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Characterization Theorem for the Numerical Solution of Fuzzy Differential Inclusions (FDIs)

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Abstract

In this paper, we investigate the numerical solution of fuzzy differential inclusions (FDIs) using characterization results for fuzzy differential equations (FDEs) based on Hukuhara differentiability. By employing the characterization theorem introduced by Bede and the construction of solutions via differential inclusions developed by Kaleva, we establish a rigorous connection among fuzzy differential equations, fuzzy differential inclusions, and systems of ordinary differential equations (ODEs). In particular, under suitable regularity and monotonicity assumptions, it is shown that the solution of a fuzzy differential inclusion coincides with the solution of the corresponding fuzzy differential equation and can be equivalently represented by a system of ODEs defined on the α-level sets. This equivalence enables the reduction of fuzzy-valued problems to classical real-valued systems, thereby allowing the direct application of standard numerical methods for ordinary differential equations. Based on this framework, we propose a numerical approach for solving FDIs by first transforming the fuzzy problem into a parametric family of ODEs and then applying the Euler method to approximate the solutions. The validity of the proposed approach is illustrated through a numerical example, in which the approximate fuzzy solution is compared with the exact solution. The results demonstrate that classical numerical schemes can be effectively employed for fuzzy differential inclusions without reformulating them within a fully fuzzy numerical framework.

Keywords: Fuzzy differential equations, Fuzzy derivative, Differential inclusions

Mathematics Subject Classification (2020): 34A07

1 Introduction

Fuzzy differential equations (FDEs) play an essential role in modeling dynamical systems involving uncertainty, imprecision, or incomplete information, which frequently arise in science, engineering, economics, and control theory. In many practical situations, system parameters, initial conditions, or external forces cannot be described precisely by real numbers and are more appropriately represented by fuzzy numbers.

Consequently, fuzzy initial value problems (FIVPs) naturally emerge as mathematical models for such systems.

One of the main challenges in the theory of fuzzy differential equations is the definition of an appropriate notion of differentiability for fuzzy-valued functions. Among the earliest and most widely used concepts are the Hukuhara differentiability and the Seikkala derivative. Based on these notions, extensive research has been devoted to the existence, uniqueness, and qualitative properties of solutions of FDEs. In particular, Kaleva [3] established fundamental results connecting fuzzy differential equations with differential inclusions and showed that, under suitable monotonicity assumptions, solutions obtained via differential inclusions coincide with fuzzy solutions of the corresponding FDEs.

Due to the inherent complexity of fuzzy-valued problems, obtaining exact analytical solutions of FDEs is often difficult or even impossible. This difficulty has motivated the development of numerical methods for fuzzy differential equations. A significant breakthrough in this direction was achieved by Bede [1], who introduced a characterization theorem stating that, under Hukuhara differentiability, a fuzzy differential equation can be equivalently transformed into a system of ordinary differential equations defined on the α-level sets. This result implies that classical numerical methods for ordinary differential equations can be directly applied to fuzzy differential equations without the need to reformulate numerical schemes in a fully fuzzy framework.

Following this idea, Pederson and Sambandham [2] investigated numerical solutions of hybrid fuzzy differential equations using characterization theorems and demonstrated the effectiveness of classical ODE solvers in the fuzzy setting. Their work further emphasized the importance of equivalence results between fuzzy differential equations and parametric systems of ordinary differential equations for both theoretical analysis and numerical computation.

In recent years, the scope of fuzzy differential equations has been extended to more complex models, including fuzzy stochastic differential systems and generalized differentiability concepts. For instance, Siahmansouri, Yousefi, and Gholamian [4] studied fuzzy stochastic differential systems and analyzed their mathematical structure and solution behavior. Moreover, Yousefi et al. [5] employed the homotopy perturbation method to approximate solutions of fuzzy initial value problems under generalized differentiability, highlighting the ongoing interest in developing efficient analytical and numerical techniques for fuzzy dynamical systems.

