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Research article

Solving Fractional Black-Scholes and Navier-Stokes Equations Via a New t^ρ/ρ-Integral Transform and Residual Power Series

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Abstract

This paper introduces a novel approach for solving two-dimensional time-fractional Navier-Stokes and Black-Scholes equations. The method integrates a new integral transform–based on a generalized power function of the form $\frac{t^\rho}{\rho}$– with the residual power series method. This combined approach, termed the "generalized integral transform residual power series method," utilizes the Katugampola fractional derivative in the Caputo sense. The convergence of the method is rigorously established, and its efficacy, accuracy, and precision are demonstrated through illustrative examples. The results highlight the method's potential for efficiently solving complex fractional partial differential equations across various scientific and engineering disciplines.

Keywords: New $\frac{t^\rho}{\rho}$-general transform, Black-Scholes equation, Navier-Stockes, Residual power series

Mathematics Subject Classification (2020): 35R11, 35Q30, 91G20, 34A08

1 Introduction

Fractional calculus (FC) has emerged as a robust mathematical framework for modeling and analyzing complex phenomena, particularly those described by fractional differential equations (FDEs) [1–3]. While classical integer-order models are widespread, they frequently fall short in representing the memory effects and hereditary properties inherent in real-world systems. In contrast, FC has shown significant applicability across diverse fields, including quantum physics, image recognition, circuit theory, and epidemiology. The foundational aspects of FC have been thoroughly documented [4–7], emphasizing its theoretical richness.

FDEs are increasingly recognized as fundamental tools for modeling irregular behaviors, such as nonlinear seismic vibrations and fractal patterns within financial markets. Furthermore, these equations have demonstrated effectiveness in computational anatomy, biochemical processes, and various natural systems [8–11]. Because nonlinear problems occur naturally in architecture, astronomy, and engineering, they have garnered considerable attention despite their inherent complexity [12–14].

The quest for higher accuracy and computational efficiency continues to drive advancements in analytical methods. Various techniques have been developed to solve nonlinear FDEs involving the Caputo derivative, such as the modified Homotopy Analysis Transform Method (MHATM) and the Variational Iteration Method (VIM) [15]. Additionally, the residual power series method (RPSM) and the Sardar sub-equation method have been successfully applied to shallow water and Schrödinger equations, respectively [16, 17].

These analytical techniques are frequently employed to solve the time-fractional Navier–Stokes and Black–Scholes equations [18]. The Navier–Stokes equations, independently formulated by Louis Marie Henri Navier and George Gabriel Stokes, form the bedrock of fluid mechanics. Despite their utility in aerodynamics, oceanography, and biomedical engineering [19, 20], establishing the existence of global solutions remains a formidable challenge in contemporary mathematics [21]. For an incompressible fluid, the classical Navier–Stokes equations are mathematically expressed as:

$$\frac{\partial \mathbf{U}}{\partial t} + (\mathbf{U}\cdot\nabla)\mathbf{U} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{U} + \mathbf{F}, \quad \nabla\cdot\mathbf{U} = 0.$$

In this formulation, $\mathbf{U}$ denotes the fluid velocity field, p represents the pressure distribution (Pa), t is time, and $\mathbf{F}$ corresponds to the external body force per unit volume. The parameters μ and ρ refer to the dynamic viscosity and density of the fluid, respectively. This governing equation is derived from Newtons second law applied to fluid motion, and its inherently nonlinear structure reflects the intrinsic complexities of fluid dynamics [16].

In the modern financial landscape, the classical Black–Scholes model remains a fundamental benchmark for pricing options; however, standard integer-order equations often fail to account for the 'memory effects' and long-range dependencies inherent in fractal market systems. Recent literature suggests that fractional differential equations (FDEs) offer a more robust framework for capturing these irregular dynamics [22–25]. While early studies utilized the Laplace homotopy perturbation method (LHPM) for integer-order assets [26], subsequent research has successfully extended these models using the Liouville–Caputo fractional derivative [27, 28]. Because financial markets exhibit self-similar properties, replacing the ordinary derivative with the Caputo-type Katugampola fractional derivative allows for a more precise representation of asset price evolution. This approach is particularly relevant for European options, where the payoff is determined by the relationship between the strike price (E) and the asset price at expiration (S). By integrating the Katugampola framework, this study provides a powerful analytical tool to navigate the complexities of fractal dynamics in both call and put option pricing.

The versatility of the Katugampola fractional derivative is rooted in its dual-parameter structure, characterized by the fractional order r γ and the generalization parameter ρ. In the context of the proposed model, α represents the anomalous diffusion or "memory effect" of the system; in financial markets, this captures the long-range dependence of asset prices, whereas in fluid dynamics, it characterizes the sub-diffusive nature of flow in complex media. Complementing this, the scaling parameter ρ adjusts the underlying temporal geometry of the governing equations. By varying ρ, the operator generalizes several established frameworks, effectively transitioning toward the Caputo derivative as $\rho \to 1$ and toward the Hadamard derivative as $\rho \to 0^+$. This dual-parameter flexibility allows the generalized integral transform residual power series method to more accurately map the fractal dynamics of the Black–Scholes and Navier–Stokes equations compared to traditional single-parameter fractional operators.

The residual power series (RPS) method offers a robust alternative for constructing series solutions without the need for linearization or discretization [29, 30]. This technique is particularly advantageous because it expresses solutions as Taylor series, provides flexibility in point evaluation, and maintains structural consistency when transitioning between orders. Consequently, the RPS technique minimizes computational overhead while maintaining high accuracy [31, 32]. Integral transforms, such as the Fourier, Laplace [33], and Shehu transforms [34], are essential for simplifying differential problems. In this work, we introduce a novel $\frac{t^\rho}{\rho}$-generalized integral transform that unifies the generalized Laplace family. By integrating this transform with the RPS method within the Katugampola fractional framework, we solve complex models including the Black–Scholes and Navier–Stokes equations. To validate the proposed method, the study presents comprehensive numerical simulations. Through 2D and 3D visualizations and detailed comparison tables generated in Mathematica, we demonstrate the method's high degree of precision, stability, and computational efficiency.

2 Elementary Concepts

In this section, we present several fundamental definitions and properties of fractional derivatives and integrals.

Definition 1. *[15] The Riemann–Liouville fractional integral of a function $f(t)$ of order $\gamma > 0$ is defined as:*

$$ {}^{RL}_{a}\mathbf{I}_t^{\gamma} f(t) = \frac{1}{\Gamma(\gamma)} \int_a^t (t-\varepsilon)^{\gamma-1} f(\varepsilon)\mathrm{d}\varepsilon. \tag{1}$$

Definition 2. *[15] The Riemann–Liouville fractional derivatives of a function $f(t)$ of order $\gamma > 0$ defined as:*

$$ {}^{RL}_{a}\mathbf{D}^{\gamma}_{t} f(t) = \frac{1}{\Gamma(1-\gamma)} \frac{\mathrm{d}}{\mathrm{d}t} \int_a^t (t-\varepsilon)^{-\gamma} f(\varepsilon) \mathrm{d}\varepsilon. \tag{2} $$

Definition 3. *[15] The Caputo fractional derivatives of a function $f(t)$of order $\gamma > 0$ defined as:*

$$ {}^{C}_{a}\mathbf{D}^{\gamma}_{t} f(t) = \frac{1}{\Gamma(1-\gamma)} \int_a^t (t-\varepsilon)^{-\gamma} \frac{\mathrm{d}f(\varepsilon)}{\mathrm{d}t} \mathrm{d}\varepsilon. \tag{3} $$

Definition 4. *[15] The Katugampola fractional integral in the Caputo sense of the function $f(t)$ of order $\gamma > 0$ defined as:*

$$ {}_{a}\mathbf{I}^{\gamma,\rho}_{t} f(t) = \frac{1}{\Gamma(\gamma)} \int_a^t \left(\frac{t^\rho - \varepsilon^\rho}{\rho} \right)^{\gamma-1} f(\varepsilon) \frac{\mathrm{d}\varepsilon}{\varepsilon^{1-\rho}}. \tag{4} $$

Definition 5. *[15] The Katugampola fractional derivative in the Caputo sense of the function $f(t)$ with order $\gamma > 0$ defined as:*

$$ {}^{KC}_{a}\mathbf{D}^{\gamma,\rho}_{t} f(t) = {}_{a}\mathbf{I}^{\gamma,\rho}_{t} \{\eta^{(n-\gamma)} f(t)\} = \frac{1}{\Gamma(n-\gamma)} \int_a^t \left(\frac{t^\rho - \varepsilon^\rho}{\rho} \right)^{n-\gamma-1} \eta^n f(\varepsilon) \frac{\mathrm{d}\varepsilon}{\varepsilon^{1-\rho}}, \tag{5} $$

where $\eta = t^{1-\rho} \frac{\mathrm{d}}{\mathrm{d}t}$.

Lemma 1. *[16] For $\rho > 0$, $0 < \gamma, \eta \leq 1$, we have*

1. ${}^{KC}\mathbf{D}^{\gamma,\rho}_{t} c = 0.$
2. ${}^{KC}\mathbf{D}^{\gamma,\rho}_{t} \left(\frac{t^\rho}{\rho}\right)^{\kappa} = \frac{\Gamma(1+\kappa)}{\Gamma(1+\kappa-\gamma)} \left(\frac{t^\rho}{\rho}\right)^{\kappa-\gamma}.$
3. ${}^{KC}\mathbf{D}^{\gamma,\rho}_{t} \left({}^{K}\mathbf{D}^{\eta,\rho}_{t} f(t)\right) = {}^{K}\mathbf{D}^{\gamma+\eta,\rho}_{t} f(t).$
4. ${}^{KC}\mathbf{I}^{\gamma,\rho}_{t} \left({}^{K}\mathbf{D}^{\eta,\rho}_{t} f(t)\right) = f(t) - f(0).$

In this section, we introduce a new $\frac{t^\rho}{\rho}$-general integral transform which is cover most or even all type of integral transforms in the family of generalized Laplace transform.

Definition 6. *The new $\frac{t^\rho}{\rho}$-general integral transform defined as:*

$$ \mathbb{T}_{\frac{t^\rho}{\rho}} \{f(t)\}(s) = \mathscr{F}(\nu(s)) := \mu(s) \int_0^\infty e^{-\nu(s)\frac{t^\rho}{\rho}} f(t) \frac{\mathrm{d}t}{t^{1-\rho}}, \tag{6} $$

where it is valid for s and w.

The inverse $\frac{t^\rho}{\rho}$-general integral transform defined by:

$$ \mathbb{T}^{-1}_{\frac{t^\rho}{\rho}} \{\mathscr{F}(\nu(s))\}(t) = f(t) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \frac{1}{\mu(s)} e^{\nu(s)\frac{t^\rho}{\rho}} \mathscr{F}(s) \mathrm{d}s. \tag{7} $$

Definition 7. *Function $f : [0,\infty) \to \mathbb{R}$ is a $\frac{t^\rho}{\rho}$-exponential order $e^{d\frac{t^\rho}{\rho}}$, if there exist non-negative constants M, d, T such that $|f(t)| \leq M e^{d\frac{t^\rho}{\rho}}$ for $t \geq T$.*

Theorem 1. *Let for all $t \geq 0$, the function $f(t)$ be piecewise continuous function that convince $|f(t)| \leq M e^{d\frac{t^\rho}{\rho}}$, then $\mathbb{T}_\rho\{f(t)\}(s)$ be present for all $\nu(s) > d$.*

Proof. Due to definition 6, we can write

$$ \begin{aligned} \left|\mathbb{T}_\rho\{f(t)\}(s)\right| &= \left|\mu(s) \int_0^\infty e^{-\nu(s)\frac{t^\rho}{\rho}} f(t) \frac{\mathrm{d}t}{t^{1-\rho}}\right| \\ &\leq \mu(s) \int_0^\infty e^{-\nu(s)\frac{t^\rho}{\rho}} |f(t)| \frac{\mathrm{d}t}{t^{1-\rho}} \\ &\leq M\mu(s) \int_0^\infty e^{-(\nu(s)-d)\frac{t^\rho}{\rho}} \frac{\mathrm{d}t}{t^{1-\rho}} \\ &\leq \frac{M\mu(s)}{\nu(s)-d}. \end{aligned} \tag{8} $$

□

In this subsection, we establish the relationship between the $\frac{t^\rho}{\rho}$-generalized integral transform and other existing integral transforms.

