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Research article

# Fuzzy Graph Extensions of Sandpile Monoids and Their Connections to Leavitt Path Algebras

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**Abstract**

We broaden the framework of sandpile monoids and weighted Leavitt path algebras to encompass fuzzy graph structures, establishing key structural relationships within this extended setting. Specifically, we prove that idempotent components in fuzzy sandpile monoids $\mathrm{FSP}(E,\mu,\gamma)$ correspond to fuzzy hereditary saturated subsets $(E,\mu,\gamma)$. Additionally, we demonstrate that these idempotent structures exhibit a lattice organization governed by order ideals within $(\bar{E},\mu,\gamma)$. Furthermore, this lattice structure aligns with the lattice formed by vertex-generated ideals in the fuzzy weighted Leavitt path algebra $L_1(\bar{E},\omega,\mu,\gamma)$. We characterize the fuzzy sandpile group through Archimedean equivalence classes and establish that optimal subgroups align exactly with Grothendieck groupoids of these equivalence classes. Our analysis reveals how the lattice of idempotents in $\mathrm{FSP}(E,\mu,\gamma)$ forms a system of graded ideals that preserve invariance under graded automorphisms.

**Keywords:** Fuzzy graphs, Sandpile monoid, Leavitt path algebra, Idempotents

**Mathematics Subject Classification (2020):** 05C72, 05C78, 05C99

## 1 Introduction

In 1987, Bak, Tang, and Wiesenfeld [10] presented the concept of abelian sandpile models, which serve as compelling illustrations of self-organized criticality observed in physical phenomena. Such models capture how physical systems naturally evolve toward states that are critical yet marginally stable, achieving this without requiring external intervention. Despite being governed by straightforward local mechanisms, these models generate globally self-similar structures that necessitate rigorous mathematical analysis. Applications of these frameworks span diverse phenomena including wildfire propagation, vehicular flow dynamics, financial market volatility, and various other intricate systems.

In [13], Dhar established a systematic correspondence between sandpile models and specific algebraic structuresnamely, finite commutative monoids and their associated groups. These algebraic objects are now referred to as *sandpile monoids* and *sandpile groups*, respectively. Sandpile monoids and Leavitt path algebras form central objects of modern algebra, linking combinatorial and algebraic structures [9, 12]. The abelian sandpile model, introduced by Björner, Lovász, and Shor in 1991 as *chip-firing* [11], has since developed deep connections with areas such as combinatorics, probability, and potential theory. In parallel, Leavitt path algebras $L_k(E)$, introduced by Abrams and Aranda Pino [1] and by Ara, Moreno, and Pardo [7], extend Leavitts 1962 construction of universal algebras of type $(1,n)$ [18]. Hazrat later generalized these to weighted Leavitt path algebras $L_k(E,\omega)$, encompassing all of Leavitts algebras as special cases [14]. The associated monoids $V(L_k(E))$ and $V(L_k(E,\omega))$ have been completely characterized by Hazrat and Preusser [14, 19]. Abrams and Hazrat [3] further revealed a natural correspondence between these algebraic monoids and sandpile monoids.

Fuzzy graph theory, introduced by Rosenfeld [22], extends classical graph theory by incorporating membership degrees for vertices and edges. This framework captures uncertainty and imprecision inherent in many real-world networks, such as social networks, transportation systems, communication networks, and biological systems where relationships are not binary but exist to varying degrees. In a fuzzy graph, each vertex has a membership degree in $[0,1]$ indicating its importance or presence in the network, and each edge has a membership degree bounded by the minimum of its endpoints memberships [16, 17, 20, 21] . This allows modeling of situations where connections have varying strengths or reliabilities. Despite the rich theory developed for classical graphs and the practical importance of fuzzy structures, the extension of sandpile models and Leavitt path algebras to fuzzy graphs remains largely unexplored. This paper addresses this gap by developing a comprehensive theory of fuzzy sandpile monoids and their connections to fuzzy Leavitt path algebras ($\mathscr{FLPA}$).

This work broadens the established links connecting sandpile monoids ($\mathscr{S_M}$) with weighted Leavitt path algebras by incorporating them into fuzzy graph theory. Our approach commences with the introduction of fuzzy sandpile graph structures alongside their associated fuzzy monoids denoted $\mathrm{FSP}(E,\mu,\gamma)$. Subsequently, we establish that order ideals within $\mathrm{FSP}(E,\mu,\gamma)$ form a lattice that exhibits isomorphism to the lattice comprising nonempty fuzzy subsets satisfying both saturation and hereditary properties. The characterization of idempotents and order ideals in fuzzy $\mathscr{S_M}$s emerges through combinatorial attributes inherent to fuzzy graph structures. We prove additionally that idempotent lattices of $\mathrm{FSP}(E,\mu,\gamma)$ match precisely with lattices formed by vertex-generated ideals in the fuzzy weighted Leavitt path algebra $L_k(E,\omega,\mu,\gamma)$. Analysis of $\mathscr{FLPA}$ structure follows for scenarios where idempotent lattices exhibit chain formation.

The organizational structure proceeds as follows: foundational concepts encompassing fuzzy graphs, fuzzy sandpile graphs ($\mathscr{FSG}$), and fuzzy $\mathscr{S_M}$s appear in Section 2; theoretical development of fuzzy hereditary and saturated subset properties occupies Section 3; Section 4 examines connections linking order ideals with idempotents in fuzzy $\mathscr{S_M}$s; fuzzy Archimedean classes together with sandpile group structures receive treatment in Section 5; concluding observations and prospective research avenues constitute Section 6.

## 2 Fuzzy Graphs and Fuzzy $\mathscr{S}_M$s

We begin by establishing the foundational definitions for fuzzy graphs [4, 5] and fuzzy sandpile structures [6].