Despite these advances, the relationship between fuzzy differential inclusions (FDIs), fuzzy differential equations, and their corresponding systems of ordinary differential equations has not been fully explored from a unified numerical perspective. In particular, it is of interest to clarify under what conditions the numerical treatment of fuzzy differential inclusions can be reduced to the numerical solution of classical ODE systems via characterization theorems.

Motivated by these observations, the main objective of this paper is to investigate the equivalence between fuzzy differential inclusions and systems of ordinary differential equations by combining the characterization theorem of Bede with the differential inclusion framework developed by Kaleva. Under suitable regularity and monotonicity assumptions, we establish a theorem showing that the solution of a fuzzy differential inclusion can be obtained equivalently by solving a corresponding system of ordinary differential equations. This equivalence provides a theoretical justification for applying standard numerical methods, such as the Euler method, to approximate solutions of fuzzy differential inclusions. A numerical example is presented to illustrate the effectiveness of the proposed approach.

2 Preliminaries

Definition 1. *Let R_F denote the family of all non-empty, compact, convex subsets of R. Denote by E^1 the set of $\widetilde{u}: R \to [0,1]$ such that $\widetilde{u}$ satisfies $(i)-(iv)$ mentioned next:*

1. *$\widetilde{u}$ is normal that is, there exists an $y_0 \in R^n$ such that $\widetilde{u}(y_0) = 1$,*

2. *$\widetilde{u}$ is fuzzy convex,*

3. *$\widetilde{u}$ is upper semi continuous,*

4. *$[\widetilde{u}]^0 = \{\overline{y \in R : \widetilde{u}(y) > 0}\}$ is compact.*

We denote the α-level set $[\widetilde{u}]^\alpha = \{y \in R : \widetilde{u}(y) \geq \alpha\}$ for $0 < \alpha \leq 1$. Clearly the α-level sets $[\widetilde{u}]^\alpha \in R_F$.

Definition 2. *[3] Let I be a real interval. A mapping $\widetilde{y}: I \longrightarrow E^1$ is called a fuzzy process and its α-level set is denoted by*

$$[\widetilde{y}]^\alpha = [\underline{y}^\alpha, \overline{y}^\alpha] \quad t \in I, \ \ 0 < \alpha \leq 1.$$

Let $\widetilde{x}$, $\widetilde{y} \in E^1$. If there exists $\widetilde{z} \in E^1$ such that $\widetilde{x} = \widetilde{y} + \widetilde{z}$, then $\widetilde{z}$ is called the Hukuhara difference of $\widetilde{x}$ and $\widetilde{y}$ and it is denoted by $\widetilde{x} \ominus \widetilde{y}$. In this paper the "$\ominus$" sign stands always for Hukuhara difference and let us remark that $\widetilde{x} \ominus \widetilde{y} \neq \widetilde{x} + (-1)\widetilde{y}$.

Definition 3. *[3] The α-level set of a triangular fuzzy number as $\widetilde{T} = (x^l, x^c, x^r)$ in E^1 is given by*

$$[\widetilde{T}]^\alpha = [x^c - (1-\alpha)(x^c - x^l), \ x^c + (1-\alpha)(x^r - x^c)],$$

where $x^l \leq x^c \leq x^r$.
Let us recall the definition of Hukuhara differentiability.

Definition 4. *Let $d_H(A,B)$ be the Hausdorff distance between sets $A, B \in R_F$. The supremum metric d_∞ on E^1 is defined by*

$$d_\infty(\widetilde{U}, \widetilde{V}) = \sup\{d_H([\widetilde{U}]^\alpha, [\widetilde{V}]^\alpha) : \alpha \in [0,1]\},$$

and (E^1, d_∞) is a complete metric space, for more details see [2].