1. If $\mu(s) = 1$ and $\nu(s) = s$ then, the new generalized integral transform is generalized Laplace-transform [33,35].

$$\mathscr{L}_{\frac{t^\rho}{\rho}}\{f(t)\}(s) = \int_0^\infty e^{-s\frac{t^\rho}{\rho}} f(t)\frac{\mathrm{d}t}{t^{1-\rho}} = \mathscr{L}\{f(\rho t)^{\frac{1}{\rho}}\}(s): \tag{9}$$

2. If $\mu(s) = 1$ and $\nu(s) = \frac{s}{w}$ then, the new generalized integral transform is generalized Shehu-transform [16]:

$$\mathbb{S}_{\frac{t^\rho}{\rho}}\{f(t)\}(s,w) = \int_0^\infty e^{-\frac{s}{w}\frac{t^\rho}{\rho}} f(t)\frac{\mathrm{d}t}{t^{1-\rho}}. \tag{10}$$

3. If $\mu(s) = 1$ and $\nu(s) = s^{\frac{1}{\rho}}$ then, the new generalized integral transform is generalized η-Laplace transform:

$$\mathscr{L}_{\frac{t^\rho}{\rho}}\{f(t)\}(s,\rho) = \int_0^\infty e^{-s^{\frac{1}{\rho}}\frac{t^\rho}{\rho}} f(t)\frac{\mathrm{d}t}{t^{1-\rho}}. \tag{11}$$

4. If $\mu(s) = \frac{1}{s}$ and $\nu(s) = \frac{1}{s}$ then, the new generalized integral transform is generalized Sumudu transform:

$$\mathscr{S}_{\frac{t^\rho}{\rho}}\{f(t)\}(s) = \frac{1}{s}\int_0^\infty e^{-\frac{1}{s}\frac{t^\rho}{\rho}} f(t)\frac{\mathrm{d}t}{t^{1-\rho}}. \tag{12}$$

5. If $\mu(s) = \frac{1}{s}$ and $\nu(s) = s$ then, the new generalized integral transform is generalized Aboodh transform:

$$\mathbb{A}_{\frac{t^\rho}{\rho}}\{f(t)\}(s) = \frac{1}{s}\int_0^\infty e^{-s\frac{t^\rho}{\rho}} f(t)\frac{\mathrm{d}t}{t^{1-\rho}}. \tag{13}$$

6. If $\mu(s) = s$ and $\nu(s) = s^2$ then, the new generalized integral transform is generalized Pourreza transform:

$$\mathbb{P}_{\frac{t^\rho}{\rho}}\{f(t)\}(s) = s\int_0^\infty e^{-s^2\frac{t^\rho}{\rho}} f(t)\frac{\mathrm{d}t}{t^{1-\rho}}. \tag{14}$$

7. If $\mu(s) = s$ and $\nu(s) = \frac{1}{s}$ then, the new generalized integral transform is generalized Elzaki transform:

$$\mathbb{E}_{\frac{t^\rho}{\rho}}\{f(t)\}(s) = s\int_0^\infty e^{-\frac{1}{s}\frac{t^\rho}{\rho}} f(t)\frac{\mathrm{d}t}{t^{1-\rho}}. \tag{15}$$

8. If $\mu(s) = s^2$ and $\nu(s) = s$ then, the new generalized integral transform is generalized Mohand transform:

$$\mathbb{M}_{\frac{t^\rho}{\rho}}\{f(t)\}(s) = s^2\int_0^\infty e^{-s\frac{t^\rho}{\rho}} f(t)\frac{\mathrm{d}t}{t^{1-\rho}}. \tag{16}$$

9. If $\mu(s) = 1$ and $\nu(s) = \frac{1}{s}$ then, the new generalized integral transform is generalized Kamal transform:

$$\mathbb{K}_{\frac{t^\rho}{\rho}}\{f(t)\}(s) = \int_0^\infty e^{-\frac{1}{s}\frac{t^\rho}{\rho}} f(t)\frac{\mathrm{d}t}{t^{1-\rho}}. \tag{17}$$

10. If $\mu(s) = \frac{1}{s^2}$ and $\nu(s) = \frac{1}{s}$ then, the new generalized integral transform is generalized Sawi transform:

$$\mathbf{L}_{\frac{t^\rho}{\rho}}\{f(t)\}(s) = \frac{1}{s^2}\int_0^\infty e^{-\frac{1}{s}\frac{t^\rho}{\rho}} f(t)\frac{\mathrm{d}t}{t^{1-\rho}}. \tag{18}$$

In this section, we derive and establish several key properties of the proposed generalized integral transform. These results are fundamental to its mathematical foundation and demonstrate its efficacy in solving fractional differential equations.

Proposition 1. *(Linearity) If the $\frac{t^\rho}{\rho}$–generalized integral transform of functions $u(t)$ and $v(t)$ exists for $s > c_1$ and $s > c_2$ respectively, then the $\frac{t^\rho}{\rho}$–generalized integral transform $au(t)+bv(t)$, for any constants a and b exists and*

$$\mathbb{T}_{\frac{t^\rho}{\rho}}\{au(t)+bv(t)\}(s) = a\mathbb{T}_{\frac{t^\rho}{\rho}}\{u(t)\}(s) + b\mathbb{T}_{\frac{t^\rho}{\rho}}\{v(t)\}(s). \tag{19}$$

Proof. Linearity properties follows directly from Definition 6. □

Proposition 2. *(Shifting property of $\frac{t^\rho}{\rho}$–generalized integral transform) Suppose $\mathbb{T}_{\frac{t^\rho}{\rho}}\{f(t)\}(s)=\mathscr{F}(s)$ then, we have*

$$\mathbb{T}_{\frac{t^\rho}{\rho}}\left\{\exp\left(\alpha\frac{t^\rho}{\rho}\right)f(t)\right\}(s)=\mathscr{F}(\nu(s)-\alpha). \tag{20}$$

Proof. According to definition 6, we have

$$\mathbb{T}_{\frac{t^\rho}{\rho}}\left\{\exp\left(\alpha\frac{t^\rho}{\rho}\right)f(t)\right\}(s)=\mu(s)\int_0^\infty e^{-(\nu(s)-\alpha)\frac{t^\rho}{\rho}}f(t)\frac{dt}{t^{1-\rho}}=\mathscr{F}(\nu(s)-\alpha). \tag{21}$$

□

Lemma 2. *If $\mathbb{T}_{\frac{t^\rho}{\rho}}\{f(t)\}(s)=\mathscr{F}(\nu(s))$ then, we have the following result*

1. $\mathbb{T}_{\frac{t^\rho}{\rho}}\{1\}(s)=\frac{\mu(s)}{\nu(s)},\quad \nu(s)>0.$
2. $\mathbb{T}_{\frac{t^\rho}{\rho}}\{t^\rho\}(s)=\mu(s)\rho^{\frac{\rho}{\rho}}\frac{\Gamma(1+\frac{\rho}{\rho})}{\nu(s)^{1+\frac{\rho}{\rho}}},\quad \rho\in\mathbb{R},\nu(s)>0.$
3. $\mathbb{T}_{\frac{t^\rho}{\rho}}\left\{\left(\frac{t^\rho}{\rho}\right)^\alpha\right\}(s)=\mu(s)\frac{\Gamma(1+\alpha)}{\nu(s)^{1+\alpha}},\quad \rho,\alpha\in\mathbb{R},\nu(s)>0.$
4. $\mathbb{T}_{\frac{t^\rho}{\rho}}\left\{e^{\lambda\frac{t^\rho}{\rho}}\right\}(s)=\frac{\mu(s)}{\nu(s)-\lambda},\quad \nu(s)>\lambda.$

Definition 8. *Let $u(t)$ and $v(t)$ be two piecewise continuous functions, and let $\mathbb{T}_{\frac{t^\rho}{\rho}}\{f(t)\}(s)=\mathscr{F}(\nu(s))$. Then the $\frac{t^\rho}{\rho}$-convolution of $u(t)$ and $v(t)$ is defined as follows:*

$$(u*_{\frac{t^\rho}{\rho}}v)(t)=\int_0^t u\left((t^\rho-\tau^\rho)^{\frac{1}{\rho}}\right)v(\tau)\frac{d\tau}{\tau^{1-\rho}}. \tag{22}$$

Lemma 3. *Let $u(t)$ and $v(t)$ be two piecewise continuous functions. If $\mathbb{T}_{\frac{t^\rho}{\rho}}\{f(t)\}(s)=\mathscr{F}(\nu(s))$, then we obtain*

$$(u*_{\frac{t^\rho}{\rho}}v)(t)=(v*_{\frac{t^\rho}{\rho}}u)(t). \tag{23}$$

Proof. By definition of $\frac{t^\rho}{\rho}$–generalized integral transform and substituting $t^\rho-\tau^\rho=\varepsilon^\rho$, we have

$$\begin{aligned}(u*_{\frac{t^\rho}{\rho}}v)(t)&=\int_0^t u\left((t^\rho-\tau^\rho)^{\frac{1}{\rho}}\right)v(\tau)\frac{d\tau}{\tau^{1-\rho}}\\&=-\int_t^0 u(\varepsilon)v\left((t^\rho-\varepsilon^\rho)^{\frac{1}{\rho}}\right)\frac{d\varepsilon}{\varepsilon^{1-\rho}}\\&=\int_0^t u(\varepsilon)v\left((t^\rho-\varepsilon^\rho)^{\frac{1}{\rho}}\right)\frac{d\varepsilon}{\varepsilon^{1-\rho}}\\&=(v*_{\frac{t^\rho}{\rho}}u)(t).\end{aligned} \tag{24}$$

□

Theorem 2. *If $\mathbb{T}_{\frac{t^\rho}{\rho}}\{u(t)\}(s)=\mathscr{U}(\nu(s))$ and $\mathbb{T}_{\frac{t^\rho}{\rho}}\{v(t)\}(s)=\mathscr{V}(\nu(s))$ then, we have*

$$\mathbb{T}_{\frac{t^\rho}{\rho}}\left\{(u*_{\frac{t^\rho}{\rho}}v)(t)\right\}=\frac{1}{\mu(s)}\mathbb{T}_{\frac{t^\rho}{\rho}}\{u(t)\}\,\mathbb{T}_{\frac{t^\rho}{\rho}}\{v(t)\}. \tag{25}$$

Proof. Based on Definition 6, and by reversing the order of integration, we obtain

$$\begin{aligned}\mathbb{T}_{\frac{t^\rho}{\rho}}\left\{(u*_{\frac{t^\rho}{\rho}}v)(t)\right\}&=\mu(s)\int_0^\infty e^{-\nu(s)\frac{t^\rho}{\rho}}\left(\int_0^t u\left((t^\rho-\tau^\rho)^{\frac{1}{\rho}}\right)v(\tau)\frac{d\tau}{\tau^{1-\rho}}\right)\frac{dt}{t^{1-\rho}}\\&=\mu(s)\int_0^\infty\int_\tau^\infty e^{-\nu(s)\frac{t^\rho}{\rho}}u\left((t^\rho-\tau^\rho)^{\frac{1}{\rho}}\right)v(\tau)\frac{dt}{t^{1-\rho}}\frac{d\tau}{\tau^{1-\rho}}.\end{aligned} \tag{26}$$

By substituting $t^\rho - \tau^\rho = \varepsilon^\rho$, we have

$$\begin{aligned}\mathbb{T}_{\frac{t^\rho}{\rho}}\left\{(u *_{\frac{t^\rho}{\rho}} v)(t)\right\} &= \mu(s)\int_0^\infty\int_0^\infty e^{-\nu(s)\frac{\tau^\rho+\varepsilon^\rho}{\rho}}u(\varepsilon)\,v(\tau)\frac{\mathrm{d}\varepsilon}{\varepsilon^{1-\rho}}\frac{\mathrm{d}\tau}{\tau^{1-\rho}} \\ &= \left(\mu(s)\int_0^\infty e^{-\nu(s)\frac{\tau^\rho}{\rho}}v(\tau)\frac{\mathrm{d}\tau}{\tau^{1-\rho}}\right)\left(\int_0^\infty e^{-\nu(s)\frac{\varepsilon^\rho}{\rho}}u(\varepsilon)\frac{\mathrm{d}\varepsilon}{\tau^{1-\varepsilon}}\right) \\ &= \frac{1}{\mu(s)}\mathbb{T}_{\frac{t^\rho}{\rho}}\{u(t)\}\mathbb{T}_{\frac{t^\rho}{\rho}}\{v(t)\}.\end{aligned} \tag{27}$$

□

Theorem 3. *(Derivative properties) Assume $\mathbb{T}_{\frac{t^\rho}{\rho}}\{f(t)\}(s) = \mathscr{F}(\nu(s))$ be a $\frac{t^\rho}{\rho}$–generalized integral transform, then we have the following result*

1. $\mathbb{T}_{\frac{t^\rho}{\rho}}\{\eta f(t)\}(s) = \nu(s)\mathbb{T}_{\frac{t^\rho}{\rho}}\{f(t)\}(s) - \mu(s)f(0)$.
2. $\mathbb{T}_{\frac{t^\rho}{\rho}}\{\eta^n f(t)\}(s) = \nu^n(s)\mathbb{T}_{\frac{t^\rho}{\rho}}\{f(t)\}(s) - \mu(s)\sum_{k=0}^{n-1}\nu(s)^{n-k-1}\eta^k f(0)$.