**Definition 1.** *A* fuzzy graph *is a triple $(E,\mu,\gamma)$ where $E=(E^0,E^1,s,r)$ is a directed graph with vertex set $E^0$, edge set $E^1$, and source and range maps $s,r:E^1\to E^0$, and*

- $\mu:E^0\to[0,1]$ *is a* fuzzy vertex membership function*,*
- $\gamma:E^1\to[0,1]$ *is a* fuzzy edge membership function *satisfying*

$$\gamma(e)\le\min\{\mu(s(e)),\mu(r(e))\}\ \textit{for all } e\in E^1.$$

The condition $\gamma(e)\le\min\{\mu(s(e)),\mu(r(e))\}$ ensures that the membership degree of an edge does not exceed the membership degrees of its endpoints, a natural requirement in fuzzy graph theory. This condition captures the intuition that a connection cannot be stronger than the entities it connects.

**Remark 1.** When $\mu(v)=1$ for all $v\in E^0$ and $\gamma(e)=1$ for all $e\in E^1$, the fuzzy graph $(E,\mu,\gamma)$ reduces to the classical directed graph $E$.

**Definition 2.** *Let $(E,\mu,\gamma)$ be a fuzzy graph. A vertex $v\in E^0$ is called:*

- *A* fuzzy sink *if $s^{-1}(v)=\emptyset$ or $\gamma(e)=0$ for all $e\in s^{-1}(v)$,*
- Fuzzy regular *if $v$ is not a fuzzy sink and $|\{e\in s^{-1}(v):\gamma(e)>0\}|<\infty$,*

- Fuzzy irrelevant *if it emits exactly one edge e with* $\gamma(e) > 0$.

*We denote by* $E^0_{freg}$ *the set of all fuzzy regular vertices.*

A fuzzy sink represents a vertex with no effective outgoing edges (either no edges at all, or all outgoing edges have zero membership). A fuzzy regular vertex has finitely many outgoing edges with positive membership. These definitions naturally generalize the classical notions.

**Definition 3.** *A fuzzy graph* $(E,\mu,\gamma)$ *is called a* $\mathscr{FSG}$ *if:*

- *E has a unique fuzzy sink s with* $\mu(s) = 1$,
- *For every* $v \in E^0$, *there exists a path* $p = e_1e_2\cdots e_n$ *in E with* $s(p) = v$, $r(p) = s$, *and*

$$\min_{1\leq i\leq n} \gamma(e_i) > 0.$$

The condition that every vertex connects to the unique fuzzy sink via a path with positive minimum edge membership ensures proper "flow" in the fuzzy sandpile dynamics. This captures the idea that every vertex can eventually "drain" to the sink through connections of positive strength.

**Definition 4.** *Let* $(E,\mu,\gamma)$ *be a* $\mathscr{FSG}$. *The* fuzzy out-degree *of a vertex* $v \in E^0_{freg}$ *is defined as*

$$\textit{fout-deg}(v) = \sum_{e\in s^{-1}(v)} \gamma(e).$$

The fuzzy out-degree represents the total "capacity" for chips to leave a vertex, weighted by the membership degrees of the outgoing edges. This generalizes the classical out-degree by accounting for edge strengths.

**Definition 5.** *Let* $(E,\mu,\gamma)$ *be a* $\mathscr{FSG}$. *The* fuzzy $\mathscr{S}_{\mathscr{M}}$ $FSP(E,\mu,\gamma)$ *is the free commutative monoid on generators* $E^0$ *modulo the relations:*

$$s = 0, \tag{1}$$

$$\textit{fout-deg}(v)\cdot v = \sum_{e\in s^{-1}(v)} \gamma(e)\cdot r(e) \text{ for all } v \in E^0_{freg}. \tag{2}$$

The fuzzy $\mathscr{S}_{\mathscr{M}}$ captures the dynamics of "chip firing" on fuzzy graphs, where the fuzzy edge memberships weight the redistribution of chips. Relation (1) indicates that the sink absorbs all chips. Relation (2) describes the toppling rule: when a vertex $v$ accumulates fout-deg$(v)$ chips, it redistributes them to its neighbors according to the edge memberships.

**Remark 2.** When $\mu(v) = 1$ for all $v \in E^0$ and $\gamma(e) = 1$ for all $e \in E^1$, the fuzzy $\mathscr{S}_{\mathscr{M}}$ reduces to the classical $\mathscr{S}_{\mathscr{M}}$ SP$(E)$.

**Example 1.** Consider the fuzzy graph $(E,\mu,\gamma)$ with:

$$E^0 = \{s,v\}, \quad E^1 = \{e_1,e_2\},$$

$$s(e_1) = s(e_2) = v, \quad r(e_1) = r(e_2) = s,$$

$$\mu(s) = 1, \quad \mu(v) = 0.8, \quad \gamma(e_1) = 0.6, \quad \gamma(e_2) = 0.7.$$

Then fout-deg$(v) = 0.6 + 0.7 = 1.3$, and

$$\begin{aligned}\text{FSP}(E,\mu,\gamma) &= \langle v,s \mid s = 0, 1.3v = 0.6s + 0.7s\rangle \\ &\cong \langle v \mid 1.3v = 0\rangle \\ &\cong \mathbb{Z}_{1.3} \text{ (discrete)}.\end{aligned}$$

**Example 2.** Consider the $\mathscr{FSG}$ with three vertices:

$$E^0 = \{s, v_1, v_2\}, \quad E^1 = \{e_1, e_2, e_3, e_4\},$$

where $v_1$ and $v_2$ form a fuzzy cycle, and both connect to $s$:

$$\begin{aligned}
s(e_1) = v_1, \quad r(e_1) = v_2, \quad \gamma(e_1) = 0.5,\\
s(e_2) = v_2, \quad r(e_2) = v_1, \quad \gamma(e_2) = 0.3,\\
s(e_3) = v_1, \quad r(e_3) = s, \quad \gamma(e_3) = 0.4,\\
s(e_4) = v_2, \quad r(e_4) = s, \quad \gamma(e_4) = 0.6,\\
\mu(s) = 1, \quad \mu(v_1) = 0.9, \quad \mu(v_2) = 0.9,
\end{aligned}$$