Definition 5. *[3] Let $\widetilde{f}: T \to E^1$ and $y_0 \in T \subseteq R$. We say that $\widetilde{f}$ is Hukuhara differentiable at y_0 if there exists an element $\widetilde{f}' \in E^1$ such that for all $h > 0$ sufficiently small, there are $\widetilde{f}(y_0 + h) \ominus \widetilde{f}(y_0)$, $\widetilde{f}(y_0) \ominus \widetilde{f}(y_0 - h))$ and the limits (in the metric d_∞)*

$$\lim_{h \to 0}\left(\frac{\widetilde{f}(y_0 + h) \ominus \widetilde{f}(y_0)}{h}\right) = \lim_{h \to 0}\left(\frac{\widetilde{f}(y_0) \ominus \widetilde{f}(y_0 - h)}{h}\right) = \widetilde{f}'(y_0). \tag{1}$$

The fuzzy set $\widetilde{f}'(y_0)$ is called the Hukuhara derivative of $\widetilde{f}$ at y_0.
Recall that $\widetilde{U} \ominus \widetilde{V} = \widetilde{W} \in E^1$ are defined on α-level sets, where $[\widetilde{U}]^\alpha \ominus [\widetilde{V}]^\alpha = [\widetilde{W}]^\alpha$ for all $\alpha \in [0,1]$. By consideration of definition of the metric d_∞ all the α-level set $[\widetilde{f}(0)]^\alpha$ are Hukuhara differentiable at y_0 with Hukuhara derivatives $[\widetilde{f}'(y_0)]^\alpha$ for each $\alpha \in [0,1]$, when $\widetilde{f}: T \to E^1$ is Hukuhara differentiable at y_0 with Hukuhara derivative $\widetilde{f}'(y_0)$.

Remark 1. [3] If $\widetilde{f}: T \longrightarrow E^1$ is Hukuhara differentiable and its Hukuhara derivative $\widetilde{f}'$ is integrable over $[0,1]$, then

$$\widetilde{f}(t) = \widetilde{f}(t_0) + \int_{t_0}^{t} \widetilde{f}'(s) ds,$$

for all $0 \leq t_0 \leq t \leq 1$.
The Seikkala derivative $y'(t)$ of a fuzzy process $\widetilde{y}$ is defined by

$$[\widetilde{y}'(t)]^\alpha = [(\underline{y}^\alpha)'(t), (\overline{y}^\alpha)'(t)], \quad 0 < \alpha \leq 1,$$

provided that its equation defines a fuzzy number $\widetilde{y}'(t) \in E^1$.

3 Characterization Theorems for the Solutions of FDEs by using ODEs

We review one of the main results presented in Bede [1].

Theorem 1. *[3] Let $F : (a,b) \to R_F$ be Hukuhara differentiable and denote $[F(t)]^\alpha = [\underline{F}^\alpha(t), \overline{F}^\alpha(t)]$. Then the boundary functions $\underline{F}^\alpha(t), \overline{F}^\alpha(t)$ are differentiable and we have*

$$[F'(t)]^\alpha = [(\underline{F}^\alpha(t))', (\overline{F}^\alpha(t))'], \quad \alpha \in [0,1].$$

Let us consider the fuzzy initial value problem (FIVP)

$$x' = f(t,x), \qquad x(t_0) = x_0, \tag{2}$$

where $f : [t_0, t_0 + a] \times R_F \to R_F$ and $x_0 \in R_F$. Then the above theorem shows us a way to translate the FIVP Eq. (2) into a system of ODE. Let $[x(t)]^\alpha = [\underline{x}^\alpha(t), \overline{x}^\alpha(t)]$. Now, suppose $x(t)$ is Hukuhara differentiable from Theorem 1, $[x'(t)]^\alpha = [(\underline{x}^\alpha(t))', (\overline{x}^\alpha(t))']$. Clearly, Eq. (2) translates into the following system of ODEs