Proof. By applying equation (6) and integration by parts, we obtain

$$\mathbb{T}_{\frac{t^\rho}{\rho}}\{\eta f(t)\}(s) = \mu(s)\int_0^\infty e^{-\nu(s)\frac{t^\rho}{\rho}}f'(t)\mathrm{d}t = \nu(s)\mathbb{T}_{\frac{t^\rho}{\rho}}\{f(t)\}(s) - \mu(s)f(0). \tag{28}$$

To prove part (2) of Theorem 3, we proceed by induction. Assuming that equation (28) holds for $n = k$, then for $n = k+1$, we obtain

$$\begin{aligned}\mathbb{T}_{\frac{t^\rho}{\rho}}\left\{\eta^{(k+1)}f(t)\right\}(s) &= \mathbb{T}_{\frac{t^\rho}{\rho}}\left\{\left(\eta(\eta^k f(t))\right)\right\}(s) = \nu(s)\mathbb{T}_{\frac{t^\rho}{\rho}}\left\{\eta^k f(t)\right\}(s) - \mu(s)\eta^k f(0) \\ &= \nu(s)\left[\nu(s)^k\mathbb{T}_{\frac{t^\rho}{\rho}}\{f(t)\} - \mu(s)\sum_{j=0}^{k-1}\nu(s)^{k-j-1}\eta^j f(0)\right] - \mu(s)\eta^k f(0) \\ &= \nu(s)^{k+1}\mathbb{T}_{\frac{t^\rho}{\rho}}\{f(t)\}(s) - \mu(s)\sum_{j=0}^{k}\nu(s)^{k-j}\eta^j f(0).\end{aligned}$$

This implies that the second case of Theorem 3 holds for $n = k+1$. □

Theorem 4. *Suppose $\mathbb{T}_{\frac{t^\rho}{\rho}}\{f(t)\}(s) = \mathscr{F}(s)$. then, we have the new $\frac{t^\rho}{\rho}$–generalized integral transform of Katugapola fractional integral of Caputo sense as follows*

$$\mathbb{T}_{\frac{t^\rho}{\rho}}\left\{{}^{KC}\mathbf{I}_t^{\gamma,\rho}f(t)\right\}(s) = \frac{1}{\mu(s)\nu(s)^\gamma}\mathbb{T}_{\frac{t^\rho}{\rho}}\{f(t)\}(s). \tag{29}$$

Proof. According to (6), we have

$$\begin{aligned}\mathbb{T}_{\frac{t^\rho}{\rho}}\left\{{}^{KC}\mathbf{I}_t^{\gamma,\rho}f(t)\right\}(s) &= \mu(s)\int_0^\infty e^{-\nu(s)\frac{t^\rho}{\rho}}\left[\frac{1}{\Gamma(\gamma)}\int_0^t\left(\frac{t^\rho-\varepsilon^\rho}{\rho}\right)^{\gamma-1}f(\varepsilon)\frac{\mathrm{d}\varepsilon}{\varepsilon^{1-\rho}}\right]\frac{\mathrm{d}t}{t^{1-\rho}} \\ &= \mu(s)\int_0^\infty e^{-\nu(s)\frac{t^\rho}{\rho}}\left[\frac{\rho^{1-\gamma}}{\Gamma(\gamma)}\int_0^t\left((t^\rho-\varepsilon^\rho)^{\frac{1}{\rho}}\right)^{\rho(\gamma-1)}f(\varepsilon)\frac{\mathrm{d}\varepsilon}{\varepsilon^{1-\rho}}\right]\frac{\mathrm{d}t}{t^{1-\rho}} \\ &= \mu(s)\frac{\rho^{1-\gamma}}{\Gamma(\gamma)}\int_0^\infty e^{-\nu(s)\frac{t^\rho}{\rho}}\left[t^{\rho(\gamma-1)} *_{\frac{t^\rho}{\rho}} f(t)\right]\frac{\mathrm{d}t}{t^{1-\rho}} \\ &= \frac{\rho^{1-\alpha}}{\Gamma(\gamma)}\mathbb{T}_{\frac{t^\rho}{\rho}}\left\{t^{\rho(\gamma-1)} *_{\frac{t^\rho}{\rho}} f(t)\right\} \\ &= \frac{\rho^{1-\alpha}}{\Gamma(\gamma)\mu(s)}\mathbb{T}_{\frac{t^\rho}{\rho}}\left\{t^{\rho(\gamma-1)}\right\}\mathbb{T}_{\frac{t^\rho}{\rho}}\{f(t)\} \\ &= \frac{\rho^{1-\gamma}}{\Gamma(\gamma)\mu(s)}\mathbb{T}_{\frac{t^\rho}{\rho}}\left\{t^{\rho(\gamma-1)}\right\}\mathbb{T}_{\frac{t^\rho}{\rho}}\{f(t)\} \\ &= \frac{1}{\mu(s)\nu(s)^\gamma}\mathbb{T}_{\frac{t^\rho}{\rho}}\{f(t)\}.\end{aligned} \tag{30}$$

□

Theorem 5. *Suppose $\mathbb{T}_{\frac{t^\rho}{\rho}}\{f(t)\}(s)=\mathscr{F}(s)$ then, we have the new $\frac{t^\rho}{\rho}$–generalized integral transform of Katugapola fractional derivatives of Caputo sense as follows*

$$\mathbb{T}_{\frac{t^\rho}{\rho}}\left\{{}^{KC}\mathbf{D}_t^{\gamma,\rho}f(t)\right\}(s)=\frac{\nu^\gamma(s)}{\mu(s)}\mathbb{T}_{\frac{t^\rho}{\rho}}\{f(t)\}(s)-\sum_{k=0}^{n-1}\nu(s)^{\gamma-k-1}\eta^k f(0). \tag{31}$$

Proof. According to definition (6), Theorem 3 and Theorem 4, we have

$$\begin{aligned}\mathbb{T}_{\frac{t^\rho}{\rho}}\left\{{}^{KC}\mathbf{D}_t^{\gamma,\rho}f(t)\right\}(s)&=\mathbb{T}_{\frac{t^\rho}{\rho}}\left\{\mathbf{I}_t^{n-\gamma,\rho}\eta^n f(t)\right\}(s)=\frac{1}{\mu(s)\nu(s)^{n-\gamma}}\mathbb{T}_{\frac{t^\rho}{\rho}}\{\eta^n f(t)\}\\&=\frac{1}{\mu(s)\nu(s)^{n-\gamma}}\left[\nu^n(s)\mathbb{T}_{\frac{t^\rho}{\rho}}\{f(t)\}(s)-\mu(s)\sum_{k=0}^{n-1}\nu(s)^{n-k-1}\eta^k f(0)\right]\\&=\frac{\nu(s)^\gamma}{\mu(s)}\mathbb{T}_{\frac{t^\rho}{\rho}}\{f(t)\}(s)-\sum_{k=0}^{n-1}\nu(s)^{\gamma-k-1}\eta^k f(0).\end{aligned} \tag{32}$$

□

Definition 9. *[15] The Mittag-Leffler functions are defined as follows:*

1. $E_\alpha(z)=\sum_{i=0}^\infty\frac{z^k}{\Gamma(k\alpha+1)},\quad z\in\mathbb{R},Re(\alpha)>0$
2. $E_{\alpha,\beta}(z)=\sum_{i=0}^\infty\frac{z^k}{\Gamma(k\alpha+\beta)},\quad z\in\mathbb{R},Re(\alpha)>0$

Lemma 4. *Suppose $Re(\alpha)>0$ and $\left|\frac{\lambda}{\nu^\alpha(s)}\right|<1$, then, we have*

1. $\mathbb{T}_{\frac{t^\rho}{\rho}}\left\{E_\alpha\left(\lambda\left(\frac{t^\rho}{\rho}\right)^\alpha\right)\right\}(s)=\frac{\mu(s)\nu(s)^{\alpha-1}}{\nu(s)^{\alpha-1}-\lambda}$.
2. $\mathbb{T}_{\frac{t^\rho}{\rho}}\left\{\left(\frac{t^\rho}{\rho}\right)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda\left(\frac{t^\rho}{\rho}\right)^\alpha\right)\right\}(s)=\frac{\mu(s)}{\nu(s)^\alpha-\lambda}$.

Proof. According to Definition (6) and Lemma 2, we have

$$\begin{aligned}\mathbb{T}_{\frac{t^\rho}{\rho}}\left\{E_\alpha\left(\lambda\left(\frac{t^\rho}{\rho}\right)^\alpha\right)\right\}(s)&=\sum_{i=0}^\infty\frac{\lambda^i}{\Gamma(i\alpha+1)\rho^{i\alpha}}\mathbb{T}_{\frac{t^\rho}{\rho}}\left\{t^{i\rho\alpha}\right\}\\&=\sum_{i=0}^\infty\frac{\lambda^i}{\Gamma(i\alpha+1)\rho^{i\alpha}}\mu(s)\rho^{i\alpha}\frac{\Gamma(1+i\alpha)}{\nu(s)^{1+i\alpha}}\\&=\frac{\mu(s)}{\nu(s)}\sum_{i=0}^\infty\left(\frac{\lambda}{\nu(s)^\alpha}\right)^i=\frac{\mu(s)\nu(s)^{\alpha-1}}{\nu(s)^\alpha-\lambda}.\end{aligned} \tag{33}$$

The proof of the second part proceeds as follows:

$$\begin{aligned}\mathbb{T}_{\frac{t^\rho}{\rho}}\left\{\left(\frac{t^\rho}{\rho}\right)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda\left(\frac{t^\rho}{\rho}\right)^\alpha\right)\right\}(s)&=\sum_{i=0}^\infty\frac{\lambda^i}{\Gamma(i\alpha+\alpha)\rho^{i\alpha+\alpha-1}}\mathbb{T}_{\frac{t^\rho}{\rho}}\left\{t^{t^{i\rho\alpha+\rho\alpha-\rho}}\right\}\\&=\sum_{i=0}^\infty\frac{\lambda^i}{\Gamma(i\alpha+\alpha)\rho^{i\alpha+\alpha-1}}\mu(s)\rho^{i\alpha+\alpha-1}\frac{\Gamma(\alpha+i\alpha)}{\nu(s)^{\alpha+i\alpha}}\\&=\frac{\mu(s)}{\nu(s)^\alpha}\sum_{i=0}^\infty\left(\frac{\lambda}{\nu(s)^\alpha}\right)^i=\frac{\mu(s)}{\nu(s)^\alpha-\lambda}.\end{aligned} \tag{34}$$

