \|Q(\acute{\mathfrak{s}}) - Q(\mathfrak{s})\| \\ &\leq \frac{a}{\Gamma_q(\gamma+\delta+1)} + \left[\Phi_1\left(\theta k_1 + \frac{\lambda(\Gamma_q(\beta+1)+1)k_2}{\Gamma_q(\beta+1)}\right) + \eta\Phi_2\right]\|\mathfrak{s} - \acute{\mathfrak{s}}\|_W, \end{aligned}$$

where $\Phi_i, i = 1,2$, are given by (13). Then

$$\|\acute{\mathfrak{s}} - \mathfrak{s}\|_W \leq a\left[\Gamma_q(\gamma+\delta+1)\left(1 - \left[\Phi_1\left(\theta k_1 + \frac{\lambda(\Gamma_q(\beta+1)+1)k_2}{\Gamma_q(\beta+1)}\right) + \eta\Phi_2\right]\right)\right]^{-1},$$

If we put

$$A_{\varphi^{\diamond},\psi^{\diamond},\mathfrak{m}} := \left[\Gamma_q(\gamma+\delta+1)\left(1 - \left[\Phi_1\left(\theta k_1 + \frac{\lambda(\Gamma_q(\beta+1)+1)k_2}{\Gamma_q(\beta+1)}\right) + \eta\Phi_2\right]\right)\right]^{-1},$$

we obtain $\|\acute{\mathfrak{s}} - \mathfrak{s}\|_E \leq A_{\varphi^{\diamond},\psi^{\diamond},\mathfrak{m}}a$. Hence, the $q-\mathbb{F}$L-E$\mathbb{P}$ (1) is stable in UH sense. □

Theorem 4. *Consider the functions φ, ψ and $\mathfrak{m}$ with the same definitions as in Theorem 1. Also, suppose that* (H_1)*, Eq.* (11) *are valid and assume that*

H_3) *there exists a function $g \in C(\Omega, \mathbb{R}_+)$ which is nondecreasing and $b_g > 0$ such that*

$$\mathbb{I}_q^{\gamma+\delta}\left[g(\tilde{\tilde{\tau}})\right] \leq b_g \cdot g(\tilde{\tilde{\tau}}), \qquad \tilde{\tilde{\tau}} \in \Omega. \tag{19}$$

Then $q-\mathbb{F}$L-E$\mathbb{P}$ (1) *is* UHR *stable with respect to g.*

Proof. From Remark 1, we see that

$$\mathfrak{s}(\tilde{\tau}) = \mathbb{I}_q^{\gamma+\delta}\left[h_{\acute{\mathfrak{s}}}(\tilde{\tau})\right] - \mathbb{I}_q^{\delta}\left[\frac{\eta}{\tilde{\tau}^{\varpi}}\acute{\mathfrak{s}}(\tilde{\tau})\right] + \frac{c_0\tilde{\tau}^{\delta}}{\Gamma_q(\delta+1)} + \frac{c_1\tilde{\tau}^{\delta+1}}{\Gamma_q(\delta+2)} + c_2\tilde{\tau}^{\delta-1} + \mathbb{I}_q^{\gamma+\delta}\left[f(\tilde{\tau})\right],$$

for $c_j \in \mathbb{R}$, $j = 0, 1, 2$, where $|f(\tilde{\tau})| \leq a \cdot g(\tilde{\tau})$, $\tilde{\tau} \in \Omega$ and $\acute{\mathfrak{s}} \in W$ is a solution of the inequality (3). Let the unique solution of (18) is $\mathfrak{s} \in W$. Thus, by employing Lemma 6, we get

$$|\acute{\mathfrak{s}}(\tilde{\tau}) - \mathcal{Q}\acute{\mathfrak{s}}(\tilde{\tau})| = \left|\mathbb{I}_q^{\gamma+\delta}\left[f(\tilde{\tau})\right]\right| \leq a\mathbb{I}_q^{\gamma+\delta}\left[f(\tilde{\tau})\right] \leq a \cdot b_g \cdot g(\tilde{\tau}),$$

and so,

$$\begin{aligned}|\acute{\mathfrak{s}}(\tilde{\tau}) - \mathfrak{s}(\tilde{\tau})| &= \left|\acute{\mathfrak{s}}(\tilde{\tau}) - \mathbb{I}_q^{\gamma+\delta}\left[h(\tilde{\tau})\right] - \mathbb{I}_q^{\delta}\left[\frac{\eta}{\tilde{\tau}^{\varpi}}\mathfrak{s}(\tilde{\tau})\right] + \frac{e_0\tilde{\tau}^{\delta}}{\Gamma_q(\delta+1)} + \frac{e_1\tilde{\tau}^{\delta+1}}{\Gamma_q(\delta+2)} + e_2\tilde{\tau}^{\delta-1}\right| \\ &= |\acute{\mathfrak{s}}(\tilde{\tau}) - \mathcal{Q}\acute{\mathfrak{s}}(\tilde{\tau}) + \mathcal{Q}\acute{\mathfrak{s}}(\tilde{\tau}) - \mathcal{Q}\mathfrak{s}(\tilde{\tau})| \leq |\acute{\mathfrak{s}}(\tilde{\tau}) - \mathcal{Q}\acute{\mathfrak{s}}(\tilde{\tau})| + |\mathcal{Q}\acute{\mathfrak{s}}(\tilde{\tau}) - \mathcal{Q}\mathfrak{s}(\tilde{\tau})|.\end{aligned}$$