$$\begin{aligned} (\underline{x}^\alpha(t))' &= \underline{f}^\alpha(t, \underline{x}^\alpha(t), \overline{x}^\alpha(t)), \\ (\overline{x}^\alpha(t))' &= \overline{f}^\alpha(t, \underline{x}^\alpha(t), \overline{x}^\alpha(t)), \\ \underline{x}^\alpha(0) &= \underline{x}_0^\alpha, \\ \overline{x}^\alpha(0) &= \overline{x}_0^\alpha, \end{aligned} \tag{3}$$

where

$$[f(t,x)]^\alpha = [\underline{f}^\alpha(t, \underline{x}^\alpha(t), \overline{x}^\alpha(t)), \overline{f}^\alpha(t, \underline{x}^\alpha(t), \overline{x}^\alpha(t))].$$

Then, in [3], Kaleva states that if we ensure that the solution $[\underline{x}^\alpha(t), \overline{x}^\alpha(t)]$ of the system (3) are valid level sets of a fuzzy number valued function and if the derivatives $[(\underline{x}^\alpha(t))', (\overline{x}^\alpha(t))']$ are valid level sets of a fuzzy-valued function, then by using the stacking Theorem we can construct the solution of the FIVP (2).

In the following corollary we show that the converse result holds as well and the FIVP Eq. (2) will be equivalent to the system Eq. (3).

4 Construction of Solutions Via Differential Inclusions

In this section, we review construction of solutions by differential inclusions from [3].

We consider the fuzzy differential equation Eq. (2), where $f : [t_0, t_0 + a] \times R_F \to R_F$ is obtained by Zadeh's extension principle from a continuous function $h : [t_0, t_0 + a] \times R \to R$.

Now, $f(t,x)$ can be computed levelwise, i.e., for each $\alpha \in [0,1]$

$$[f(t,x)]^\alpha = h(t, [x]^\alpha),$$

for all $t \in [0,a]$, $x \in R_F$. we interpret the fuzzy initial value problem Eq. (2) as a set of differential inclusions:

$$\begin{cases} y'^\alpha(t) = h(t, y^\alpha(t)), \quad 0 \le \alpha \le 1, \\ y^\alpha(0) \in [x_0]^\alpha, \end{cases} \tag{4}$$

Under suitable assumptions, the attainable sets

$$\pounds^\alpha(t) = \{y^\alpha(t) | \, y^\alpha \text{ is a solution of Eq. (4)}\},$$

are α-levels of a fuzzy set $x(t)$, which we call a solution of Eq. (2). If we assume the uniqueness of the solutions for initial value problems

$$\begin{cases} y'^\alpha(t) = h(t, y^\alpha(t)), \\ y^\alpha(0) = y_0, \end{cases} \tag{5}$$

it follows that $\pounds^\alpha(t) = [z^\alpha(t), r^\alpha(t)]$, where

$$\begin{cases} z'^\alpha(t) = h(t, z^\alpha(t)), \\ z^\alpha(0) = [\underline{z}_0]^\alpha \\ r'^\alpha(t) = h(t, r^\alpha(t)), \\ r^\alpha(0) = [\overline{r}_0]^\alpha. \end{cases}$$

The relation between the fuzzy solutions and solution via differential inclusion has been studied recently by Kaleva [3]. And also, supposing that h is nondecreasing, the following result has been obtained:

Theorem 2. *If h is nondecreasing with respect to the second argument then, using the form Eq. (1), the fuzzy solution of Eq. (2) and the solution via differential inclusions are identical.*

5 A Theorem for the Solution of FDIs by using ODEs

In the section, we give a Theorem by combination of Theorem 2 and Corollary 1.