□

3 Fractional Black-Scholes and Navier-Stokes Differential Equations

In this section, we introduce time-fractional Black-Scholes differential equations,

$${}^{KC}\mathbf{D}_t^{\gamma,\rho}\Phi(\varphi,\varsigma,t)+\frac{1}{2}\delta_1^2\varphi^2\frac{\partial^2\Phi}{\partial\varphi^2}+\frac{1}{2}\delta_2^2\varsigma^2\frac{\partial^2\Phi}{\partial\varsigma^2}+\varpi\delta_1\delta_2\varphi\varsigma\frac{\partial^2\Phi}{\partial\varphi\partial\varsigma}+r\left[\varphi\frac{\partial\Phi}{\partial\varphi}+\varsigma\frac{\partial\Phi}{\partial\varsigma}\right]-r\Phi=0, \tag{35}$$

with the initial conditions

$$\Phi(\varphi,\varsigma,0)=\max(\beta_1\varphi+\beta_2\varsigma-K,0), \tag{36}$$

and time-fractional Navier-Stokes differential system

$$\begin{aligned} {}^{KC}\mathbf{D}_t^{\gamma,\rho}\Phi(\varphi,\varsigma,t)+\Phi\frac{\partial\Phi}{\partial\varphi}+\Psi\frac{\partial\Phi}{\partial\varsigma}&=\rho\left(\frac{\partial^2\Phi}{\partial\varphi^2}+\frac{\partial^2\Phi}{\partial\varsigma^2}\right)+\hbar, \quad \varphi,\varsigma,t>0,\\ {}^{KC}\mathbf{D}_t^{\gamma,\rho}\Psi(\varphi,\varsigma,t)+\Phi\frac{\partial\Psi}{\partial\varphi}+\Psi\frac{\partial\Psi}{\partial\varsigma}&=\rho\left(\frac{\partial^2\Psi}{\partial\varphi^2}+\frac{\partial^2\Psi}{\partial\varsigma^2}\right)-\hbar, \quad \varphi,\varsigma,t>0, \end{aligned} \tag{37}$$

with subject to the initial conditions

$$\begin{aligned} \Phi(\varphi,\varsigma,0)&=-\sin(\varphi+\varsigma),\\ \Psi(\varphi,\varsigma,0)&=\sin(\varphi+\varsigma). \end{aligned} \tag{38}$$

Next, we introduce some definitions of the fractional order power series with the parameter ρ, which is initially given by [16, 36, 37].

Definition 10. *[16] A generalized fractional power series with respect to the variable t and parameterized by ρ is defined as follows:*

$$\Psi(\varphi,\varsigma,t)=\sum_{k=0}^{\infty}\psi_k(\varphi,\varsigma)\left(\frac{t^\rho}{\rho}\right)^{k\gamma}, \tag{39}$$

where $t\geq 0$, α is the fractional order, and the coefficients $\psi_k(\varphi,\varsigma)$ are functions of φ and ς. These coefficients determine the contribution of each term in the infinite series, with the index i ranging over all non-negative integers. The series is constructed around $t=0$.

Theorem 6. *[16] (Taylor Expansion of Fractional Order α) Assume that $\psi(\varphi,\varsigma,t)$ admits a generalized fractional Taylor series expansion in terms of the parameter ρ, given by:*

$$\Psi(\varphi,\varsigma,t)=\sum_{k=0}^{\infty}\psi_k(\varphi,\varsigma)\left(\frac{t^\rho}{\rho}\right)^{k\gamma}. \tag{40}$$

If $\psi(\varphi,\varsigma,t)$ and ${}^{KC}\mathbf{D}_t^{\alpha,\rho}\psi(\varphi,\varsigma,t)$ are continuous functions on the interval $[0,M]$, and ${}^{KC}\mathbf{D}_t^{\alpha,\rho}\psi(\varphi,\varsigma,t)$ is n-times differentiable on $(0,M)$, then the coefficients $\psi_i(\varphi,\varsigma)$ of the fractional-order power series in terms of the parameter ρ are given by:

$$\psi_k(\varphi,\varsigma,0)=\frac{{}^{KC}\mathbf{D}_t^{k\gamma,\rho}\psi(\varphi,\varsigma,0)}{\Gamma(1+k\gamma)}. \tag{41}$$

Theorem 7. *Suppose that $\psi(\varphi,\varsigma,t)$ is a power series of fractional-order with a parameter ρ around $t=0$, and ${}^{KC}\mathbf{D}_t^{\gamma,\rho}\psi(\varphi,\varsigma,0)$ exists on $[0,a]\times[0,b]\times[0,M]$. Then, $\Psi(\varphi,\varsigma,\mu(s),\nu(s))=\mathbb{T}_{\frac{t^\rho}{\rho}}\{\psi(\varphi,\varsigma,t)\}(s)$ can be represented by the following form*

$$\Psi(\varphi,\varsigma,\mu(s),\nu(s))=\sum_{k=0}^{\infty}\phi_k(\varphi,\varsigma,0)\frac{\mu(s)}{\nu(s)^{1+k\gamma}}, \tag{42}$$

where, $\phi_k(\varphi,\varsigma,0)={}^{KC}\mathbf{D}_t^{k\gamma,\rho}\psi(\varphi,\varsigma,0)$.

Proof. To prove this theorem, we apply Theorem 6 along with the properties of the proposed $\frac{t^\rho}{\rho}$-generalized integral transform. □

Theorem 8. *Assume that $\Phi(\varphi,\varsigma,t)$ and its second partial derivatives be continues on $[0,a]\times[0,b]\times[0,M]$, also*

$$G(\Phi(\varphi,\varsigma,t))=r\Phi-\frac{1}{2}\delta_1^2\varphi^2\frac{\partial^2\Phi}{\partial\varphi^2}-\frac{1}{2}\delta_2^2\varsigma^2\frac{\partial^2\Phi}{\partial\varsigma^2}-\varpi\delta_1\delta_2\varphi\varsigma\frac{\partial^2\Phi}{\partial\varphi\partial\varsigma}-r\left[\varphi\frac{\partial\Phi}{\partial\varphi}+\varsigma\frac{\partial\Phi}{\partial\varsigma}\right],$$

satisfies the Lipschitz condition with constant M. If $\frac{N}{\Gamma(1+\gamma)}\left(\frac{M^\rho}{\rho}\right)^\gamma<1$, then the fractional partial differential equation (35) has a unique solution of $\Phi(p_1,p_1,t)$ on $[0,a]\times[0,b]\times[0,M]$.

Proof. By applying the fractional integral ${}^{KC}\mathbf{I}_t^{\gamma,\rho}$ to both sides of equation (35), we obtain:

$$\begin{aligned}\Phi(\varphi,\varsigma,t)-\Phi(\varphi,\varsigma,0) = &\frac{1}{\Gamma(\gamma)}\int_0^t \left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{\gamma-1} \\ &\times\left[r\Phi-\frac{1}{2}\delta_1^2\varphi^2\frac{\partial^2\Phi}{\partial\varphi^2}-\frac{1}{2}\delta_2^2\varsigma^2\frac{\partial^2\Phi}{\partial\varsigma^2}-\varpi\delta_1\delta_2\varphi\varsigma\frac{\partial^2\Phi}{\partial\varphi\partial\varsigma}-r\left[\varphi\frac{\partial\Phi}{\partial\varphi}+\varsigma\frac{\partial\Phi}{\partial\varsigma}\right]\right]\frac{\mathrm{d}\tau}{\tau^{1-\rho}}.\end{aligned} \tag{43}$$

Now, we define the new operator $P:\mathbf{C}([0,a]\times[0,b]\times[0,M])\to\mathbf{C}([0,a]\times[0,b]\times[0,M])$ as follow

$$\begin{aligned}P(\Phi(\varphi,\varsigma,t)) =&g(\varphi,\varsigma)+\frac{1}{\Gamma(\gamma)}\int_0^t \left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{\gamma-1} \\ &\times\left[r\Phi-\frac{1}{2}\delta_1^2\varphi^2\frac{\partial^2\Phi}{\partial\varphi^2}-\frac{1}{2}\delta_2^2\varsigma^2\frac{\partial^2\Phi}{\partial\varsigma^2}-\varpi\delta_1\delta_2\varphi\varsigma\frac{\partial^2\Phi}{\partial\varphi\partial\varsigma}-r\left[\varphi\frac{\partial\Phi}{\partial\varphi}+\varsigma\frac{\partial\Phi}{\partial\varsigma}\right]\right]\frac{\mathrm{d}\tau}{\tau^{1-\rho}}.\end{aligned} \tag{44}$$

For any $\Phi,\Psi\in\mathbf{C}([0,a]\times[0,b]\times[0,M]$, we have

$$\begin{aligned}|P(\Phi(\varphi,\varsigma,t))-P(\Psi(\varphi,\varsigma,t))| = &\frac{1}{\Gamma(\gamma)}\int_0^t \left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{\gamma-1} \\ \times&\left[r\Phi-\frac{1}{2}\delta_1^2\varphi^2\frac{\partial^2\Phi}{\partial\varphi^2}-\frac{1}{2}\delta_2^2\varsigma^2\frac{\partial^2\Phi}{\partial\varsigma^2}-\varpi\delta_1\delta_2\varphi\varsigma\frac{\partial^2\Phi}{\partial\varphi\partial\varsigma}-r\left[\varphi\frac{\partial\Phi}{\partial\varphi}+\varsigma\frac{\partial\Phi}{\partial\varsigma}\right]\right]\frac{\mathrm{d}\tau}{\tau^{1-\rho}} \\ -&\frac{1}{\Gamma(\gamma)}\int_0^t \left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{\gamma-1} \\ \times&\left[r\Psi-\frac{1}{2}\delta_1^2\varphi^2\frac{\partial^2\Psi}{\partial\varphi^2}-\frac{1}{2}\delta_2^2\varsigma^2\frac{\partial^2\Psi}{\partial\varsigma^2}-\varpi\delta_1\delta_2\varphi\varsigma\frac{\partial^2\Psi}{\partial\varphi\partial\varsigma}-r\left[\varphi\frac{\partial\Psi}{\partial\varphi}+\varsigma\frac{\partial\Psi}{\partial\varsigma}\right]\right]\frac{\mathrm{d}\tau}{\tau^{1-\rho}} \\ =&\left|\frac{1}{\Gamma(\gamma)}\int_0^t \left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{\gamma-1}[G(\Phi(\varphi,\varsigma,t)-G(\Psi(\varphi,\varsigma,t)]\frac{\mathrm{d}\tau}{\tau^{1-\rho}}\right|.\end{aligned} \tag{45}$$

By applying the Lipschitz condition with the constant N, we obtain

$$\begin{aligned}|P(\Phi(\varphi,\varsigma,t))-P(\Psi(\varphi,\varsigma,t))| &\le \left|\frac{1}{\Gamma(\gamma)}\int_0^t \left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{\gamma-1} N[\Phi(\varphi,\varsigma,t)-\Psi(\varphi,\varsigma,t)]\frac{\mathrm{d}\tau}{\tau^{1-\rho}}\right| \\ &\le ||\Phi-\Psi||\frac{N}{\Gamma(1+\gamma)}\left(\frac{t^\rho}{\rho}\right)^{\gamma} \\ &\le \frac{N}{\Gamma(1+\gamma)}\left(\frac{M^\rho}{\rho}\right)^{\gamma}||\Phi-\Psi|| < ||\Phi-\Psi||.\end{aligned} \tag{46}$$

□

The proof of the uniqueness of the solution for equation (37) follows a similar argument. In this section, we propose a new technique for solving linear and nonlinear fractional differential equations by employing the fractional Katugampola operator in the Caputo sense. Specifically, we consider the following general fractional differential equation:

$${}^{KC}\mathbf{D}_t^{\gamma,\rho}f(\varphi,\varsigma,t)+R[f(\varphi,\varsigma,t)]+N[f(\varphi,\varsigma,t)]=g(\varphi,\varsigma,t), \tag{47}$$

with subject to the initial condition

$$f(\varphi,\varsigma,0)=h(\varphi,\varsigma), \tag{48}$$

where ${}^{KC}\mathbf{D}_t^{\gamma,\rho}$ denotes the Katugampola operator in the Caputo sense of order γ, $R[f(\varphi,\varsigma,t)$ denotes the linear part of equation (47), $N[f(\varphi,\varsigma,t)$ denotes the nonlinear part of equation (47) and $g(\varphi,\varsigma,t)$ is a specified function.