Thus, by (H_1) and (H_3), we get

$$\|\acute{\mathfrak{s}} - \mathfrak{s}\|_W \leq ab_g g(\tilde{\tau}) + \left[\Phi_1\left(\theta k_1 + \frac{\lambda(\Gamma_q(\beta+1)+1)k_2}{\Gamma_q(\beta+1)}\right) + \eta\Phi_2\right]\|\mathfrak{s} - \acute{\mathfrak{s}}\|_W.$$

Then, we have

$$\|\acute{\mathfrak{s}} - \mathfrak{s}\|_E \leq \frac{b_g}{1 - \left[\Phi_1\left(\theta k_1 + \frac{\lambda(\Gamma_q(\beta+1)+1)k_2}{\Gamma_q(\beta+1)}\right) + \eta\Phi_2\right]} a g(\tilde{\tau}).$$

for $\tilde{\tau} \in \Omega$. If we take

$$A_{\varphi^\circ, \psi^\circ, \mathfrak{m}, g} := b_g\left\{1 - \left[\Phi_1\left(\theta k_1 + \frac{\lambda(\Gamma_q(\beta+1)+1)k_2}{\Gamma_q(\beta+1)}\right) + \eta\Phi_2\right]\right\}^{-1},$$

we obtain $\|\acute{\mathfrak{s}} - \mathfrak{s}\|_W \leq A_{\varphi^\circ, \psi^\circ, \mathfrak{m}, g} a g(\tilde{\tau})$, for $\tilde{\tau} \in \Omega$. Therefore, the $q-\mathbb{F}$L-E$\mathbb{P}$ (1) is stable in the UHR sense. □

5 Application

Consider the following $q-\mathbb{F}$L-E$\mathbb{P}$,

$$\begin{cases} {}_{R.L}\mathbb{D}_q^{3/2}\left[{}_C\mathbb{D}_q^{\ln 2/2} + \frac{1}{200\tilde{\tau}^{1/10^2}}\right]\mathfrak{s}(\tilde{\tau}) = \frac{\frac{1}{2} + \arctan(\pi\tilde{\tau}+2)}{2\pi} - \frac{1}{40e}\left(\frac{2\sin\mathfrak{s}(\tilde{\tau})}{35(e^{2+\tilde{\tau}}+2)} + \frac{\sin\left(2\pi{}_C\mathbb{D}_q^{1/4}\mathfrak{s}(\tilde{\tau})\right)}{35\pi(e^2+2)}\right) \\ \qquad - \frac{1}{53\sqrt{\pi}}\left(\frac{|\mathfrak{s}(\tilde{\tau})|}{32(e^{\tilde{\tau}+2}+3)(1+|\mathfrak{s}(\tilde{\tau})|)} + \frac{\sin^2\left(\mathbb{I}_q^{5/4}\mathfrak{s}(\tilde{\tau})\right)}{64(e^2+3)}\right), & \tilde{\tau} \in J_0, \\ \left[{}_C\mathbb{D}_q^{\ln 2/2} + \frac{\sqrt{2}}{15}\right]\mathfrak{s}(1) = 0, \ \frac{\sqrt{3}}{11}\mathfrak{s}\left(\frac{1}{5}\right) = \frac{\ln 3}{13}\mathfrak{s}(1), \mathfrak{s}(0) = 0, \end{cases} \tag{20}$$

and the following $q-$fractional inequalities

$$\begin{aligned} &\left|{}_{R.L}\mathbb{D}_q^{3/2}\left[{}_C\mathbb{D}_q^{\ln 2/2} + \frac{\sqrt{2}}{15\tilde{\tau}^{1/10^2}}\right]\mathfrak{s}(\tilde{\tau}) - \left[\mathfrak{m}(\tilde{\tau}) - \frac{1}{40e}\varphi^\diamond_{\mathfrak{s}}(\tilde{\tau}) - \frac{1}{53\sqrt{\pi}}\psi^\diamond_{\mathfrak{s}}(\tilde{\tau})\right]\right| \leq a, \\ &\left|{}_{R.L}\mathbb{D}_q^{3/2}\left[{}_C\mathbb{D}_q^{\ln 2/2} + \frac{\sqrt{2}}{15\tilde{\tau}^{1/10^2}}\right]\mathfrak{s}(\tilde{\tau}) - \left[\mathfrak{m}(\tilde{\tau}) - \frac{1}{40e}\varphi^\diamond_{\mathfrak{s}}(\tilde{\tau}) - \frac{1}{53\sqrt{\pi}}\psi^\diamond_{\mathfrak{s}}(\tilde{\tau})\right]\right| \leq a g(\tilde{\tau}), \end{aligned}$$

for $q = \left\{\frac{\sqrt{e}}{9}, \frac{1}{2}, \frac{\sqrt{8e}}{5}\right\}$, where $\mathfrak{m}(\tilde{\tau}) = \frac{1}{2\pi}\left[\frac{1}{2} + \arctan(\pi\tilde{\tau}+2)\right]$, and

$$\begin{aligned} \varphi^\diamond_{\mathfrak{s}}(\tilde{\tau}) &= \frac{2\sin(\mathfrak{s})}{35(e^{2+\tilde{\tau}}+2)} + \frac{\sin\left(2\pi{}_C\mathbb{D}_q^{1/4}\mathfrak{s}\right)}{35\pi(e^2+2)}, \\ \psi^\diamond_{\mathfrak{s}}(\tilde{\tau}) &= \frac{1}{53\sqrt{\pi}}\left[\frac{|\mathfrak{s}|}{32(e^{\tilde{\tau}+2}+3)(1+|\mathfrak{s}|)} + \frac{\sin^2\left(\mathbb{I}_q^{5/4}\mathfrak{s}\right)}{64(e^2+3)}\right]. \end{aligned}$$