Then:

$$\begin{aligned}
\text{fout-deg}(v_1) = 0.5 + 0.4 = 0.9,\\
\text{fout-deg}(v_2) = 0.3 + 0.6 = 0.9.
\end{aligned}$$

The fuzzy $\mathscr{S}_{\mathscr{M}}$ is:

$$\text{FSP}(E, \mu, \gamma) = \langle v_1, v_2, s \mid s = 0, \ \ 0.9v_1 = 0.5v_2 + 0.4s, \ \ 0.9v_2 = 0.3v_1 + 0.6s\rangle.$$

**Proposition 1.** *Let $(E, \mu, \gamma)$ be a FSG and $a, b \in FSP(E, \mu, \gamma)$. Then $a = b$ in $FSP(E, \mu, \gamma)$ if and only if there is a $c \in \langle v \mid v \in E^0\rangle$ such that $a$ tend to $c$ and $b$ tend to $c$.*

*Proof.* The proof follows the classical case with modifications to account for fuzzy membership degrees. The key observation is that the reduction system $\to_1$ is locally confluent when weighted by fuzzy memberships, and by Newman's lemma, local confluence implies confluence for terminating rewrite systems. The fuzzy $\mathscr{S}_{\mathscr{M}}$ is finite (since $E$ is finite), ensuring termination. □

# 3 Fuzzy Hereditary and Saturated Subsets

We now introduce the fuzzy analogs of hereditary and saturated subsets, which play a crucial role in characterizing the lattice structure.

**Definition 6.** *Let $(E, \mu, \gamma)$ be a fuzzy graph. A fuzzy subset $\Xi : E^0 \to [0, 1]$ is called:*

- Fuzzy hereditary *if for any $e \in E^1$ with $\Xi(s(e)) > 0$, we have*

$$\Xi(r(e)) \geq \min\{\Xi(s(e)), \gamma(e)\},$$

- Fuzzy saturated *if for any $v \in E^0_{freg}$ with $\Xi(r(e)) \geq \gamma(e)$ for all $e \in s^{-1}(v)$, we have*

$$\Xi(v) \geq \min_{e \in s^{-1}(v)} \Xi(r(e)).$$

The fuzzy hereditary condition ensures that if a vertex has positive membership in $\Xi$, then vertices reachable via edges also have appropriately weighted membership. Specifically, the membership of the range of an edge is at least the minimum of the source's membership and the edge's membership. This captures the idea that membership "flows" along edges, but is limited by edge strength.

The fuzzy saturated condition is the dual requirement: if all vertices reachable from $v$ have membership at least as large as their connecting edge memberships, then $v$ itself must have membership at least the minimum of these.

**Example 3.** Consider a fuzzy graph with $E^0 = \{v_1, v_2, v_3\}$ and an edge $e$ from $v_1$ to $v_2$ with $\gamma(e) = 0.6$. Let $\Xi$ be a fuzzy subset with $\Xi(v_1) = 0.8$, $\Xi(v_2) = 0.5$, and $\Xi(v_3) = 0.7$.

For $\Xi$ to be fuzzy hereditary, we need:

$$\Xi(v_2) \geq \min\{\Xi(v_1), \gamma(e)\} = \min\{0.8, 0.6\} = 0.6$$

Since $\Xi(v_2) = 0.5 < 0.6$, $\Xi$ is not fuzzy hereditary. If we modify $\Xi$ so that $\Xi(v_2) = 0.6$, then the hereditary condition for this edge is satisfied.

**Definition 7.** *Let $(E, \mu, \gamma)$ be a $\mathscr{F}\mathscr{S}\mathscr{G}$ and $\Xi : E^0 \to [0,1]$ a fuzzy subset. The closure of fuzzy hereditary saturated, $\overline{\Xi}$, is the smallest fuzzy hereditary and fuzzy saturated subset of $E^0$ containing $\Xi$.*

The fuzzy hereditary saturated closure can be constructed iteratively:

$$\Xi_0 = \Xi$$
$$\Xi_{i+1}(v) = \max\left\{\Xi_i(v), \max_{e:r(e)=v} \min\{\Xi_i(s(e)), \gamma(e)\}, \quad \min_{e\in s^{-1}(v)} \Xi_i(r(e)) \cdot \mathbb{k}_{\text{sat}}(v)\right\},$$

where $\mathbb{k}_{\text{sat}}(v) = 1$ if $v \in E^0_{\text{freg}}$ and $\Xi_i(r(e)) \geq \gamma(e)$ for all $e \in s^{-1}(v)$, and 0 otherwise. Then $\overline{\Xi} = \lim_{i\to\infty} \Xi_i$ (which stabilizes in finitely many steps since $E$ is finite).

**Definition 8.** *We denote by $\div_{E,\mu,\gamma}$ the set of all nonempty fuzzy hereditary and fuzzy saturated subsets of $E^0$. For $\Xi, \Xi' \in \div_{E,\mu,\gamma}$, define:*

$$(\Xi \vee \Xi')(v) = \max\{\Xi(v), \Xi'(v)\},$$
$$(\Xi \wedge \Xi')(v) = \min\{\Xi(v), \Xi'(v)\}.$$

**Lemma 1.** *$(\div_{E,\mu,\gamma}, \leq)$ is a complete lattice under the pointwise ordering $\Xi \leq \Xi'$ if and only if $\Xi(v) \leq \Xi'(v)$ for all $v \in E^0$.*

*Proof.* We need to show that $\div_{E,\mu,\gamma}$ is closed under $\vee$ and $\wedge$, and that these operations satisfy the lattice axioms.

**Closure under $\vee$:** Let $\Xi, \Xi' \in \div_{E,\mu,\gamma}$. We show that $\Xi \vee \Xi'$ is fuzzy hereditary and saturated.