In the proposed generalized $\frac{t^\rho}{\rho}$-transform residual power series method, we assume that the solution to the fractional differential equation (47) takes the following form:

$$f(\varphi,\varsigma,t)=\sum_{k=0}^{\infty}\frac{\phi_k(\varphi,\varsigma)}{\Gamma(1+k\gamma)}\left(\frac{t^\rho}{\rho}\right)^{k\gamma}, \tag{49}$$

where the functions $\phi_k(\varphi,\varsigma)$ obtained using the following steps. Clearly $f(\varphi,\varsigma,0)=\phi_0(\varphi,\varsigma)$.

1. By applying the $\frac{t^\rho}{\rho}$-general integral transform to both sides of (47) with respect to t, we obtain

$$\begin{aligned}&\mathbb{T}_{\frac{t^\rho}{\rho}}\left\{{}^{KC}\mathbf{D}_t^{\gamma,\rho} f(\varphi,\varsigma,t)\right\}(s)=\mathbb{T}_{\frac{t^\rho}{\rho}}\left\{g(\varphi,\varsigma,t)-R[f(\varphi,\varsigma,t)]-N[f(\varphi,\varsigma,t)]\right\}\\&\frac{\nu^\gamma(s)}{\mu(s)}\mathbb{T}_{\frac{t^\rho}{\rho}}\left\{f(\varphi,\varsigma,t)\right\}(s)-\mu^{\gamma-1}(s)f(\varphi,\varsigma,0)\\&=\mathbb{T}_{\frac{t^\rho}{\rho}}\left\{g(\varphi,\varsigma,t)-R[f(\varphi,\varsigma,t)]-N[f(\varphi,\varsigma,t)]\right\},\end{aligned}\tag{50}$$

or

$$\begin{aligned}&F(\varphi,\varsigma,\mu(s),\nu(s))=\frac{\mu(s)}{\nu(s)}h(\varphi,\varsigma)\\&+\frac{\mu(s)}{\nu^\gamma(s)}\mathbb{T}_{\frac{t^\rho}{\rho}}\left\{g(\varphi,\varsigma,t)-R\left[\mathbb{T}^{-1}_{\frac{t^\rho}{\rho}}\left\{F(\varphi,\varsigma,t)\right\}\right]-N\left[\mathbb{T}^{-1}_{\frac{t^\rho}{\rho}}\left\{F(\varphi,\varsigma,t)\right\}\right]\right\},\end{aligned}\tag{51}$$

where $F(\varphi,\varsigma,\mu(s),\nu(s))=\mathbb{T}_{\frac{t^\rho}{\rho}}\left\{f(\varphi,\varsigma,t)\right\}(s)$.

2. By applying the proposed $\frac{t^\rho}{\rho}$-generalized integral transform residual power series method, the function $F(\varphi,\varsigma,\mu(s),\nu(s))$ can be expressed as follows:

$$F(\varphi,\varsigma,\mu(s),\nu(s))=\sum_{k=0}^{\infty}\phi_k(\varphi,\varsigma)\frac{\mu(s)}{\nu^{1+k\gamma}(s)},\quad \mu(s)>0,\nu(s)>0.\tag{52}$$

The infinite series (52), is called the new $\frac{t^\rho}{\rho}$–generalized series solution, and

$$\lim_{\nu(s)\to\infty}\nu(s)F(\varphi,\varsigma,\mu(s),\nu(s))=\phi_0(\varphi,\varsigma).$$

Then the first k terms of the new $\frac{t^\rho}{\rho}$–generalized series solution, represented by $F_k(\varphi,\varsigma,\mu(s),\nu(s))$, and we have

$$F_k(\varphi,\varsigma,\mu(s),\nu(s))=\phi_0(\varphi,\varsigma)\frac{\mu(s)}{\nu(s)}+\sum_{k=1}^{n}\phi_k(\varphi,\varsigma)\frac{\mu(s)}{\nu(s)^{1+k\gamma}},\quad \mu(s)>0,\nu(s)>0.\tag{53}$$

To calculate the coefficients $\phi_k(\varphi,\varsigma)$ in the fractional power series (53), we define the following $\frac{t^\rho}{\rho}$-general residual function corresponding to Equation (51):

$$\begin{aligned}&\mathbb{T}_{\frac{t^\rho}{\rho}}Res_F(\varphi,\varsigma,\mu(s),\nu(s))=F(\varphi,\varsigma,\mu(s),\nu(s))-\frac{\mu(s)}{\nu(s)}h(\varphi,\varsigma)\\&-\frac{\mu(s)}{\nu^\gamma(s)}\mathbb{T}_{\frac{t^\rho}{\rho}}\left\{g(\varphi,\varsigma,t)-R\left[\mathbb{T}^{-1}_{\frac{t^\rho}{\rho}}\left\{F(\varphi,\varsigma,t)\right\}\right]-N\left[\mathbb{T}^{-1}_{\frac{t^\rho}{\rho}}\left\{F(\varphi,\varsigma,t)\right\}\right]\right\}.\end{aligned}\tag{54}$$

and the k-th new $\frac{t^\rho}{\rho}$- general residual function as follow

$$\begin{aligned}&\mathbb{T}_{\frac{t^\rho}{\rho}}Res_{F_k}(\varphi,\varsigma,\mu(s),\nu(s))=F_k(\varphi,\varsigma,\mu(s),\nu(s))-\frac{\mu(s)}{\nu(s)}h(\varphi,\varsigma)\\&-\frac{\mu(s)}{\nu^\gamma(s)}\mathbb{T}_{\frac{t^\rho}{\rho}}\left\{g(\varphi,\varsigma,t)-R\left[\mathbb{T}^{-1}_{\frac{t^\rho}{\rho}}\left\{F_k(\varphi,\varsigma,t)\right\}\right]-N\left[\mathbb{T}^{-1}_{\frac{t^\rho}{\rho}}\left\{F_k(\varphi,\varsigma,t)\right\}\right]\right\}.\end{aligned}\tag{55}$$

3. Based on the properties of $\mathbb{T}_{\frac{t^\rho}{\rho}}Res_F(\varphi,\varsigma,t)$ and $\mathbb{T}_{\frac{t^\rho}{\rho}}Res_{F_i}(\varphi,\varsigma,t)$ employed in this study, the following results hold:

   (a) $\mathbb{T}_{\frac{t^\rho}{\rho}}Res_F(\varphi,\varsigma,\mu(s),\nu(s))=0$,
   $\lim_{i\to\infty}\mathbb{T}_{\frac{t^\rho}{\rho}}Res_{F_k}(\varphi,\varsigma,\mu(s),\nu(s))=\mathbb{T}_{\frac{t^\rho}{\rho}}Res_F(\varphi,\varsigma,\mu(s),\nu(s)),\quad \mu(s)>0,\nu(s)>0$,

   (b) $\lim_{\nu(s)\to\infty}\nu(s)\mathbb{T}_{\frac{t^\rho}{\rho}}Res_F(\varphi,\varsigma,\mu(s),\nu(s))=0$,
   implies $\lim_{\nu(s)\to\infty}\nu(s)\mathbb{T}_{\frac{t^\rho}{\rho}}Res_{F_i}(\varphi,\varsigma,\mu(s),\nu(s))=0$,

   (c) $\lim_{\nu(s)\to\infty}\nu^{1+k\gamma}(s)\mathbb{T}_{\frac{t^\rho}{\rho}}Res_F(\varphi,\varsigma,\mu(s),\nu(s))=\lim_{\nu(s)\to\infty}\nu^{1+k\gamma}(s)\mathbb{T}_{\frac{t^\rho}{\rho}}Res_{F_k}(\varphi,\varsigma,\mu(s),\nu(s))=0,\quad \text{for}\quad 0<\gamma\le 1,\ k\in\mathbb{N}$.

Therefore, the coefficient functions $\phi_k(\varphi,\varsigma)$ for all $n\in\mathbb{N}$ can be recursively obtained by following equation:

$$\lim_{\nu(s)\to\infty}\left(\nu^{1+k\gamma}(s)\mathbb{T}_{\frac{t^\rho}{\rho}}Res_F(\varphi,\varsigma,\mu(s),\nu(s))\right)=0,\quad 0<\gamma\le 1,\, k=1,2,3,\ldots. \tag{56}$$

4. To obtain the solution to the fractional differential equation (47), subject to the initial condition (48), the inverse $\frac{t^\rho}{\rho}$-general integral transform is applied to the function $F(\varphi,\varsigma,\mu(s),\nu(s))$.

In the following theorem, we analyze the convergence of the proposed $\frac{t^\rho}{\rho}$-generalized residual power series method.

Theorem 9. *(Convergent of new $\frac{t^\rho}{\rho}$–generalized residual power series method) For any $(\varphi,\varsigma,t)\in[0,a]\times[0,b]\times[0,M]$, the solution of partial fractional differential equation (47) with initial condition (48) is convergent to the series $\sum_{k=0}^{\infty}\frac{\phi_k(\varphi,\varsigma)}{\Gamma(1+k\gamma)}\left(\frac{t^\rho}{\rho}\right)^{k\gamma}$.*

Proof. For any $(\varphi,\varsigma,t)\in[0,a]\times[0,b]\times[0,M]$, we suppose that

$$f(\varphi,\varsigma,t)=\sum_{k=0}^{\infty}\frac{\phi_k(\varphi,\varsigma)}{\Gamma(1+k\gamma)}\left(\frac{t^\rho}{\rho}\right)^{k\gamma}.$$

Then

$$F(\varphi,\varsigma,\mu(s),\nu(s))=\sum_{k=0}^{\infty}\phi_k(\varphi,\varsigma)\frac{\mu(s)}{\nu(s)^{1+k\gamma}},$$

for $\nu(s)>0$ with

$$F(\varphi,\varsigma,\mu(s),\nu(s))=\mathbb{T}_{\frac{t^\rho}{\rho}}\left\{f(\varphi,\varsigma,t)\right\}.$$

According to (55), we have

$$\sum_{k=0}^{n}\left\{\mathbb{T}_{\frac{t^\rho}{\rho}}Res_{F_{k+1}}(\varphi,\varsigma,\mu(s),\nu(s))-\mathbb{T}_{\frac{t^\rho}{\rho}}Res_{F_k}(\varphi,\varsigma,\mu(s),\nu(s))\right\}=\mathbb{T}_{\frac{t^\rho}{\rho}}Res_{F_{k+1}}(\varphi,\varsigma,\mu(s),\nu(s))-\mathbb{T}_{\frac{t^\rho}{\rho}}Res_{F_0}(\varphi,\varsigma,\mu(s),\nu(s)). \tag{57}$$

On the other hand, we have

$$\begin{aligned}
&\mathbb{T}_{\frac{t^\rho}{\rho}}Res_{F_{k+1}}(\varphi,\varsigma,\mu(s),\nu(s))-\mathbb{T}_{\frac{t^\rho}{\rho}}Res_{F_k}(\varphi,\varsigma,\mu(s),\nu(s))\\
=&F_{k+1}(\varphi,\varsigma,\mu(s),\nu(s))-\frac{\mu(s)}{\nu(s)}h(\varphi,\varsigma)\\
&-\frac{\mu(s)}{\nu(s)^\gamma}\mathbb{T}_{\frac{t^\rho}{\rho}}\left\{g(\varphi,\varsigma,t)-R\left[\mathbb{T}^{-1}_{\frac{t^\rho}{\rho}}\left\{F_{k+1}(\varphi,\varsigma,\mu(s),\nu(s))\right\}\right]-N\left[\mathbb{T}^{-1}_{\frac{t^\rho}{\rho}}\left\{F_{k+1}(\varphi,\varsigma,\mu(s),\nu(s))\right\}\right]\right\}\\
&-F_k(\varphi,\varsigma,\mu(s),\nu(s))+\frac{\mu(s)}{\nu(s)}h(\varphi,\varsigma)\\
&+\frac{\mu(s)}{\nu(s)^\gamma}\mathbb{T}_{\frac{t^\rho}{\rho}}\left\{g(\varphi,\varsigma,\mu(s),\nu(s))-R\left[\mathbb{T}^{-1}_{\frac{t^\rho}{\rho}}\left\{F_k(\varphi,\varsigma,\mu(s),\nu(s))\right\}\right]-N\left[\mathbb{T}^{-1}_{\frac{t^\rho}{\rho}}\left\{F_k(\varphi,\varsigma,t)\right\}\right]\right\}\\
=&\phi_{k+1}(\varphi,\varsigma)\frac{\mu(s)}{\nu(s)^{1+(k+1)\gamma}}\\
&-\frac{\mu(s)}{\nu(s)^\gamma}\mathbb{T}_{\frac{t^\rho}{\rho}}\left\{-R\left[\mathbb{T}^{-1}_{\frac{t^\rho}{\rho}}\left\{\frac{\phi_{k+1}(\varphi,\varsigma)\mu(s)}{\nu(s)^{1+(k+1)\gamma}}\right\}\right]-N\left[\mathbb{T}^{-1}_{\frac{t^\rho}{\rho}}\left\{F_{k+1}(\varphi,\varsigma,\mu(s),\nu(s))\right\}\right]\right\}\\
&-\frac{\mu(s)}{\nu(s)^\gamma}\mathbb{T}_{\frac{t^\rho}{\rho}}\left\{N\left[\mathbb{T}^{-1}_{\frac{t^\rho}{\rho}}\left\{F_k(\varphi,\varsigma,\mu(s),\nu(s))\right\}\right]\right\}
\end{aligned} \tag{58}$$