It goes without saying $\gamma = \frac{3}{2} \in (1,2)$, $\delta = \frac{\ln 2}{2} \in (0,1)$, $\eta = \frac{1}{200} > 0$, $\theta = \frac{1}{40e} > 0$, $\varpi = \frac{1}{10^2} > 0$, $\lambda = \frac{1}{53\sqrt{\pi}} > 0$, $\vartheta_1 = \frac{\sqrt{3}}{11}$, $\vartheta_2 = \frac{\ln 3}{13}$, $\omega = \frac{1}{5} \in (0,1)$, $\alpha = \frac{1}{4} < 0.3466 \simeq \frac{\ln 2}{2} = \delta$, $\beta = \frac{5}{4} \in (1,2)$. For $(\mathfrak{s}_j, \acute{\mathfrak{s}}_j) \in \mathbb{R}^2$, $j = 1,2$ and $\tilde{\tau} \in \Omega$, we have

$$|\varphi(\tilde{\tau}, \mathfrak{s}_1, \acute{\mathfrak{s}}_1) - \varphi(\tilde{\tau}, \mathfrak{s}_2, \acute{\mathfrak{s}}_2)| \leq \frac{2}{35\pi(e^2+2)}\left(|\mathfrak{s}_1 - \mathfrak{s}_2| + |\acute{\mathfrak{s}}_1 - \acute{\mathfrak{s}}_2|\right),$$
$$|\psi(\tilde{\tau}, \mathfrak{s}_1, \acute{\mathfrak{s}}_1) - \psi(\tilde{\tau}, \mathfrak{s}_2, \acute{\mathfrak{s}}_2)| \leq \frac{1}{32(e^2+3)}\left(|\mathfrak{s}_1 - \mathfrak{s}_2| + |\acute{\mathfrak{s}}_1 - \acute{\mathfrak{s}}_2|\right).$$

Hence, the condition (H_1) holds with $k_1 = \frac{2}{35\pi(e^2+2)}$, and $k_2 = \frac{1}{32(e^2+3)}$. Observe that, from (4) we have

$$\Delta = \frac{\sqrt{3}}{11}\left[\left(\frac{\ln 2}{2}+1\right)\left(\frac{1}{5}\right)^{\ln 2/2} - \left(\frac{1}{5}\right)^{\ln 2/2+1}\right] - \frac{\ln 3}{13}\left(\frac{\ln 2}{2}\right) = 0.0741 \neq 0,$$

and from relations (12) and (13), it follows that

Table 1. Numerical results of Π_j, Θ_j, Φ_j for $j = 1,2$ of $q - \mathbb{FL}$-$\mathbb{EP}$ (20) for three different values of q.

n	Π_1	Π_2	Θ_1	Θ_2	$\Gamma_q(2-\alpha)$	Φ_1	Φ_2
				$q = \sqrt{e}/9$			
1	13.4456	6.7980	10.8794	5.9476	0.9291	25.1555	13.1996
2	13.5890	6.8171	10.9935	5.9737	0.9237	25.4910	13.2844
3	13.6153	6.8206	11.0144	5.9784	0.9227	25.5528	13.3000
4	13.6202	6.8212	11.0183	5.9793	0.9225	25.5642	13.3029
5	13.6211	6.8213	11.0190	5.9795	0.9225	25.5663	13.3034
6	13.6212	6.8213	11.0191	5.9795	0.9225	25.5666	13.3035
7	13.6212	6.8214	11.0191	5.9795	0.9225	25.5667	13.3035
8	13.6213	6.8214	11.0191	5.9795	0.9225	25.5667	13.3035
9	13.6213	6.8214	11.0191	5.9795	0.9225	25.5667	13.3035

$$\Pi_1 = \frac{1}{\Gamma_q\left(\frac{3}{2}+\frac{\ln 2}{2}+1\right)} + \frac{\left(\frac{\ln 2}{2}+2\right)\left|\frac{\ln 3}{13}\right|}{|\Delta|\Gamma_q\left(\frac{3}{2}+\frac{\ln 2}{2}+1\right)} + \frac{\left(\frac{\ln 2}{2}+2\right)\left|\frac{\sqrt{3}}{11}\right|\left(\frac{1}{5}\right)^{(3/2+\ln 2/2)}}{|\Delta|\Gamma_q\left(\frac{3}{2}+\frac{\ln 2}{2}+1\right)}$$
$$+ \frac{2\left(\frac{\ln 2}{2}+2\right)\left[\left(\left|\frac{\sqrt{3}}{11}\right|\left(\frac{1}{5}\right)^{(\ln 2/2+1)} + \left|\frac{\ln 3}{13}\right|\right)+|\Delta|\right]}{|\Delta|\Gamma_q\left(\frac{\ln 2}{2}+2\right)\Gamma_q\left(\frac{3}{2}+1\right)} \simeq \begin{cases} 13.6213, & q = \sqrt{e}/9, \\ 15.9686, & q = 1/2, \\ 309.0617, & q = \sqrt{8e}/5, \end{cases}$$