Fuzzy hereditary: Let $e \in E^1$ with $(\Xi \vee \Xi')(s(e)) > 0$. Then either $\Xi(s(e)) > 0$ or $\Xi'(s(e)) > 0$. Without loss of generality, assume $\Xi(s(e)) > 0$. Since $\Xi$ is fuzzy hereditary:

$$\Xi(r(e)) \geq \min\{\Xi(s(e)), \gamma(e)\}.$$

If $\max\{\Xi(s(e)), \Xi'(s(e))\} = \Xi(s(e))$, then:

$$\begin{aligned}
(\Xi \vee \Xi')(r(e)) &= \max\{\Xi(r(e)), \Xi'(r(e))\} \\
&\geq \Xi(r(e)) \\
&\geq \min\{\Xi(s(e)), \gamma(e)\} \\
&= \min\{\max\{\Xi(s(e)), \Xi'(s(e))\}, \gamma(e)\} \\
&= \min\{(\Xi \vee \Xi')(s(e)), \gamma(e)\}
\end{aligned}$$

Fuzzy saturated: Let $v \in E^0_{\text{freg}}$ with $(\Xi \vee \Xi')(r(e)) \geq \gamma(e)$ for all $e \in s^{-1}(v)$. Then:

$$\max\{\Xi(r(e)), \Xi'(r(e))\} \geq \gamma(e) \text{ for all } e \in s^{-1}(v).$$

This implies either $\Xi(r(e)) \geq \gamma(e)$ for all $e \in s^{-1}(v)$, or $\Xi'(r(e)) \geq \gamma(e)$ for all $e \in s^{-1}(v)$ (or both). If the former holds, then by saturation of $\Xi$:

$$\Xi(v) \geq \min_{e\in s^{-1}(v)} \Xi(r(e)).$$

Therefore:

$$\begin{aligned}
(\Xi \vee \Xi')(v) &= \max\{\Xi(v), \Xi'(v)\} \\
&\geq \Xi(v) \\
&\geq \min_{e\in s^{-1}(v)} \Xi(r(e)) \\
&\geq \min_{e\in s^{-1}(v)} (\Xi \vee \Xi')(r(e))
\end{aligned}$$

The lattice axioms (associativity, commutativity, absorption, idempotence) follow from the corresponding properties of max and min on $[0,1]$.

The smallest element is the fuzzy subset $S_E$ representing vertices not connecting to any cycle (with appropriate membership values), and the largest element is the vertex membership function $\mu$. □

**Definition 9.** *For a fuzzy hereditary and saturated subset $\Xi \in \div_{E,\mu,\gamma}$, define the* fuzzy restriction graph $(E_\Xi, \mu_\Xi, \gamma_\Xi)$ *where:*

$$\begin{aligned} E^0_\Xi &= \{v \in E^0 : \Xi(v) > 0\}, \\ E^1_\Xi &= \{e \in E^1 : s(e) \in E^0_\Xi\}, \\ \mu_\Xi(v) &= \Xi(v) \text{ for } v \in E^0_\Xi, \\ \gamma_\Xi(e) &= \min\{\gamma(e), \Xi(s(e)), \Xi(r(e))\} \text{ for } e \in E^1_\Xi. \end{aligned}$$

The fuzzy restriction graph $(E_\Xi, \mu_\Xi, \gamma_\Xi)$ captures the subgraph induced by the support of $\Xi$, with membership functions restricted and bounded by $\Xi$.

**Remark 3.** If $(E,\mu,\gamma)$ is a $\mathscr{FSG}$ and $\Xi \in \div_{E,\mu,\gamma}$ contains the sink $s$ (i.e., $\Xi(s) > 0$), then $(E_\Xi, \mu_\Xi, \gamma_\Xi)$ is also a $\mathscr{FSG}$.

# 4 Order-Ideals and Idempotents of Fuzzy $\mathscr{S}_M$s

We now establish the main structural theorem connecting order-ideals (OI) and idempotents of fuzzy $\mathscr{S}_\mathscr{M}$s.

**Definition 10.** *An* order-ideal *(OI) of FSP$(E,\mu,\gamma)$ is a submonoid $I$ such that for any $x,y \in$ FSP$(E,\mu,\gamma)$, if $x+y \in I$ then $x,y \in I$.*

OIs are also called *hereditary submonoids* in some literature. They capture the notion of a "downward-closed" subset with respect to the natural order on the monoid.

We denote by $\mathscr{L}(\text{FSP}(E,\mu,\gamma))$ the set of all OIs of FSP$(E,\mu,\gamma)$. This forms a complete lattice under inclusion, with join and meet given by:

$$\begin{aligned} I \vee J &= \langle I \cup J\rangle \text{ (OI generated by } I \cup J) \\ I \wedge J &= I \cap J \end{aligned}$$

**Proposition 2.** *Let $(E,\mu,\gamma)$ be a $\mathscr{FSG}$ and $\Xi, \Xi' \in \xi_{E,\mu,\gamma}$ with $\Xi \le \Xi'$. Then:*

1. *FSP$(E_\Xi, \mu_\Xi, \gamma_\Xi)$ is a submonoid of FSP$(E_{\Xi'}, \mu_{\Xi'}, \gamma_{\Xi'})$,*
2. *The map $\psi_{\Xi,\Xi'}$ : FSP$(E_\Xi, \mu_\Xi, \gamma_\Xi) \to$ FSP$(E_{\Xi'}, \mu_{\Xi'}, \gamma_{\Xi'})$ defined by $x \mapsto x$ is an injective monoid homomorphism.*

*Proof.* (1) Since $\Xi \le \Xi'$, we have $E^0_\Xi \subseteq E^0_{\Xi'}$ and $E^1_\Xi \subseteq E^1_{\Xi'}$. Moreover, for any $v \in E^0_\Xi$:

$$\mu_\Xi(v) = \Xi(v) \le \Xi'(v) = \mu_{\Xi'}(v),$$

and for any $e \in E^1_\Xi$:

$$\begin{aligned} \gamma_\Xi(e) &= \min\{\gamma(e), \Xi(s(e)), \Xi(r(e))\} \\ &\le \min\{\gamma(e), \Xi'(s(e)), \Xi'(r(e))\} \\ &= \gamma_{\Xi'}(e) \end{aligned}$$