Then, we have

$$\begin{aligned}
&\sum_{k=0}^{n}\left\{\mathbb{T}_{\frac{t^\rho}{\rho}} Res_{F_{k+1}}(\varphi,\varsigma,\mu(s),\nu(s)) - \mathbb{T}_{\frac{t^\rho}{\rho}} Res_{F_k}(\varphi,\varsigma,\mu(s),\nu(s))\right\}\\
=&\mathbb{T}_{\frac{t^\rho}{\rho}} Res_{F_1}(\varphi,\varsigma,\mu(s),\nu(s)) - \mathbb{T}_{\frac{t^\rho}{\rho}} Res_{F_0}(\varphi,\varsigma,\mu(s),\nu(s))\\
&+\sum_{k=1}^{n}\left\{\mathbb{T}_{\frac{t^\rho}{\rho}} Res_{F_{k+1}}(\varphi,\varsigma,\mu(s),\nu(s)) - \mathbb{T}_{\frac{t^\rho}{\rho}} Res_{F_k}(\varphi,\varsigma,\mu(s),\nu(s))\right\}\\
=&\mathbb{T}_{\frac{t^\rho}{\rho}} Res_{F_1}(\varphi,\varsigma,\mu(s),\nu(s)) - \mathbb{T}_{\frac{t^\rho}{\rho}} Res_{F_0}(\varphi,\varsigma,\mu(s),\nu(s))\\
&+\frac{\phi_2(\varphi,\varsigma)\mu(s)}{\nu(s)^{1+2\gamma}}\\
&-\frac{\mu(s)}{\nu(s)^{\gamma}}\mathbb{T}_{\frac{t^\rho}{\rho}}\left\{-R\left[\mathbb{T}^{-1}_{\frac{t^\rho}{\rho}}\left\{\frac{\phi_2(\varphi,\varsigma)\mu(s)}{\nu(s)^{1+2\gamma}}\right\}\right]-N\left[\mathbb{T}^{-1}_{\frac{t^\rho}{\rho}}\{F_2(\varphi,\varsigma,t)\}\right]+N\left[\mathbb{T}^{-1}_{\frac{t^\rho}{\rho}}\{F_1(\varphi,\varsigma,t)\}\right]\right\}\\
&+\frac{\phi_3(\varphi,\varsigma)\mu(s)}{\nu(s)^{1+3\gamma}}\\
&-\frac{\mu(s)}{\nu(s)^{\gamma}}\mathbb{T}_{\frac{t^\rho}{\rho}}\left\{-R\left[\mathbb{T}^{-1}_{\frac{t^\rho}{\rho}}\left\{\frac{\phi_3(\varphi,\varsigma)\mu(s)}{\nu(s)^{1+3\gamma}}\right\}\right]-N\left[\mathbb{T}^{-1}_{\frac{t^\rho}{\rho}}\{F_3(\varphi,\varsigma,t)\}\right]+N\left[\mathbb{T}^{-1}_{\frac{t^\rho}{\rho}}\{F_2(\varphi,\varsigma,t)\}\right]\right\}\\
&\vdots\\
&+\frac{\phi_{n+1}(\varphi,\varsigma)\mu(s)}{\nu(s)^{1+(n+1)\gamma}}\\
&-\frac{\mu(s)}{\nu(s)^{\gamma}}\mathbb{T}_{\frac{t^\rho}{\rho}}\left\{-R\left[\mathbb{T}^{-1}_{\frac{t^\rho}{\rho}}\left\{\frac{\phi_{n+1}(\varphi,\varsigma)\mu(s)}{\nu(s)^{1+(n+1)\gamma}}\right\}\right]-N\left[\mathbb{T}^{-1}_{\frac{t^\rho}{\rho}}\{F_{n+1}(\varphi,\varsigma,t)\}\right]+N\left[\mathbb{T}^{-1}_{\frac{t^\rho}{\rho}}\{F_n(\varphi,\varsigma,t)\}\right]\right\},
\end{aligned}\tag{59}$$

or

$$\begin{aligned}
&\sum_{k=0}^{n}\left\{\mathbb{T}_{\frac{t^\rho}{\rho}} Res_{F_{k+1}}(\varphi,\varsigma,\mu(s),\nu(s)) - \mathbb{T}_{\frac{t^\rho}{\rho}} Res_{F_i}(\varphi,\varsigma,\mu(s),\nu(s))\right\}\\
=&\mathbb{T}_{\frac{t^\rho}{\rho}} Res_{F_1}(\varphi,\varsigma,\mu(s),\nu(s)) - \mathbb{T}_{\frac{t^\rho}{\rho}} Res_{F_0}(\varphi,\varsigma,\mu(s),\nu(s)) + \sum_{k=2}^{n+1}\frac{\phi_k(\varphi,\varsigma)\mu(s)}{\nu(s)^{1+k\gamma}}\\
&-\frac{\mu(s)}{\nu(s)^{\gamma}}\mathbb{T}_{\frac{t^\rho}{\rho}}\left\{-R\left[\mathbb{T}^{-1}_{\frac{t^\rho}{\rho}}\left\{\sum_{k=2}^{n+1}\frac{\phi_k(\varphi,\varsigma)\mu(s)}{\nu(s)^{1+k\gamma}}\right\}\right]-N\left[\mathbb{T}^{-1}_{\frac{t^\rho}{\rho}}\{F_{k+1}(\varphi,\varsigma,t)\}\right]+N\left[\mathbb{T}^{-1}_{\frac{t^\rho}{\rho}}\{F_1(\varphi,\varsigma,t)\}\right]\right\}.
\end{aligned}\tag{60}$$

Using (57), we obtain:

$$\begin{aligned}
&\mathbb{T}_{\frac{t^\rho}{\rho}} Res_{F_{n+1}}(\varphi,\varsigma,\mu(s),\nu(s)) - \mathbb{T}_{\frac{t^\rho}{\rho}} Res_{F_0}(\varphi,\varsigma,\mu(s),\nu(s))\\
&=\mathbb{T}_{\frac{t^\rho}{\rho}} Res_{F_1}(\varphi,\varsigma,\mu(s),\nu(s)) - \mathbb{T}_{\frac{t^\rho}{\rho}} Res_{F_0}(\varphi,\varsigma,\mu(s),\nu(s)) + \sum_{k=2}^{n+1}\frac{\phi_k(\varphi,\varsigma)\mu(s)}{\nu(s)^{1+k\gamma}}\\
&\quad-\frac{\mu(s)}{\nu(s)^{\gamma}}\mathbb{T}_{\frac{t^\rho}{\rho}}\left\{-R\left[\mathbb{T}^{-1}_{\frac{t^\rho}{\rho}}\left\{\sum_{k=2}^{n+1}\frac{\phi_k(\varphi,\varsigma)\mu(s)}{\nu(s)^{1+k\gamma}}\right\}\right]\right\}(\varphi,\varsigma,\mu(s),\nu(s))\\
&\quad+\frac{\mu(s)}{\nu(s)^{\gamma}}\mathbb{T}_{\frac{t^\rho}{\rho}}\left\{N\left[\mathbb{T}^{-1}_{\frac{t^\rho}{\rho}}\{F_{k+1}(\varphi,\varsigma,t)\}\right]\right\}(\varphi,\varsigma,\mu(s),\nu(s))\\
&\quad-\frac{\mu(s)}{\nu(s)^{\gamma}}\mathbb{T}_{\frac{t^\rho}{\rho}}\left\{N\left[\mathbb{T}^{-1}_{\frac{t^\rho}{\rho}}\{F_1(\varphi,\varsigma,t)\}\right]\right\}(\varphi,\varsigma,\mu(s),\nu(s)),
\end{aligned}\tag{61}$$

or

$$\begin{aligned}
&\mathbb{T}_{\frac{t^\rho}{\rho}} Res_{F_{n+1}}(\varphi,\varsigma,\mu(s),\nu(s)) = F_1(\varphi,\varsigma,\mu(s),\nu(s)) - \frac{\mu(s)}{\nu(s)}h(\varphi,\varsigma)\\
&-\frac{\mu(s)}{\nu(s)^\gamma}\mathbb{T}_{\frac{t^\rho}{\rho}}\left\{g(\varphi,\varsigma,t) - R\left[\mathbb{T}^{-1}_{\frac{t^\rho}{\rho}}\{F_1(\varphi,\varsigma,t)\}\right]\right\}(\varphi,\varsigma,\mu(s),\nu(s)) + \frac{\mu(s)}{\nu(s)^\gamma}\mathbb{T}_{\frac{t^\rho}{\rho}}\left\{N\left[\mathbb{T}^{-1}_{\frac{t^\rho}{\rho}}\{F_1(\varphi,\varsigma,t)\}\right]\right\}\\
&+\sum_{k=2}^{n+1}\frac{\phi_k(\varphi,\varsigma)\mu(s)}{\nu(s)^{1+k\gamma}} - \frac{\mu(s)}{\nu(s)^\gamma}\mathbb{T}_{\frac{t^\rho}{\rho}}\left\{-R\left[\mathbb{T}^{-1}_{\frac{t^\rho}{\rho}}\left\{\sum_{k=2}^{n+1}\frac{\phi_k(\varphi,\varsigma)\mu(s)}{\nu(s)^{1+k\gamma}}\right\}\right]\right\}\\
&+\frac{\mu(s)}{\nu(s)^\gamma}\mathbb{T}_{\frac{t^\rho}{\rho}}\left\{N\left[\mathbb{T}^{-1}_{\frac{t^\rho}{\rho}}\{F_{k+1}(\varphi,\varsigma,t)\}\right]\right\}(\varphi,\varsigma,\mu(s),\nu(s)) - \frac{\mu(s)}{\nu(s)^\gamma}\mathbb{T}_{\frac{t^\rho}{\rho}}\left\{N\left[\mathbb{T}^{-1}_{\frac{t^\rho}{\rho}}\{F_1(\varphi,\varsigma,t)\}\right]\right\},\\
&=\sum_{k=0}^{n+1}\frac{\phi_k(\varphi,\varsigma)\mu(s)}{\nu(s)^{1+k\gamma}} - \frac{\mu(s)}{\nu(s)}h(\varphi,\varsigma) - \frac{\mu(s)}{\nu(s)^\gamma}\mathbb{T}_{\frac{t^\rho}{\rho}}\left\{g(\varphi,\varsigma,t) - R\left[\mathbb{T}^{-1}_{\frac{t^\rho}{\rho}}\left\{\sum_{k=0}^{n+1}\frac{\phi_k(\varphi,\varsigma)\mu(s)}{\nu(s)^{1+k\gamma}}\right\}\right]\right\}\\
&+\frac{\mu(s)}{\nu(s)^\gamma}\mathbb{T}_{\frac{t^\rho}{\rho}}\left\{N\left[\mathbb{T}^{-1}_{\frac{t^\rho}{\rho}}\left\{\sum_{k=0}^{n+1}\frac{\phi_k(\varphi,\varsigma)\mu(s)}{\nu(s)^{1+k\gamma}}\right\}\right]\right\},
\end{aligned} \tag{62}$$