$$\Pi_2 = \frac{\Gamma_q\left(1-\frac{1}{100}\right)}{\Gamma_q\left(\frac{\ln 2}{2}-\frac{1}{100}+1\right)} + \frac{\left(\frac{\ln 2}{2}+2\right)\left|\frac{\ln 3}{13}\right|}{|\Delta|\Gamma_q\left(\frac{\ln 2}{2}+1\right)} + \frac{\left(\frac{\ln 2}{2}+2\right)\left|\frac{\sqrt{3}}{11}\right|\Gamma_q\left(1-\frac{1}{100}\right)\left(\frac{1}{5}\right)^{(\ln 2/2-1/100)}}{|\Delta|\Gamma_q\left(\frac{\ln 2}{2}-\frac{1}{100}+1\right)} \simeq \begin{cases} 6.8214, & q = \sqrt{e}/9, \\ 8.2661, & q = 1/2, \\ 36.0227, & q = \sqrt{8e}/5, \end{cases}$$

$$\Theta_1 = \frac{1}{\Gamma_q\left(\frac{3}{2}+\frac{\ln 2}{2}\right)} + \frac{\left(\left(\frac{\ln 2}{2}+1\right)\left[\frac{\ln 2}{2}\right]_q + \left[\frac{\ln 2}{2}+1\right]_q\right)\left|\frac{\ln 3}{13}\right|}{|\Delta|\Gamma_q\left(\frac{3}{2}+\frac{\ln 2}{2}+1\right)} + \frac{\left(\left(\frac{\ln 2}{2}+1\right)\left[\frac{\ln 2}{2}\right]_q + [\delta+1]_q\right)\left|\frac{\sqrt{3}}{11}\right|\left(\frac{1}{5}\right)^{3/2+\ln 2/2}}{|\Delta|\Gamma_q\left(\frac{3}{2}+\frac{\ln 2}{2}+1\right)}$$
$$+ \frac{2\left(\left(\frac{\ln 2}{2}+1\right)\left[\frac{\ln 2}{2}\right]_q + \left[\frac{\ln 2}{2}+1\right]_q\right)\left[\left(\left|\frac{\sqrt{3}}{11}\right|\left(\frac{1}{5}\right)^{\ln 2/2+1} + \left|\frac{\ln 3}{13}\right|\right)+|\Delta|\right]}{|\Delta|\Gamma_q\left(\frac{\ln 2}{2}+2\right)\Gamma_q\left(\frac{3}{2}+1\right)} \simeq \begin{cases} 11.0191, & q = \sqrt{e}/9, \\ 12.9734, & q = 1/2, \\ 244.0082, & q = \sqrt{8e}/5, \end{cases}$$

$$\Theta_2 = \frac{\Gamma_q\left(1-\frac{1}{100}\right)}{\Gamma_q\left(\frac{\ln 2}{2}-\frac{1}{100}\right)} + \frac{\left(\left(\frac{\ln 2}{2}+1\right)\left[\frac{\ln 2}{2}\right]_q + \left[\frac{\ln 2}{2}+1\right]_q\right)\left|\frac{\ln 3}{13}\right|}{|\Delta|\Gamma_q\left(\frac{\ln 2}{2}+1\right)}$$
$$+ \frac{\left(\left(\frac{\ln 2}{2}+1\right)\left[\frac{\ln 2}{2}\right]_q + \left[\frac{\ln 2}{2}+1\right]_q\right)\left|\frac{\sqrt{3}}{11}\right|\Gamma_q\left(1-\frac{1}{100}\right)\left(\frac{1}{5}\right)^{\frac{\ln 2}{2}-\frac{1}{100}}}{|\Delta|\Gamma_q\left(\frac{\ln 2}{2}+1\right)\Gamma_q\left(\frac{\ln 2}{2}-\frac{1}{100}+1\right)} \simeq \begin{cases} 5.9795, & q = \sqrt{e}/9, \\ 8.2510, & q = 1/2, \\ 50.0181, & q = \sqrt{8e}/5, \end{cases}$$

and

Table 2. Numerical results of Π_j, Θ_j, Φ_j for $j = 1,2$ of $q - \mathbb{FL}$-$\mathbb{EP}$ (20) whenever $q = \frac{1}{2}$.

n	Π_1	Π_2	Θ_1	Θ_2	$\Gamma_q(2-\alpha)$	Φ_1	Φ_2
				$q = 1/2$			
1	10.8838	7.4582	8.9766	7.1858	0.8003	22.1008	16.4374
2	13.2380	7.8594	10.8312	7.7083	0.7273	28.1311	18.4585
3	14.5537	8.0621	11.8644	7.9772	0.6947	31.6317	19.5448
4	15.2484	8.1640	12.4092	8.1135	0.6793	33.5155	20.1075
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
10	15.9572	8.2645	12.9645	8.2489	0.6647	35.4616	20.6745
11	15.9629	8.2653	12.9690	8.2499	0.6646	35.4774	20.6791
12	15.9658	8.2657	12.9712	8.2505	0.6645	35.4853	20.6814
13	15.9672	8.2659	12.9723	8.2508	0.6645	35.4893	20.6825
14	15.9679	8.2660	12.9729	8.2509	0.6645	35.4912	20.6831
15	15.9683	8.2661	12.9732	8.2510	0.6645	35.4922	20.6834
16	15.9684	8.2661	12.9733	8.2510	0.6645	35.4927	20.6835
17	15.9685	8.2661	12.9734	8.2510	0.6645	35.4930	20.6836
18	15.9686	8.2661	12.9734	8.2510	0.6645	35.4931	20.6836
19	15.9686	8.2661	12.9734	8.2510	0.6645	35.4931	20.6836

$$\Phi_1 \simeq \begin{cases} 25.5667, & q = \sqrt{e}/9, \\ 35.4931, & q = 1/2, \\ 2615.1523, & q = \sqrt{8e}/5, \end{cases} \qquad \Phi_2 \simeq \begin{cases} 13.3035, & q = \sqrt{e}/9, \\ 20.6836, & q = 1/2, \\ 511.0205, & q = \sqrt{8e}/5, \end{cases}$$