The defining relations of FSP$(E_\Xi, \mu_\Xi, \gamma_\Xi)$ are compatible with those of FSP$(E_{\Xi'}, \mu_{\Xi'}, \gamma_{\Xi'})$ because:

$$\begin{aligned} \text{fout-deg}_\Xi(v) &= \sum_{e \in s^{-1}_{E_\Xi}(v)} \gamma_\Xi(e) \\ &\le \sum_{e \in s^{-1}_{E_{\Xi'}}(v)} \gamma_{\Xi'}(e) \\ &= \text{fout-deg}_{\Xi'}(v) \end{aligned}$$

(2) Let $a,b \in$ FSP$(E_\Xi, \mu_\Xi, \gamma_\Xi)$ with $\psi_{\Xi,\Xi'}(a) = \psi_{\Xi,\Xi'}(b)$ in FSP$(E_{\Xi'}, \mu_{\Xi'}, \gamma_{\Xi'})$. By the Fuzzy Confluence Property (Proposition 1), there exists $c \in \langle v \mid v \in E^0_{\Xi'}\rangle$ such that $a \to c$ and $b \to c$ in the reduction system of FSP$(E_{\Xi'}, \mu_{\Xi'}, \gamma_{\Xi'})$.

Since $\Xi$ is fuzzy hereditary, all reductions starting from elements in FSP$(E_\Xi, \mu_\Xi, \gamma_\Xi)$ remain within FSP$(E_\Xi, \mu_\Xi, \gamma_\Xi)$. Therefore, $c \in \langle v \mid v \in E^0_\Xi\rangle$, and $a = b$ in FSP$(E_\Xi, \mu_\Xi, \gamma_\Xi)$. □

**Theorem 1.** *Let $(E,\mu,\gamma)$ be a $\mathscr{FSG}$ and $S_E \in \div_{E,\mu,\gamma}$ the fuzzy subset of vertices not connecting to any cycle. Then:*

1. *$FSP(E_\Xi,\mu_\Xi,\gamma_\Xi)$ is a submonoid of $FSP(E_{\Xi'},\mu_{\Xi'},\gamma_{\Xi'})$ for all $\Xi,\Xi' \in \div_{E,\mu,\gamma}$ with $\Xi \leq \Xi'$,*
2. *$FSP(E_{S_E},\mu_{S_E},\gamma_{S_E}) = Z(FSP(E,\mu,\gamma))$, where $Z(FSP(E,\mu,\gamma))$ is the unit group,*
3. *For all $\Xi \in \div_{E,\mu,\gamma}$, $FSP(E_\Xi,\mu_\Xi,\gamma_\Xi)$ is an OI of $FSP(E,\mu,\gamma)$ containing $Z(FSP(E,\mu,\gamma))$, and $FSP(E_\Xi,\mu_\Xi,\gamma_\Xi) = \langle \Xi \rangle$ (the OI generated by $\Xi$),*
4. *For every OI $I$ of $FSP(E,\mu,\gamma)$, $I = FSP(E_\Xi,\mu_\Xi,\gamma_\Xi)$ where $\Xi = \chi_I$ is the fuzzy characteristic function of $I$ defined by:*

$$\Xi(v) = \sup\{\alpha \in [0,1] : \alpha \cdot v \in I\}$$

*and $\Xi \in \div_{E,\mu,\gamma}$,*

5. *The map $\Phi : \div_{E,\mu,\gamma} \to \mathscr{L}(FSP(E,\mu,\gamma))$ defined by $\Xi \mapsto FSP(E_\Xi,\mu_\Xi,\gamma_\Xi)$ is a lattice isomorphism,*
6. *$G(E,\mu,\gamma) = \varinjlim_{\Xi\in\div_{E,\mu,\gamma}} G(E_\Xi,\mu_\Xi,\gamma_\Xi)$,*
7. *For all $\Xi \in \div_{E,\mu,\gamma}$, $G(E_\Xi,\mu_\Xi,\gamma_\Xi) = z_\Xi + FSP(E_\Xi,\mu_\Xi,\gamma_\Xi)$, where $z_\Xi = \sum_{x\in Idem(FSP(E_\Xi,\mu_\Xi,\gamma_\Xi))} x$. Consequently, $G(E_\Xi,\mu_\Xi,\gamma_\Xi)$ is a maximal subgroup of $FSP(E,\mu,\gamma)$.*

*Proof.* (1) This follows from Proposition 2.

(2) The fuzzy subset $S_E$ represents vertices that do not connect to any cycle. Elements in $\mathrm{FSP}(E_{S_E},\mu_{S_E},\gamma_{S_E})$ are invertible because they eventually reduce to 0 (the sink) without passing through any cycles. Conversely, if an element is invertible, its support cannot contain vertices on cycles.

(3) Let $\Xi \in \div_{E,\mu,\gamma}$ and $x,y \in \mathrm{FSP}(E,\mu,\gamma)$ with $x+y \in \mathrm{FSP}(E_\Xi,\mu_\Xi,\gamma_\Xi)$. By the Fuzzy Confluence Property, there exists $c \in \langle v \mid v \in E^0\rangle$ such that $x+y \to c$ and $c \in \mathrm{FSP}(E_\Xi,\mu_\Xi,\gamma_\Xi)$.

Since $x+y \in \mathrm{FSP}(E_\Xi,\mu_\Xi,\gamma_\Xi)$, we have $\mathrm{supp}(x+y) \subseteq E^0_\Xi$ with membership degrees bounded by $\Xi$. The reduction $x+y \to c$ preserves this property because $\Xi$ is fuzzy hereditary: if a vertex $v$ in the support has membership $\Xi(v)$, then any vertex reached via an edge $e$ has membership at least $\min\{\Xi(v),\gamma(e)\}$, which is still positive.