By utilizing the properties of $\mathbb{T}_{\frac{t^\rho}{\rho}} Res_{F_{n+1}}(\varphi,\varsigma,\mu(s),\nu(s))$ as $n\to\infty$, we obtain

$$\begin{aligned}
\sum_{k=0}^{\infty}\frac{\phi_k(\varphi,\varsigma)\mu(s)}{\nu(s)^{1+k\gamma}} =& \frac{\mu(s)}{\nu(s)}h(\varphi,\varsigma) + \frac{\mu(s)}{\nu(s)^\gamma}\mathbb{T}_{\frac{t^\rho}{\rho}}\left\{g(\varphi,\varsigma,t) - R\left[\mathbb{T}^{-1}_{\frac{t^\rho}{\rho}}\left\{\sum_{k=0}^{\infty}\frac{\phi_k(\varphi,\varsigma)\mu(s)}{\nu(s)^{1+k\gamma}}\right\}\right]\right\}\\
&-\frac{\mu(s)}{\nu(s)^\gamma}\mathbb{T}_{\frac{t^\rho}{\rho}}\left\{N\left[\mathbb{T}^{-1}_{\frac{t^\rho}{\rho}}\left\{\sum_{k=0}^{\infty}\frac{\phi_k(\varphi,\varsigma)\mu(s)}{\nu(s)^{1+k\gamma}}\right\}\right]\right\}.
\end{aligned} \tag{63}$$

□

These results demonstrate that the infinite series $\sum_{k=0}^{\infty}\frac{\phi_k(\varphi,\varsigma)\mu(s)}{\nu^{1+k\gamma}(s)}$ is the solution to the equivalent form of (53). Therefore, by applying the inverse proposed $\frac{t^\rho}{\rho}$-generalized transform to both sides of (63), we obtain the solution to the nonlinear fractional differential equation (47) subject to the initial condition (48).

4 Analytical Solution of Time-Fractional Black-Scholes and Navier-Stokes Equations

In this section, we apply our proposed method to the time-fractional Black–Scholes and Navier–Stokes equations.

Example 1. Consider the following time-fractional Black–Scholes equation with two assets

$$^{KC}\mathbf{D}_t^{\gamma,\rho}\Phi(\varphi,\varsigma,t) = r\Phi - \frac{1}{2}\delta_1^2\varphi^2\frac{\partial^2\Phi}{\partial\varphi^2} - \frac{1}{2}\delta_2^2\varsigma^2\frac{\partial^2\Phi}{\partial\varsigma^2} - \varpi\delta_1\delta_2\varphi\varsigma\frac{\partial^2\Phi}{\partial\varphi\partial\varsigma} - r\left[\varphi\frac{\partial\Phi}{\partial\varphi} + \varsigma\frac{\partial\Phi}{\partial\varsigma}\right], \tag{64}$$

with the following initial condition

$$\Phi(\varphi,\varsigma,0) = \phi_0(\varphi,\varsigma) = \max(\beta_1\varphi + \beta_2\varsigma - K, 0), \quad \varphi,\varsigma\in[0,\infty), t\in[0,M]. \tag{65}$$

We assume that the following series is the solution to equation (64) about the point $t=0$:

$$\Phi(\varphi,\varsigma,t) = \sum_{k=0}^{\infty}\frac{\phi_k}{\Gamma(1+k\gamma)}\left(\frac{t^\rho}{\rho}\right)^{k\gamma}, \tag{66}$$

where $\phi_k = \phi_k(\varphi,\varsigma)$, is coefficient function with respect to φ and ς.

1. By applying new $\frac{t^\rho}{\rho}$- general transform on both sides of equation (64), we have

$$\begin{aligned}
\mathbb{T}_{\frac{t^\rho}{\rho}}\left\{^{KC}\mathbf{D}_t^{\gamma,\rho}\Phi(\varphi,\varsigma,t)\right\} =& \ \mathbb{T}_{\frac{t^\rho}{\rho}}\left\{r\Phi - \frac{1}{2}\delta_1^2\varphi^2\frac{\partial^2\Phi}{\partial\varphi^2} - \frac{1}{2}\delta_2^2\varsigma^2\frac{\partial^2\Phi}{\partial\varsigma^2} - \varpi\delta_1\delta_2\varphi\varsigma\frac{\partial^2\Phi}{\partial\varphi\partial\varsigma} - r\left[\varphi\frac{\partial\Phi}{\partial\varphi} + \varsigma\frac{\partial\Phi}{\partial\varsigma}\right]\right\}\\
&= \Theta(\varphi,\varsigma,\mu(s),\nu(s)) - \frac{\mu(s)}{\nu(s)}h(\varphi,\varsigma) - \frac{\mu(s)}{\nu^\gamma(s)}\Big\{r\Theta - \frac{1}{2}\delta_1^2\varphi^2\frac{\partial^2\Theta}{\partial\varphi^2} - \frac{1}{2}\delta_2^2\varsigma^2\frac{\partial^2\Theta}{\partial\varsigma^2}\\
&-\varpi\delta_1\delta_2\varphi\varsigma\frac{\partial^2\Theta}{\partial\varphi\partial\varsigma} - r\left[\varphi\frac{\partial\Theta}{\partial\varphi} + \varsigma\frac{\partial\Theta}{\partial\varsigma}\right]\Big\} = 0,
\end{aligned} \tag{67}$$

where $\Theta(\varphi,\varsigma,\mu(s),\nu(s)) = \mathbb{T}_{\frac{t^\rho}{\rho}}\{\Phi(\varphi,\varsigma,t)\}(\varphi,\varsigma,\mu(s),\nu(s))$ for $\mu(s)>0, \nu(s)>0$.

2. Suppose the solution of equation (67) be the following form

$$\Theta(\varphi,\varsigma,\mu(s),\nu(s)) = \sum_{k=0}^{\infty} \frac{\phi_k \mu(s)}{\nu(s)^{1+k\gamma}}. \tag{68}$$

Therefore, the first n-th terms of series (68) will be written as follow

$$\Theta_n(\varphi,\varsigma,\mu(s),\nu(s)) = \max(\beta_1\varphi+\beta_2\varsigma-K,0)\frac{\mu(s)}{\nu(s)} + \sum_{k=1}^{n} \frac{\phi_k \mu(s)}{\nu(s)^{1+k\gamma}}, \tag{69}$$

Hence, according to (54), we have

$$\begin{aligned}\mathbb{T}_{\frac{t^\rho}{\rho}} Res_\Theta(\varphi,\varsigma,\mu(s),\nu(s)) =& \Theta(\varphi,\varsigma,\mu(s),\nu(s)) - \max(\beta_1\varphi+\beta_2\varsigma-K,0)\frac{\mu(s)}{\nu(s)} \\ &- \frac{\mu(s)}{\nu^\gamma(s)}\Big\{r\Theta - \frac{1}{2}\delta_1^2\varphi^2\frac{\partial^2\Theta}{\partial\varphi^2} - \frac{1}{2}\delta_2^2\varsigma^2\frac{\partial^2\Theta}{\partial\varsigma^2} \\ &- \varpi\delta_1\delta_2\varphi\varsigma\frac{\partial^2\Theta}{\partial\varphi\partial\varsigma} - r\left[\varphi\frac{\partial\Theta}{\partial\varphi} + \varsigma\frac{\partial\Theta}{\partial\varsigma}\right]\Big\},\end{aligned} \tag{70}$$

and the n-th of equation (70) will be obtained as follow

$$\begin{aligned}\mathbb{T}_{\frac{t^\rho}{\rho}} Res_{\Theta_n}(\varphi,\varsigma,\mu(s),\nu(s)) =& \Theta_n(\varphi,\varsigma,\mu(s),\nu(s)) - \max(\beta_1\varphi+\beta_2\varsigma-K,0)\frac{\mu(s)}{\nu(s)} \\ &- \frac{\mu(s)}{\nu(s)^\gamma}\Big\{r\Theta_n - \frac{1}{2}\delta_1^2\varphi^2\frac{\partial^2\Theta_n}{\partial\varphi^2} - \frac{1}{2}\delta_2^2\varsigma^2\frac{\partial^2\Theta_n}{\partial\varsigma^2} \\ &- \varpi\delta_1\delta_2\varphi\varsigma\frac{\partial^2\Theta_n}{\partial\varphi\partial\varsigma} - r\left[\varphi\frac{\partial\Theta_n}{\partial\varphi} + \varsigma\frac{\partial\Theta_n}{\partial\varsigma}\right]\Big\}.\end{aligned} \tag{71}$$

3. for $n=1$, we have

$$\Theta_1(\varphi,\varsigma,\mu(s),\nu(s)) = \max(\beta_1\varphi+\beta_2\varsigma-K,0)\frac{\mu(s)}{\nu(s)} + \frac{\phi_1\mu(s)}{\nu(s)^{1+\gamma}}, \tag{72}$$

Therefore, we have

$$\begin{aligned}&\mathbb{T}_{\frac{t^\rho}{\rho}} Res_{\Theta_1}(\varphi,\varsigma,\mu(s),\nu(s)) = \Theta_1(\varphi,\varsigma,\mu(s),\nu(s)) - \max(\beta_1\rho+\beta_2\varsigma-K,0)\frac{\mu(s)}{\nu(s)} \\ &- \frac{\mu(s)}{\nu(s)^\gamma}\Big\{r\Theta_1 - \frac{1}{2}\delta_1^2\varphi^2\frac{\partial^2\Theta_1}{\partial\varphi^2} - \frac{1}{2}\delta_2^2\varsigma^2\frac{\partial^2\Theta_1}{\partial\varsigma^2} - \varpi\delta_1\delta_2\varphi\varsigma\frac{\partial^2\Theta_1}{\partial\varphi\partial\varsigma} - r\left[\varphi\frac{\partial\Theta_1}{\partial\varphi} + \varsigma\frac{\partial\Theta_1}{\partial\varsigma}\right]\Big\} \\ &= \max(\beta_1\rho+\beta_2\varsigma-K,0)\frac{\mu(s)}{\nu(s)} + \frac{\phi_1\mu(s)}{\nu(s)^{1+\gamma}} - \max(\beta_1\rho+\beta_2\varsigma-K,0)\frac{\mu(s)}{\nu(s)} \\ &- \frac{\mu(s)}{\nu(s)^\gamma}\left\{r\max(\beta_1\varphi+\beta_2\varsigma-K,0)\frac{\mu(s)}{\nu(s)} + r\frac{\phi_1\mu(s)}{\nu(s)^{1+\gamma}} - \frac{1}{2}\delta_1^2\varphi^2\frac{\partial^2\phi_1}{\partial\varphi^2}\frac{\mu(s)}{\nu(s)^{1+\gamma}}\right\} \\ &+ \frac{\mu(s)}{\nu(s)^\gamma}\left\{\frac{1}{2}\delta_1^2\varsigma^2\frac{\partial^2\phi_1}{\partial\varsigma^2}\frac{\mu(s)}{\nu(s)^{1+\gamma}} + \varpi\delta_1\delta_2\varphi\varsigma\frac{\partial^2\phi_1}{\partial\varphi\partial\varsigma}\frac{\mu(s)}{\nu(s)^{1+\gamma}}\right\} \\ &+ \frac{\mu(s)}{\nu(s)^\gamma}\left\{r\varphi\beta_1\frac{\mu(s)}{\nu(s)} + r\varphi\frac{\partial\phi_1}{\partial\varphi}\frac{\mu(s)}{\nu(s)^{1+\gamma}} + r\varsigma\beta_2\frac{\mu(s)}{\nu(s)} + r\varsigma\frac{\partial\phi_1}{\partial\varsigma}\frac{\mu(s)}{\nu(s)^{1+\gamma}}\right\},\end{aligned} \tag{73}$$