Tables 1, 2 and 3 show the numerical results of Π_j, Θ_j, Φ_j for $j = 1,2$ of $q - \mathbb{FL}$-$\mathbb{EP}$ (20) whenever $q = \frac{\sqrt{e}}{9}, \frac{1}{2}, \frac{\sqrt{8e}}{5}$. We can see graphical representation of these variables for three cases of q in Figures 1a,1b, 1c, 1d, 1e and 1f respectively. We remark that

$$A := \theta k_1 + \lambda \frac{(\Gamma_q(\beta+1)+1)k_2}{\Gamma_q(\beta+1)} \simeq \begin{cases} 0.0001, & q = \sqrt{e}/9 \\ 0.0001, & q = 1/2 \\ 0.0001, & q = \sqrt{8e}/5 \end{cases} < \begin{Bmatrix} 0.9335, & q = \sqrt{e}/9, \\ 0.8967, & q = 1/2, \\ 0.0422, & q = \sqrt{8e}/5, \end{Bmatrix} \simeq [1 - \eta\Phi_2]\,\Phi_1^{-1} =: B.$$

Table 4 shows the numerical results of A and B for $q - \mathbb{FL}$-$\mathbb{EP}$ (20) with $q = \frac{\sqrt{e}}{9}, \frac{1}{2}, \frac{\sqrt{8e}}{5}$. We can see the graphical representation of these variables for three cases of q in Figures 2a, 2b and 2c, respectively. Therefore, by Theorem 1, $q - \mathbb{FL}$-$\mathbb{EP}$ (20) has a unique solution on Ω.

Furthermore, by Theorem 3, $q - \mathbb{FL}$-$\mathbb{EP}$ (20) is stable in the UH sense with

$$\|\tilde{\mathfrak{s}} - \mathfrak{s}\|_W \leq \frac{a\tilde{\tau}^{(\gamma+\delta)}}{\Gamma_q(\gamma+\delta+1)} = \frac{a\tilde{\tau}^{(3/2+\ln 2/2)}}{\Gamma_q\left(\frac{3}{2}+\frac{\ln 2}{2}+1\right)} \simeq \begin{Bmatrix} 0.8769 \times a, & q = \sqrt{e}/9, \\ 0.8350 \times a, & q = 1/2, \\ 3.2661 \times a, & q = \sqrt{8e}/5, \end{Bmatrix} \cdot \tilde{\tau}^{(3/2+\ln 2/2)},$$

$a > 0$. Let $g(\tilde{\tau}) = \tilde{\tau}^{\sqrt{3}/4}$, then

$$\mathbb{I}_q^{\gamma+\delta}[g(\tilde{\tau})] = \mathbb{I}_q^{3/2+\frac{\ln 2}{2}}\left[\tilde{\tau}^{\sqrt{3}/4}\right] \leq \frac{\Gamma_q\left(\frac{\sqrt{3}}{4}\right)}{\Gamma_q\left(\gamma+\delta+\frac{\sqrt{3}}{4}\right)}\tilde{\tau}^{\sqrt{3}/4}$$
$$= \frac{\Gamma_q\left(\frac{\sqrt{3}}{4}\right)}{\Gamma_q\left(\frac{3}{2}+\frac{\ln 2}{2}+\frac{\sqrt{3}}{4}\right)}\tilde{\tau}^{\sqrt{3}/4} \simeq \begin{Bmatrix} 0.7660, & q = \sqrt{e}/9, \\ 0.5300, & q = 1/2, \\ 0.3556, & q = \sqrt{8e}/5, \end{Bmatrix}\tilde{\tau}^{\sqrt{3}/4} = b_g \cdot g(\tilde{\tau}).$$

Table 5 shows the numerical results of UH and UHR stabilities of $q - \mathbb{FL}$-$\mathbb{EP}$ (20) for $q = \frac{\sqrt{e}}{9}, \frac{1}{2}, \frac{\sqrt{8e}}{5}$. Also, one can see the graphical representation of these stabilities for three cases of q in Figures 3a and 3b. Then, the condition (19) is satisfied with $g(\tilde{\tau}) = \tilde{\tau}^{\sqrt{3}/4}$ and

$$b_g = \frac{\Gamma_q\left(\frac{\sqrt{3}}{4}\right)}{\Gamma_q\left(\gamma+\delta+\frac{\sqrt{3}}{4}\right)}.$$

Table 3. Numerical results of Π_j, Θ_j, Φ_j for $j = 1,2$ of $q-\mathbb{F}$L-E$\mathbb{P}$ (20) whenever $q = \frac{\sqrt{8e}}{5}$.