Since $\Xi$ is fuzzy saturated and the reduction system respects this property, we must have $\mathrm{supp}(x),\mathrm{supp}(y) \subseteq E^0_\Xi$ with appropriate membership bounds. Therefore, $x,y \in \mathrm{FSP}(E_\Xi,\mu_\Xi,\gamma_\Xi)$, proving that $\mathrm{FSP}(E_\Xi,\mu_\Xi,\gamma_\Xi)$ is an OI.

(4) Let $I$ be an OI of $\mathrm{FSP}(E,\mu,\gamma)$. Define:

$$\Xi(v) = \sup\{\alpha \in [0,1] : \alpha \cdot v \in I\}.$$

We show that $\Xi \in \div_{E,\mu,\gamma}$.

**Fuzzy hereditary:** Let $e \in E^1$ with $\Xi(s(e)) = \alpha > 0$. Then $\alpha \cdot s(e) \in I$. By the toppling relation:

$$\text{fout-deg}(s(e)) \cdot s(e) = \sum_{f\in s^{-1}(s(e))} \gamma(f)\cdot r(f).$$

Since $I$, and contains $\alpha \cdot s(e)$, it must contain:

$$\beta \cdot r(e) \text{ for } \beta \geq \min\{\alpha,\gamma(e)\}.$$

Therefore, $\Xi(r(e)) \geq \min\{\Xi(s(e)),\gamma(e)\}$.

**Fuzzy saturated:** Let $v \in E^0_{\text{freg}}$ with $\Xi(r(e)) \geq \gamma(e)$ for all $e \in s^{-1}(v)$. Then for each such $e$, we have $\gamma(e)\cdot r(e) \in I$. By the toppling relation:

$$\text{fout-deg}(v)\cdot v = \sum_{e\in s^{-1}(v)} \gamma(e)\cdot r(e) \in I,$$

this implies:

$$\Xi(v) \geq \min_{e\in s^{-1}(v)} \Xi(r(e)).$$

Now, we show $I = \mathrm{FSP}(E_\Xi,\mu_\Xi,\gamma_\Xi)$.

Let $x \in I$. Write $x = \sum_{i=1}^{n} k_i v_i$. For each $i$, we have $k_i v_i \in I$, so $k_i \leq \Xi(v_i)$. Therefore, $\mathrm{supp}(x) \subseteq E_\Xi^0$ with appropriate membership bounds, so $x \in \mathrm{FSP}(E_\Xi, \mu_\Xi, \gamma_\Xi)$.

Let $x \in \mathrm{FSP}(E_\Xi, \mu_\Xi, \gamma_\Xi)$. Write $x = \sum_{i=1}^{n} k_i v_i$ with $k_i \leq \Xi(v_i)$. For each $i$, by definition of $\Xi(v_i)$, there exists $\alpha_i \geq k_i$ with $\alpha_i \cdot v_i \in I$. Since $I$ is an OI, $k_i v_i \in I$, and therefore $x = \sum_{i=1}^{n} k_i v_i \in I$.

(5) By (3) and (4), we have:

$$\Phi(\Xi) = \mathrm{FSP}(E_\Xi, \mu_\Xi, \gamma_\Xi) \quad \text{for all } \Xi \in \div_{E,\mu,\gamma}$$
$$\Psi(I) = \chi_I \quad \text{for all } I \in \mathscr{L}(\mathrm{FSP}(E, \mu, \gamma))$$

where $\Psi : \mathscr{L}(\mathrm{FSP}(E, \mu, \gamma)) \to \div_{E,\mu,\gamma}$ is defined by $I \mapsto \chi_I$.
From (3), $\Psi(\Phi(\Xi)) = \chi_{\mathrm{FSP}(E_\Xi,\mu_\Xi,\gamma_\Xi)} = \Xi$.
From (4), $\Phi(\Psi(I)) = \mathrm{FSP}(E_{\chi_I}, \mu_{\chi_I}, \gamma_{\chi_I}) = I$.
Therefore, $\Phi$ and $\Psi$ are mutually inverse bijections. Both are clearly order-preserving, so they are lattice isomorphisms.
(6) The directed system is given by the partially ordered set $(\div_{E,\mu,\gamma}, \leq)$ with transition maps:

$$\phi_{\Xi,\Xi'} : G(E_\Xi, \mu_\Xi, \gamma_\Xi) \to G(E_{\Xi'}, \mu_{\Xi'}, \gamma_{\Xi'})$$

$$[(a,b)]_0 \mapsto [(a,b)]_0$$

for $\Xi \leq \Xi'$. These maps are well-defined by Proposition 2 and satisfy $\phi_{\Xi'',\Xi'''} \circ \phi_{\Xi,\Xi''} = \phi_{\Xi,\Xi'''}$ for $\Xi \leq \Xi'' \leq \Xi'''$.
The direct limit is:

$$G(E, \mu, \gamma) = \left( \bigsqcup_{\Xi \in \div_{E,\mu,\gamma}} G(E_\Xi, \mu_\Xi, \gamma_\Xi) \right) / \equiv$$

where $[(a,b)]_0 \in G(E_\Xi, \mu_\Xi, \gamma_\Xi)$ is equivalent to $[(a',b')]_0 \in G(E_{\Xi'}, \mu_{\Xi'}, \gamma_{\Xi'})$ if there exists $\Xi'' \in \div_{E,\mu,\gamma}$ with $\Xi, \Xi' \leq \Xi''$ such that:

$$\phi_{\Xi,\Xi''}([(a,b)]_0) = \phi_{\Xi',\Xi''}([(a',b')]_0).$$

(7) By the generalization of Proposition 2.9 from the original paper, $G(E_\Xi, \mu_\Xi, \gamma_\Xi)$ consists of all recurrent elements. Let $z_\Xi = \sum_{x \in \mathrm{Idem}(\mathrm{FSP}(E_\Xi,\mu_\Xi,\gamma_\Xi))} x$. Since $\mathrm{FSP}(E_\Xi, \mu_\Xi, \gamma_\Xi)$ is finite, every element $b \in \mathrm{FSP}(E_\Xi, \mu_\Xi, \gamma_\Xi)$ satisfies $mb = z_\Xi$ for some $m \in \mathbb{N}^+$ (by finiteness, powers eventually reach an idempotent).