Theorem 3. *Let us consider the FIVP Eq. (2) where* $f : [t_0, t_0 + a] \times R_F \to R_F$ *and a set of differential inclusions Eq. (4) where a continuous function* $h : [t_0, t_0 + a] \times R \to R$ *are such that*

1. $[f(t,x)]^\alpha = [\underline{f}^\alpha(t, \underline{x}^\alpha(t), \overline{x}^\alpha(t)), \overline{f}^\alpha(t, \underline{x}^\alpha(t), \overline{x}^\alpha(t))]$

2. *If h is nondecreasing with respect to the second argument and using the form Eq. (1).*

3. *There exist* $L > 0$ *such that*
$$|\underline{f}^\alpha(t, x_1, y_1) - \underline{f}^\alpha(t_1, x_1, y_1)| \leq L\, max\{|x_2 - x_1|, |y_2 - y_1|\},$$
and
$$|\underline{f}^\alpha(t, x_1, y_1) - \underline{f}^\alpha(t_1, x_1, y_1)| \leq L\, max\{|x_2 - x_1|, |y_2 - y_1|\},$$
for all $\alpha \in [0,1]$.

Then the FDIs (4) and the system of ODE (3) are equivalent.

Proof. According to condition (2), and by using Theorem 2, we conclude that the fuzzy solution of (2) and the solution via differential inclusions are identical. Therefore FDIs Eq. (4) and the fuzzy solution of (2) are identical. Clearly conditions (1) and (3) satisfy in the hypothesis Corollary (2) [2]. Then FIVP (2) and the system of ODEs (3) are equivalent. □

6 Numerical Example

In this section, we illustrate the applicability of the proposed theoretical framework and the equivalence between fuzzy differential inclusions and systems of ordinary differential equations through a numerical example. In particular, we demonstrate how classical numerical methods for ODEs can be employed to approximate solutions of fuzzy differential inclusions.

Example 1. Consider the following fuzzy differential equation IVP:

$$\begin{cases} x'(t) = x(t), \\ [x(0)]^\alpha \in [0.75 + 0.25\alpha, 1.125 - 0.125\alpha], \quad 0 \leq \alpha \leq 1, \end{cases} \tag{6}$$

The corresponding crisp ordinary differential equation is linear and admits a unique solution. According to the characterization theorem and the results established in previous sections, the fuzzy solution can be constructed levelwise. The exact fuzzy solution of problem (6) is given by

$$x^\alpha(t) = [(0.75 + 0.25\alpha)e^t, (1.125 - 0.125\alpha)e^t].$$

If we use the Euler method with $N = 10$ then approximate solution of this equation in $t = 1$ is as follows:

$$y^\alpha(1) = [(0.75 + 0.25\alpha)(1 + 0.1)^1 0, (1.125 - 0.125\alpha)(1 + 0.1)^1 0].$$

The results are summarized in Table 1.

Table 1. Exact and Euler approximate fuzzy solutions at $t = 1$

α	Exact solution $[x(1)]^\alpha$	Euler solution $y(1)]^\alpha$
0	2.0387	3.0581
0.5	2.3785	2.8882
1	2.7183	2.7183

As shown in Table 1, the Euler method provides a reasonable approximation of the exact fuzzy solution at $t = 1$. The numerical results preserve the fuzzy structure of the solution and maintain the ordering of the α-level sets. As expected, a discrepancy between the exact and approximate solutions is observed due to the first-order accuracy of the Euler method. Nevertheless, the approximation improves as the step size decreases, and higher-order numerical schemes could be employed to enhance accuracy.

Figure 1 illustrates the comparison between the exact fuzzy solution and the approximate fuzzy solution obtained by the Euler method. The solid curves represent the exact α-level bounds, while the dashed curves correspond to the numerical approximation.

This example confirms that, by exploiting the equivalence between fuzzy differential inclusions and systems of ordinary differential equations, classical numerical methods can be effectively applied to approximate fuzzy solutions without reformulating the problem in a fully fuzzy numerical framework.

6.1 Error Analysis

In this subsection, we analyze the numerical error associated with the Euler approximation of the fuzzy solution obtained in the previous section. Since the fuzzy solution is constructed levelwise, the error analysis is also performed on the α-level sets by comparing the exact and approximate solutions of the corresponding ordinary differential equations.