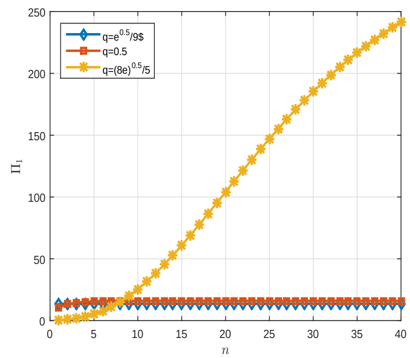
n	Π_1	Π_2	Θ_1	Θ_2	$\Gamma_q(2-\alpha)$	Φ_1	Φ_2
				$q = \sqrt{8e}/5$			
1	0.4211	5.7269	0.7759	7.6808	1.6519	0.8908	10.3767
2	0.9622	7.3235	1.4169	9.1054	1.1059	2.2435	15.5572
3	1.8810	8.8978	2.3652	10.7468	0.8149	4.7835	22.0861
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
58	288.6660	35.9762	228.4976	49.1910	0.1087	2390.5025	488.4603
59	290.0324	36.0227	229.5596	49.2649	0.1085	2405.3940	489.9923
60	291.3115	36.0661	230.5537	49.3338	0.1083	2419.3543	491.4242
61	292.5085	36.1066	231.4839	49.3981	0.1082	2432.4370	492.7624
62	293.6284	36.1444	232.3542	49.4581	0.1080	2444.6932	494.0128
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
76	303.4370	36.4708	239.9758	49.9772	0.1067	2552.6947	504.8996
77	303.8418	36.4841	240.2903	49.9984	0.1066	2557.1774	505.3465
78	304.2198	36.4966	240.5839	50.0181	0.1066	2561.3642	505.7635
79	304.5726	36.5081	240.8581	50.0365	0.1065	2565.2745	506.1527
80	304.9020	36.5189	241.1140	50.0537	0.1065	2568.9260	506.5159
81	305.2094	36.5290	241.3529	50.0698	0.1065	2572.3356	506.8547
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
91	307.3539	36.5991	243.0189	50.1813	0.1062	2596.1486	509.2152
92	307.4976	36.6037	243.1305	50.1887	0.1062	2597.7461	509.3732
93	307.6316	36.6081	243.2346	50.1957	0.1061	2599.2368	509.5206
94	307.7567	36.6122	243.3318	50.2022	0.1061	2600.6277	509.6580
95	307.8734	36.6160	243.4224	50.2082	0.1061	2601.9256	509.7863
96	307.9822	36.6195	243.5070	50.2139	0.1061	2603.1365	509.9059
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
102	308.4980	36.6363	243.9077	50.2406	0.1060	2608.8761	510.4725
103	308.5649	36.6385	243.9597	50.2440	0.1060	2609.6211	510.5460
104	308.6273	36.6405	244.0082	50.2473	0.1060	2610.3160	510.6145
105	308.6856	36.6424	244.0534	50.2503	0.1060	2610.9644	510.6785
106	308.7399	36.6442	244.0956	50.2531	0.1060	2611.5691	510.7381
107	308.7906	36.6458	244.1350	50.2557	0.1060	2612.1333	510.7937
108	308.8378	36.6473	244.1717	50.2581	0.1060	2612.6596	510.8456
109	308.8819	36.6488	244.2059	50.2604	0.1060	2613.1505	510.8940
110	308.9230	36.6501	244.2379	50.2625	0.1060	2613.6084	510.9391
111	308.9614	36.6513	244.2677	50.2645	0.1060	2614.0355	510.9812
112	308.9972	36.6525	244.2955	50.2664	0.1060	2614.4340	511.0205
113	309.0305	36.6536	244.3214	50.2681	0.1060	2614.8056	511.0571
114	309.0617	36.6546	244.3456	50.2697	0.1060	2615.1523	511.0913
115	309.0907	36.6555	244.3681	50.2712	0.1060	2615.4756	511.1232

It follows from Theorem 4 that $q-\mathbb{F}$L-E$\mathbb{P}$ (20) is stable in the UHR sense with

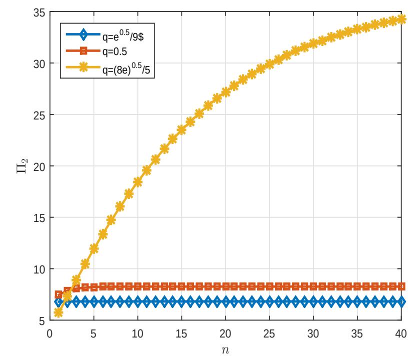
$$\|\acute{\mathfrak{s}}-\mathfrak{s}\|_W \leq \left\{\begin{array}{ll} 0.7660\times a, & q=\sqrt{e}/9, \\ 0.5300\times a, & q=1/2, \\ 0.3556\times a, & q=\sqrt{8e}/5, \end{array}\right\}\cdot g(\tilde{\tau}), \qquad a>0,\ \tilde{\tau}\in\Omega.$$

6 Conclusion

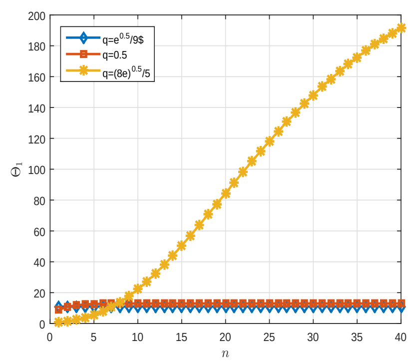
In the current research, a class of singular $q-\mathbb{F}$L-E$\mathbb{P}$ by generalization simultaneously both $q-$fractional derivatives of RL and Caputo type, with some conditions, was discussed. In fact, we tried to analyze the existence and uniqueness of solutions for the problem in our results based on famous fixed point theorems of Banach and Schaefer. Furthermore, the stability was checked for this equation such that well, one can see the accuracy of the results, including the existence of the solution, its uniqueness and stability in the example. This particular