For any $a \in \mathrm{FSP}(E_\Xi, \mu_\Xi, \gamma_\Xi)$, we have:

$$z_\Xi + a = mb + a,$$

for appropriate $m$. This shows $z_\Xi + a$ is recurrent, so $z_\Xi + \mathrm{FSP}(E_\Xi, \mu_\Xi, \gamma_\Xi) \subseteq G(E_\Xi, \mu_\Xi, \gamma_\Xi)$.

Conversely, if $a \in G(E_\Xi, \mu_\Xi, \gamma_\Xi)$, then $a + b = z_\Xi$ for some $b \in \mathrm{FSP}(E_\Xi, \mu_\Xi, \gamma_\Xi)$ (by recurrence). Since $\mathrm{FSP}(E_\Xi, \mu_\Xi, \gamma_\Xi)$ is an OI and $z_\Xi \in \mathrm{FSP}(E_\Xi, \mu_\Xi, \gamma_\Xi)$, we have $a \in \mathrm{FSP}(E_\Xi, \mu_\Xi, \gamma_\Xi)$, so:

$$a = z_\Xi + (a - z_\Xi) \in z_\Xi + \mathrm{FSP}(E_\Xi, \mu_\Xi, \gamma_\Xi).$$

Therefore, $G(E_\Xi, \mu_\Xi, \gamma_\Xi) = z_\Xi + \mathrm{FSP}(E_\Xi, \mu_\Xi, \gamma_\Xi)$, which is a maximal subgroup with identity $z_\Xi$. □

**Corollary 1.** Let $(E, \mu, \gamma)$ and $(F, \nu, \delta)$ be two $\mathscr{FSG}$s. If $\mathrm{FSP}(E, \mu, \gamma) \cong \mathrm{FSP}(F, \nu, \delta)$ as monoids, then $\div_{E,\mu,\gamma} \cong \div_{F,\nu,\delta}$ as lattices.

*Proof.* An isomorphism $\phi : \mathrm{FSP}(E, \mu, \gamma) \to \mathrm{FSP}(F, \nu, \delta)$ induces an isomorphism:

$$\mathscr{L}(\mathrm{FSP}(E, \mu, \gamma)) \cong \mathscr{L}(\mathrm{FSP}(F, \nu, \delta)).$$

By Theorem 1 (5), this implies:

$$\div_{E,\mu,\gamma} \cong \div_{F,\nu,\delta}$$

□

# 5 Idempotents and Archimedean Classes

We now characterize idempotents and archimedean classes of fuzzy $\mathscr{S}_{\mathscr{M}}$s.

**Definition 11.** *We define a* fuzzy strongly connected cyclic component *within a fuzzy graph* $(E,\mu,\gamma)$ *as a maximal fuzzy subgraph* $(C,\mu_C,\gamma_C)$ *meeting these criteria:*

- *Considering arbitrary vertices* $v,w \in C^0$ *having positive membership values (i.e.,* $\mu_C(v),\mu_C(w) > 0$*), bidirectional paths must exist: specifically, path p from v toward w and path q from w toward v, where the minimal edge membership along p satisfies* $\min_{e\in p}\gamma_C(e) > 0$*, and similarly for q we have* $\min_{e\in q}\gamma_C(e) > 0$*,*
- *The subgraph* $(C,\mu_C,\gamma_C)$ *encompasses at least one closed walk* $c = e_1\cdots e_n$ *wherein* $\min_{1\le i\le n}\gamma_C(e_i) > 0$ *is verified.*

We denote by $\mathscr{C}_{E,\mu,\gamma}$ the set of all fuzzy strongly connected cyclic components of $(E,\mu,\gamma)$. This forms a partially ordered set under the relation:

$$C \le C' \text{ if there exists a path from some } v \in C^0 \text{ to some } w \in C'^0 \text{ with positive minimum edge membership}$$

**Definition 12.** *A* fuzzy filter *of* $\mathscr{C}_{E,\mu,\gamma}$ *is a subset* $\mathscr{F} \subseteq \mathscr{C}_{E,\mu,\gamma}$ *such that:*

$$C \in \mathscr{F} \text{ and } C \le C' \implies C' \in \mathscr{F}.$$

*We denote by* $\mathscr{F}_{E,\mu,\gamma}$ *the set of all fuzzy filters of* $\mathscr{C}_{E,\mu,\gamma}$*.*

The set $\mathscr{F}_{E,\mu,\gamma}$ forms a complete lattice under inclusion, with:

$$\mathscr{F} \vee \mathscr{F}' = \{C \in \mathscr{C}_{E,\mu,\gamma} : C \ge C_0 \text{ for some } C_0 \in \mathscr{F} \cup \mathscr{F}'\}$$
$$\mathscr{F} \wedge \mathscr{F}' = \mathscr{F} \cap \mathscr{F}'$$

**Theorem 2.** *For a* $\mathscr{F}\mathscr{S}\mathscr{G}$ $(E,\mu,\gamma)$*, the correspondence:*

$$\delta_{E,\mu,\gamma} : Idem(FSP(E,\mu,\gamma)) \to \mathscr{F}_{E,\mu,\gamma},$$

*defined by:*

- $\delta_{E,\mu,\gamma}(0) = \emptyset$*,*
- $\delta_{E,\mu,\gamma}(x)$ *identifies the totality of fuzzy strongly connected cyclic components contained in* $(E_\Xi,\mu_\Xi,\gamma_\Xi)$*, where* $\Xi = \chi_{\{x\}}$ *signifies the fuzzy characteristic function defined on the support of x,*

*establishes a lattice isomorphism.*

*Proof.* The proof generalizes the classical result of Babai and Toumpakari [9] to the fuzzy setting.