Let $x(t)$ denote the exact solution of the crisp ordinary differential equation (6), and let $y(t)$ be its numerical approximation obtained by the explicit Euler method with step size $h = 0.1$. The global discretization error at time $t = 1$ is defined as $E(t) = |x(t) - y(t)|$.

For the fuzzy case, the absolute error at each α-level is given by

$$E^{\alpha}(t) = \max\left\{\left|\underline{x}^{\alpha}(t) - \underline{y}^{\alpha}(t)\right|, |\overline{x}^{\alpha}(t) - \overline{y}^{\alpha}(t)|\right\}.$$

The absolute errors at $t = 1$ can be computed explicitly.

Table 2. Absolute error of Euler approximation at $t = 1$.

α	Lower bound error	Upper bound error
0	|2.0387-1.9453| = 0.0934	|3.0581-2.9185| = 0.1396
0.5	|2.3785-2.2695| = 0.1090	|2.8882-2.7563| = 0.1319
1	|2.7183-2.5937| = 0.1246	|2.7183-2.5937| = 0.1246

From Table 2, it is observed that the numerical error increases slightly as the width of the fuzzy initial condition increases. This behavior is consistent with the linear dependence of the solution on the initial data. Moreover, the magnitude of the error confirms the first-order accuracy of the Euler method, since the global error is of order $\mathscr{O}(h)$.

It is worth noting that the error remains uniformly bounded for all $\alpha \in [0, 1]$, and the monotonic structure of the α-level sets is preserved by the numerical scheme. This indicates that the Euler method maintains the essential fuzzy properties of the solution, despite its relatively low order of accuracy.

Finally, we emphasize that higher-order numerical schemes, such as RungeKutta methods, are expected to significantly reduce the approximation error while preserving the equivalence framework established in this paper. This observation provides a natural direction for future research. For more details, see Figure 1.

7 Conclusion

In this paper, we investigated the numerical solution of fuzzy differential inclusions by establishing a rigorous connection among fuzzy differential equations, fuzzy differential inclusions, and systems of ordinary differential equations. By combining the characterization theorem for fuzzy differential equations under Hukuhara differentiability with the differential inclusion framework developed by Kaleva, we showed that, under suitable regularity and monotonicity assumptions, fuzzy differential inclusions can be equivalently represented by parametric systems of ordinary differential equations defined on the α-level sets.

This equivalence result provides a solid theoretical foundation for reducing fuzzy-valued dynamical problems to classical real-valued systems. Consequently, well-established numerical methods for ordinary differential equations can be directly applied to approximate solutions of fuzzy differential inclusions without the need to reformulate numerical schemes in a fully fuzzy environment. The proposed

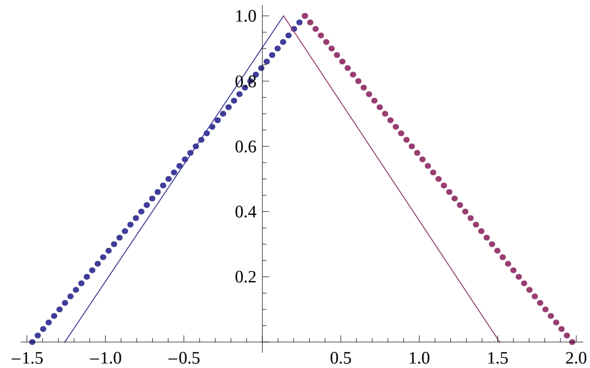

Figure 1. The exact solution FDI (solid graph) and the approximate solution FDI (dashed graph) by using Euler method

framework clarifies the mathematical relationship between fuzzy solutions obtained via differential inclusions and those derived from characterization theorems, thereby unifying two important approaches in the literature.