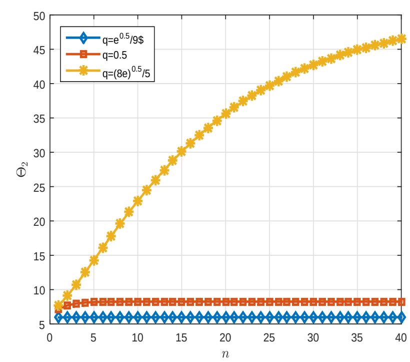
(a) Π_1



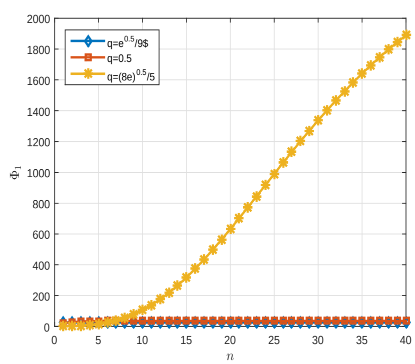
(b) Π_2



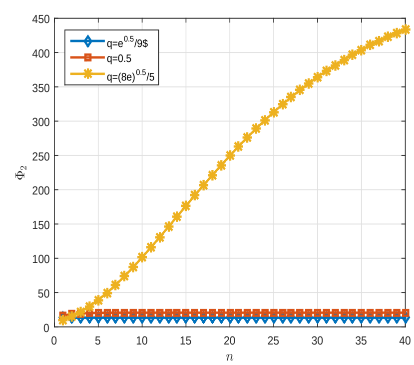
(c) Θ_1



(d) Θ_2



(e) Φ_1



(f) Φ_2

Figure 1. Graphical representation of Π_j, Θ_j, Φ_j for $j = 1,2$ of $q-\mathbb{FL}\text{-}\mathbb{EP}$ (20) for $q \in \left\{ \frac{\sqrt{e}}{9}, \frac{1}{2}, \frac{\sqrt{8e}}{5} \right\}$.

problem provided us with a powerful tool in modeling physical mathematical problems, dynamical system and quantum models. Clearly, it can be used to generalize the results presented in [12, 16, 39].

Abbreviations

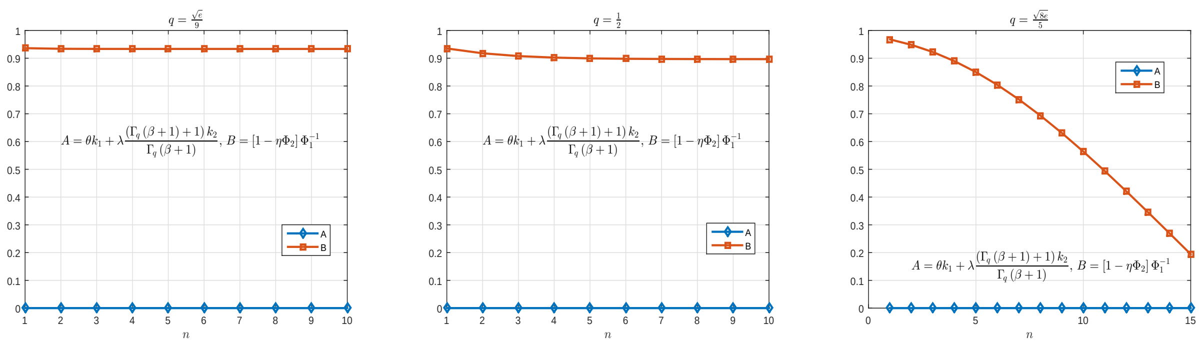
$\mathbb{DE}$: Differential equation

$q-\mathbb{DE}$: $q-$differential equation

$q-\mathbb{FL}\text{-}\mathbb{EP}$: $q-$fractional Lane-Emden problem

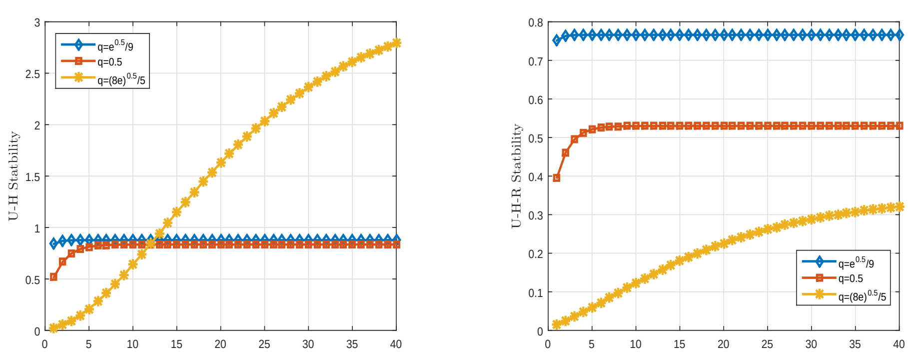
Table 4. Numerical results of A and B of $q-\mathbb{F}$L-$\mathbb{EP}$ (20) whenever $q \in \left\{ \frac{\sqrt{e}}{9}, \frac{1}{2}, \frac{\sqrt{8e}}{5} \right\}$.