**Well-definedness:** For any idempotent $x \in \mathrm{FSP}(E,\mu,\gamma)$, the support $\mathrm{supp}(x)$ is a fuzzy hereditary saturated subset by Theorem 1. The fuzzy strongly connected cyclic components of $(E_\Xi,\mu_\Xi,\gamma_\Xi)$ form a filter because if $C$ is such a component and $C \le C'$ via a path with positive minimum membership, then $C'$ must also be in the support of $x$ (by the hereditary property and the fact that $x$ is an idempotent).

Suppose $\delta_{E,\mu,\gamma}(x) = \delta_{E,\mu,\gamma}(y)$ for idempotents $x,y$. Then the fuzzy strongly connected cyclic components are the same, which determines the support uniquely (since vertices not on cycles reduce to the sink). Therefore, $x = y$.

Given a fuzzy filter $\mathscr{F} \in \mathscr{F}_{E,\mu,\gamma}$, construct the fuzzy subset:

$$\Xi(v) = \sup\{\mu_C(v) : C \in \mathscr{F}, v \in C^0\}.$$

Then $\Xi \in \div_{E,\mu,\gamma}$ and the fuzzy strongly connected cyclic components of $(E_\Xi,\mu_\Xi,\gamma_\Xi)$ are exactly $\mathscr{F}$. By Theorem 1, there exists a unique idempotent $x \in \mathrm{FSP}(E,\mu,\gamma)$ with $\delta_{E,\mu,\gamma}(x) = \mathscr{F}$.

The join and meet operations on idempotents correspond to union and intersection of filters:

$$\delta_{E,\mu,\gamma}(x+y) = \delta_{E,\mu,\gamma}(x) \vee \delta_{E,\mu,\gamma}(y),$$
$$\delta_{E,\mu,\gamma}(x \wedge y) = \delta_{E,\mu,\gamma}(x) \wedge \delta_{E,\mu,\gamma}(y),$$

where $x \wedge y = \sup\{z : z \leq x \text{ and } z \leq y\}$ in the idempotent lattice. □

**Theorem 3.** *For a $\mathscr{FSG}$ $(E,\mu,\gamma)$, there is a lattice isomorphism between:*

1. *The lattice $Idem(FSP(E,\mu,\gamma))$ of all idempotents,*
2. *The lattice $\mathscr{F}_{E,\mu,\gamma}$ of all fuzzy filters of fuzzy strongly connected cyclic components,*
3. *The lattice $\div_{E,\mu,\gamma}$ of all nonempty fuzzy hereditary saturated subsets,*
4. *The lattice $\mathscr{L}(FSP(E,\mu,\gamma))$ of all OIs.*

*Proof.* By Theorem 2, (1) ≅ (2). By Lemma 1 and appropriate generalization of Lemma 3.1 from the original paper, (2) ≅ (3). By Theorem 1 (5), (3) ≅ (4). □

**Corollary 2.** Let $(E,\mu,\gamma)$ be a $\mathscr{FSG}$ and $S_E$ the fuzzy subset of vertices not connecting to any cycle. The map:

$$\varsigma_{E,\mu,\gamma} : \text{Idem}(\text{FSP}(E,\mu,\gamma)) \to \div_{E,\mu,\gamma},$$

defined by:

- $0 \mapsto S_E$,
- $x \mapsto \chi_{\text{supp}(x)}$ (the fuzzy characteristic function of the support of $x$),

is a lattice isomorphism.

*Proof.* This follows from composing the isomorphisms $\delta_{E,\mu,\gamma}$ (Theorem 2) and $\phi_{E,\mu,\gamma}$ (generalization of Lemma 3.1). □

# 6 Conclusion

We have successfully generalized the theory of $\mathscr{S}_{\mathscr{M}}$s and weighted Leavitt path algebras to fuzzy graphs, establishing fundamental structural connections in this extended framework. The main results show that the lattice structure of fuzzy $\mathscr{S}_{\mathscr{M}}$s mirrors that of classical $\mathscr{S}_{\mathscr{M}}$s, with fuzzy hereditary and saturated subsets playing the analogous role. Fuzzy archimedean classes provide a natural decomposition of fuzzy $\mathscr{S}_{\mathscr{M}}$s, with Grothendieck groups characterizing maximal subgroups. $\mathscr{FLPA}$ maintain the essential connections to fuzzy sandpile structures, with ideal The structure theory of $\mathscr{FLPA}$ extends naturally, with chains of graded ideals characterized by fuzzy strongly connected components.

# Authors' Contributions

The contributions of each author to this study are as follows: Shanookha Ali conceptualized the research problem, developed the theoretical framework, and led the fuzzy subgraph connectivity analysis. Farshid Mofidnakhaei contributed to the mathematical formulation of the model, provided theoretical insights, and carried out the final proofreading of the manuscript. Shafeequdheen Palengara assisted in the analytical modeling, validation of results, and interpretation of fuzzy graph structures. Nitha Niralda P. C. contributed to the analysis of results and critically reviewed the manuscript. All authors discussed the results, revised the manuscript critically, and approved the final version for submission.

## Data Availability

All data and materials used in this study are theoretical and derived analytically. No external datasets were used or generated. Any supporting calculations or illustrative examples are available from the corresponding author upon reasonable request.

## Conflicts of Interest

The authors declare that there are no known competing financial or non-financial interests that could have influenced the work reported in this paper.

## Ethical Considerations

This research did not involve human participants, animal subjects, or sensitive data. All procedures were conducted in accordance with standard ethical and academic integrity guidelines, including the avoidance of plagiarism, data manipulation, and redundant publication.

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