The numerical example presented in this study demonstrates the practical applicability of the theoretical results. By employing the explicit Euler method, we constructed approximate fuzzy solutions and compared them with the corresponding exact solutions. A detailed error analysis showed that the numerical approximation preserves the fuzzy structure of the solution and that the global error behaves consistently with the first-order convergence of the Euler scheme. These findings confirm that classical numerical methods can effectively capture the behavior of fuzzy differential inclusions when applied within the proposed equivalence framework.

Although the Euler method was used for simplicity, the developed approach is not restricted to first-order schemes. Higher-order numerical methods, such as Runge?Kutta methods, multistep schemes, or adaptive step-size algorithms, can be readily incorporated to improve accuracy. Furthermore, the framework can be extended to more general settings, including fuzzy differential equations under generalized differentiability, fuzzy stochastic differential systems, and higher-dimensional or nonlinear fuzzy models. These extensions constitute promising directions for future research.

Overall, the results of this paper contribute to the theoretical and numerical analysis of fuzzy differential inclusions by providing a clear justification for the use of classical numerical techniques and by strengthening the link among fuzzy differential equations, differential inclusions, and ordinary differential equations.

8 Future Work

Future research can explore several promising directions based on the theoretical and numerical framework developed in this study:

- **Extension to higher-order and generalized fuzzy systems:**
  The present analysis can be extended to higher-order fuzzy differential equations and systems under generalized Hukuhara differentiability. Such extensions would broaden the applicability of the characterization-based approach to more complex fuzzy dynamical models.

- **Fractional fuzzy differential equations:**
  An important direction for future work is the investigation of fuzzy fractional-order differential equations and inclusions. Fractional fuzzy models frequently arise in viscoelasticity, anomalous diffusion, control theory, and finance, where memory and hereditary effects play a crucial role.

- **Hybrid numerical approaches:**
  The equivalence between fuzzy differential inclusions and ordinary differential equations allows the incorporation of advanced numerical techniques. Hybrid approaches combining classical ODE solvers with integral transforms (such as Laplace, Sumudu, or Elzaki transforms) may improve convergence properties and computational efficiency for more complex fuzzy systems.

- **Multi-dimensional fuzzy partial differential equations:**

The proposed framework can be extended to fuzzy partial differential equations, particularly multidimensional problems arising in heat transfer, fluid dynamics, and wave propagation under uncertainty. Establishing characterization results for such systems remains an open and challenging problem.

- **Error analysis and stability:**
  Although a basic error analysis was presented for the Euler method, further research is needed to develop rigorous stability and convergence results for various numerical schemes applied to fuzzy differential inclusions. Such theoretical guarantees are essential for reliable numerical simulations in practical applications.

Overall, the present work provides a foundation for future methodological developments and highlights the potential of characterization-based approaches for the numerical analysis of fuzzy dynamical systems.

Authors' Contributions

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data Availability

All data in the paper are available from the corresponding authors upon reasonable request.

Conflicts of Interest

The authors declare that there is no conflict of interest.

Ethical Considerations

The authors have diligently addressed ethical concerns, such as informed consent, plagiarism, data fabrication, misconduct, falsification, double publication, redundancy, submission, and other related matters.

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References

[1] B. Bede, Note on Numerical solutions of fuzzy differential equations by predictor corrector method, Information Sciences, 178, 1917–1922, (2008).

[2] F. Pederson, M. Sambandham, Numerical solution of hybrid fuzzy differential equation IVPs by a characterization theorem, 1, (2008).

[3] O. Kaleva, A note on fuzzy differential equations, Nonlinear Analysis, 64, 895–900, (2006).

[4] S. Siahmansouri, E. Yousefi, M. Gholamian, Fuzzy stochastic differential system, Mathematical Analysis and Convex Optimization, (2025), e731888.

[5] S. Siahmansouri, E. Yousefi, M. Gholamian, S. Neyshabouri, Application of the Homotopy Perturbation Method for Approximating Solutions to Fuzzy Initial Value Problems under Generalized Differentiable, Analytical and Numerical Solutions for Nonlinear Equations, 9(2), 184–192, (2025).