n	$\Gamma_q(\beta+1)$	A	B	$\Gamma_q(\beta+1)$	A	B	$\Gamma_q(\beta+1)$	A	B
	$q=\frac{\sqrt{e}}{9}$			$q=0.5$			$q=\sqrt{8e}/5$		
1	1.0561	0.0001	0.9363	1.3289	0.0001	0.9352	9.6823	0.0001	0.9678
2	1.0214	0.0001	0.9340	1.0520	0.0001	0.9178	4.9141	0.0001	0.9481
3	1.0153	0.0001	0.9336	0.9453	0.0001	0.9077	3.0258	0.0001	0.9222
4	1.0142	0.0001	0.9335	0.8980	0.0001	0.9023	2.0848	0.0001	0.8896
5	1.0140	0.0001	0.9335	0.8757	0.0001	0.8995	1.5465	0.0001	0.8501
6	1.0139	0.0001	0.9335	0.8649	0.0001	0.8980	1.2086	0.0001	0.8039
7	1.0139	0.0001	0.9335	0.8595	0.0001	0.8973	0.9820	0.0001	0.7515
8	1.0139	0.0001	0.9335	0.8569	0.0001	0.8969	0.8221	0.0001	0.6936
9	1.0139	0.0001	0.9335	0.8555	0.0001	0.8968	0.7049	0.0001	0.6308
10	1.0139	0.0001	0.9335	0.8549	0.0001	0.8967	0.6162	0.0001	0.5639
11	1.0139	0.0001	0.9335	0.8546	0.0001	0.8966	0.5473	0.0001	0.4936
12	1.0139	0.0001	0.9335	0.8544	0.0001	0.8966	0.4926	0.0001	0.4209
13	1.0139	0.0001	0.9335	0.8543	0.0001	0.8966	0.4485	0.0001	0.3463
14	1.0139	0.0001	0.9335	0.8543	0.0001	0.8966	0.4123	0.0001	0.2705
15	1.0139	0.0001	0.9335	0.8542	0.0001	0.8966	0.3822	0.0001	0.1942
16	1.0139	0.0001	0.9335	0.8542	0.0001	0.8966	0.3569	0.0001	0.1180
17	1.0139	0.0001	0.9335	0.8542	0.0001	0.8966	0.3355	0.0001	0.0422
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮



(a) $q=\frac{\sqrt{e}}{9}$ **(b)** $q=\frac{1}{2}$ **(c)** $q=\frac{\sqrt{8e}}{5}$

Figure 2. Graphical representation of A and B of $q-\mathbb{F}$L-$\mathbb{EP}$ (20) for $q=\frac{\sqrt{e}}{9}, \frac{1}{2}, \frac{\sqrt{8e}}{5}$.



(a) UH Stability **(b)** UHR Stability

Figure 3. 2D plot of UH and UHR Stabilities for Problem (20) whenever $q \in \left\{ \frac{\sqrt{e}}{9}, \frac{1}{2}, \frac{\sqrt{8e}}{5} \right\}$.

Table 5. Numerical results of UH and UHR stabilities for Problem (20) whenever $q = \frac{\sqrt{e}}{9}, \frac{1}{2}, \frac{\sqrt{8e}}{5}$.

	Stability of $q-\mathbb{FL}$-$\mathbb{EP}$ (20) in the sense of					
	U-H	U-H-R	U-H	U-H-R	U-H	U-H-R
n	$q = \sqrt{e}/9$		$q = 1/2$		$q = \sqrt{8e}/5$	
1	0.8413	0.7516	0.5174	0.3961	0.0240	0.0147
2	0.8703	0.7634	0.6660	0.4611	0.0528	0.0244
3	0.8757	0.7655	0.7479	0.4951	0.0936	0.0353
4	0.8766	0.7659	0.7908	0.5124	0.1460	0.0470
5	0.8768	<u>0.7660</u>	0.8127	0.5212	0.2091	0.0594
6	0.8768	0.7660	0.8239	0.5256	0.2815	0.0721
7	<u>0.8769</u>	0.7660	0.8294	0.5278	0.3621	0.0849
8	0.8769	0.7660	0.8322	0.5289	0.4494	0.0977
9	0.8769	0.7660	0.8336	0.5295	0.5421	0.1104
10	0.8769	0.7660	0.8343	0.5298	0.6389	0.1228
11	0.8769	0.7660	0.8347	0.5299	0.7386	0.1350
12	0.8769	0.7660	0.8349	<u>0.5300</u>	0.8402	0.1468
13	0.8769	0.7660	<u>0.8350</u>	0.5300	0.9427	0.1583
14	0.8769	0.7660	0.8350	0.5300	1.0453	0.1693
⋮	⋮	⋮	⋮	⋮	⋮	⋮
100	0.8769	0.7660	0.8350	0.5300	3.2639	0.3554
101	0.8769	0.7660	0.8350	0.5300	3.2644	0.3554
102	0.8769	0.7660	0.8350	0.5300	3.2649	0.3555
103	0.8769	0.7660	0.8350	0.5300	3.2653	0.3555
104	0.8769	0.7660	0.8350	0.5300	3.2657	0.3555
105	0.8769	0.7660	0.8350	0.5300	<u>3.2661</u>	<u>0.3556</u>

L-E$\mathbb{P}$: Lane-Emden problem

$\mathbb{F}$L-E$\mathbb{P}$: Fractional Lane-Emden problem

UH: Ulam-Hyers

UHR: Ulam-Hyers-Rassias

RL: Riemann-Liouville

Authors' Contributions

MH: Actualization, methodology, formal analysis, validation, investigation, initial draft and was a major contributor in writing the manuscript. **MES**: Actualization, methodology, formal analysis, validation, investigation, software, simulation, initial draft and was a major contributor in writing the manuscript. **MAMK**: Actualization, methodology, formal analysis, validation, investigation, review and edit. **MK**: Actualization, methodology, formal analysis, validation, investigation, software, simulation, review. All authors read and approved the final manuscript.

Data Availability

The manuscript has no associated data or the data will not be deposited.

Conflicts of Interest

The authors declare that there is no conflict of interest.

Ethical Considerations

The authors have diligently addressed ethical concerns, such as informed consent, plagiarism, data fabrication, misconduct, falsification, double publication, redundancy, submission, and other related matters.

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