After multiplying $\nu(s)^{1+\gamma}$ on both sides of equation (73) and taking $\lim\limits_{\nu(s)\to\infty}$, we have

$$\begin{aligned}&\lim_{\nu(s)\to\infty} \nu^{1+\gamma}(s)\mathbb{T}_{\frac{t^\rho}{\rho}} Res_{\Theta_1}(\varphi,\varsigma,\mu(s),\nu(s)) = \mu(s)\{\phi_1 - r\max(\beta_1\varphi+\beta_2\varsigma-K,0) + r(\varphi\beta_1+\varsigma\beta_2)\} = 0 \\ &\phi_1 = r\max(\beta_1\varphi+\beta_2\varsigma-K,0) - r(\varphi\beta_1+\varsigma\beta_2),\end{aligned} \tag{74}$$

Therefore, the first approximate can be written as follows

$$\Theta_1(\varphi,\varsigma,\mu(s),\nu(s)) = \max(\beta_1\varphi+\beta_2\varsigma-K,0)\frac{\mu(s)}{\nu(s)} + \{r\max(\beta_1\varphi+\beta_2\varsigma-K,0) - r(\varphi\beta_1+\varsigma\beta_2)\}\frac{\mu(s)}{\nu(s)^{1+\gamma}}, \tag{75}$$

for $n = 2$, we have

$$\Theta_2(\varphi,\varsigma,\mu(s),\nu(s)) = \max(\beta_1\varphi+\beta_2\varsigma-K,0)\frac{\mu(s)}{\nu(s)} + \frac{\phi_1\mu(s)}{\nu(s)^{1+\gamma}} + \frac{\phi_2\mu(s)}{\nu(s)^{1+2\gamma}}, \tag{76}$$

Therefore, we have

$$\begin{aligned}
&\mathbb{T}_{\frac{t^\rho}{\rho}} Res_{\Theta_2}(\varphi,\varsigma,\mu(s),\nu(s)) = \Theta_2(\varphi,\varsigma,\mu(s),\nu(s)) - \max(\beta_1\rho+\beta_2\varsigma-K,0)\frac{\mu(s)}{\nu(s)} \\
&-\frac{\mu(s)}{\nu^\gamma(s)}\left\{r\Theta_2 - \frac{1}{2}\delta_1^2\varphi^2\frac{\partial^2\Theta_2}{\partial\varphi^2} - \frac{1}{2}\delta_2^2\varsigma^2\frac{\partial^2\Theta_2}{\partial\varsigma^2} - \varpi\delta_1\delta_2\varphi\varsigma\frac{\partial^2\Theta_2}{\partial\varphi\partial\varsigma} - r\left[\varphi\frac{\partial\Theta_2}{\partial\varphi} + \varsigma\frac{\partial\Theta_2}{\partial\varsigma}\right]\right\} \\
&= \max(\beta_1\rho+\beta_2\varsigma-K,0)\frac{\mu(s)}{\nu(s)} + \frac{\phi_1\mu(s)}{\nu(s)^{1+\gamma}} + \frac{\phi_2\mu(s)}{\nu(s)^{1+2\gamma}} - \max(\beta_1\rho+\beta_2\varsigma-K,0)\frac{\mu(s)}{\nu(s)} \\
&-\frac{\mu(s)}{\nu(s)^\gamma}\left\{r\max(\beta_1\varphi+\beta_2\varsigma-K,0)\frac{\mu(s)}{\nu(s)} + r\frac{\phi_1\mu(s)}{\nu(s)^{1+\gamma}} + r\frac{\phi_2\mu(s)}{\nu(s)^{1+2\gamma}} - \frac{1}{2}\delta_1^2\varphi^2\frac{\partial^2\phi_1}{\partial\varphi^2}\frac{\mu(s)}{\nu(s)^{1+\gamma}} - \frac{1}{2}\delta_1^2\varphi^2\frac{\partial^2\phi_2}{\partial\varphi^2}\frac{\mu(s)}{\nu(s)^{1+2\gamma}}\right\} \\
&+\frac{\mu(s)}{\nu(s)^\gamma}\left\{\frac{1}{2}\delta_1^2\varsigma^2\frac{\partial^2\phi_1}{\partial\varsigma^2}\frac{\mu(s)}{\nu(s)^{1+\gamma}} + \frac{1}{2}\delta_1^2\varsigma^2\frac{\partial^2\phi_2}{\partial\varsigma^2}\frac{\mu(s)}{\nu(s)^{1+2\gamma}} + \varpi\delta_1\delta_2\varphi\varsigma\frac{\partial^2\phi_1}{\partial\varphi\partial\varsigma}\frac{\mu(s)}{\nu(s)^{1+\gamma}} + \varpi\delta_1\delta_2\varphi\varsigma\frac{\partial^2\phi_2}{\partial\varphi\partial\varsigma}\frac{\mu(s)}{\nu(s)^{1+2\gamma}}\right\} \\
&+\frac{\mu(s)}{\nu(s)^\gamma}\left\{r\varphi\beta_1\frac{\mu(s)}{\nu(s)} + r\varphi\frac{\partial\phi_1}{\partial\varphi}\frac{\mu(s)}{\nu(s)^{1+\gamma}} + r\varphi\frac{\partial\phi_2}{\partial\varphi}\frac{\mu(s)}{\nu(s)^{1+2\gamma}} + r\varsigma\beta_2\frac{\mu(s)}{\nu(s)} + r\varsigma\frac{\partial\phi_1}{\partial\varsigma}\frac{\mu(s)}{\nu(s)^{1+\gamma}} + r\varsigma\frac{\partial\phi_2}{\partial\varsigma}\frac{\mu(s)}{\nu(s)^{1+2\gamma}}\right\}.
\end{aligned} \tag{77}$$

After multiplying $\nu(s)^{1+2\gamma}$ on both sides of equation (77) and taking $\lim_{\nu(s)\to\infty}$, we have

$$\begin{aligned}
&\lim_{\nu(s)\to\infty}\nu(s)^{1+2\gamma}\mathbb{T}_{\frac{t^\rho}{\rho}} Res_{\Theta_1}(\varphi,\varsigma,\mu(s),\nu(s)) = \mu(s)\left\{\phi_2 - r^2\max(\beta_1\rho+\beta_2\varsigma-K,0) + r^2(\varphi\beta_1+\varsigma\beta_2)\right\} = 0 \\
&\phi_2 = r^2\max(\beta_1\varphi+\beta_2\varsigma-K,0) - r^2(\varphi\beta_1+\varsigma\beta_2),
\end{aligned} \tag{78}$$

Therefore, the second approximate can be written as follows

$$\begin{aligned}
\Theta_2 =& \max(\beta_1\varphi+\beta_2\varsigma-K,0)\frac{\mu(s)}{\nu(s)} \\
&+ \{r\max(\beta_1\varphi+\beta_2\varsigma-K,0) - r(\varphi\beta_1+\varsigma\beta_2)\}\frac{\mu(s)}{\nu(s)^{1+\gamma}} \\
&+ \left\{r^2\max(\beta_1\varphi+\beta_2\varsigma-K,0) - r^2(\varphi\beta_1+\varsigma\beta_2)\right\}\frac{\mu(s)}{\nu(s)^{1+2\gamma}},
\end{aligned} \tag{79}$$

Finally, the n-th approximate can be written as follows

$$\begin{aligned}
\Theta_n =& \max(\beta_1\varphi+\beta_2\varsigma-K,0)\frac{\mu(s)}{\nu(s)} \\
&+ \{r\max(\beta_1\varphi+\beta_2\varsigma-K,0) - r(\varphi\beta_1+\varsigma\beta_2)\}\frac{\mu(s)}{\nu(s)^{1+\gamma}} \\
&+ \left\{r^2\max(\beta_1\varphi+\beta_2\varsigma-K,0) - r^2(\varphi\beta_1+\varsigma\beta_2)\right\}\frac{\mu(s)}{\nu(s)^{1+2\gamma}} \\
&\vdots \\
&+ \{r^n\max(\beta_1\varphi+\beta_2\varsigma-K,0) - r^n(\varphi\beta_1+\varsigma\beta_2)\}\frac{\mu(s)}{\nu(s)^{1+n\gamma}}.
\end{aligned} \tag{80}$$

If $n \to \infty$, then we have

$$\begin{aligned}\Theta(\varphi,\varsigma,\mu(s),\nu(s)) =& \max(\beta_1\varphi+\beta_2\varsigma-K,0)\frac{\mu(s)}{\nu(s)}\\ &+\{r\max(\beta_1\varphi+\beta_2\varsigma-K,0)-r(\varphi\beta_1+\varsigma\beta_2)\}\frac{\mu(s)}{\nu(s)^{1+\gamma}}\\ &+\left\{r^2\max(\beta_1\varphi+\beta_2\varsigma-K,0)-r^2(\varphi\beta_1+\varsigma\beta_2)\right\}\frac{\mu(s)}{\nu(s)^{1+2\gamma}}\\ &\vdots\\ &+\{r^n\max(\beta_1\varphi+\beta_2\varsigma-K,0)-r^n(\varphi\beta_1+\varsigma\beta_2)\}\frac{\mu(s)}{\nu(s)^{1+n\gamma}}\\ &\vdots\end{aligned} \tag{81}$$

4. By imposing the inverse new $\frac{t^\rho}{\rho}$–generalized transform, the solution of fractional differential equation (64) obtained as follows

$$\begin{aligned}\Phi(\varphi,\varsigma,t) =& \max(\beta_1\varphi+\beta_2\varsigma-K,0)\\ &+\{r\max(\beta_1\varphi+\beta_2\varsigma-K,0)-r(\varphi\beta_1+\varsigma\beta_2)\}\frac{\left(\frac{t^\rho}{\rho}\right)^\gamma}{\Gamma(1+\gamma)}\\ &+\left\{r^2\max(\beta_1\varphi+\beta_2\varsigma-K,0)-r^2(\varphi\beta_1+\varsigma\beta_2)\right\}\frac{\left(\frac{t^\rho}{\rho}\right)^{2\gamma}}{\Gamma(1+2\gamma)}\\ &\vdots\\ &+\{r^n\max(\beta_1\varphi+\beta_2\varsigma-K,0)-r^n(\varphi\beta_1+\varsigma\beta_2)\}\frac{\left(\frac{t^\rho}{\rho}\right)^{n\gamma}}{\Gamma(1+n\gamma)}\\ &\vdots\end{aligned} \tag{82}$$

or

$$\begin{aligned}\Phi(\varphi,\varsigma,t) =& \max(\beta_1\varphi+\beta_2\varsigma-K,0)\left[1+\frac{r\left(\frac{t^\rho}{\rho}\right)^\gamma}{\Gamma(1+\gamma)}+\frac{r^2\left(\frac{t^\rho}{\rho}\right)^{2\gamma}}{\Gamma(1+2\gamma)}+\frac{r^3\left(\frac{t^\rho}{\rho}\right)^{3\gamma}}{\Gamma(1+3\gamma)}+\dots\right]\\ &-(\varphi\beta_1+\varsigma\beta_2)\left[\frac{r\left(\frac{t^\rho}{\rho}\right)^\gamma}{\Gamma(1+\gamma)}+\frac{r^2\left(\frac{t^\rho}{\rho}\right)^{2\gamma}}{\Gamma(1+2\gamma)}+\frac{r^3\left(\frac{t^\rho}{\rho}\right)^{3\gamma}}{\Gamma(1+3\gamma)}+\dots\right]\\ =& \max(\beta_1\varphi+\beta_2\varsigma-K,0)\left[\sum_{k=0}^{\infty}\frac{\left(r\left(\frac{t^\rho}{\rho}\right)^\gamma\right)^k}{\Gamma(1+k\gamma)}\right]-(\varphi\beta_1+\varsigma\beta_2)\left[\sum_{k=0}^{\infty}\frac{\left(r\left(\frac{t^\rho}{\rho}\right)^\gamma\right)^k}{\Gamma(1+k\gamma)}-1\right]\\ =& \max(\beta_1\varphi+\beta_2\varsigma-K,0)E_\gamma\left(r\left(\frac{t^\rho}{\rho}\right)^\gamma\right)-(\varphi\beta_1+\varsigma\beta_2)\left[E_\gamma\left(r\left(\frac{t^\rho}{\rho}\right)^\gamma\right)-1\right].\end{aligned} \tag{83}$$

For $\gamma=1$, then, we have

$$\Phi(\varphi,\varsigma,t) = \max(\beta_1\varphi+\beta_2\varsigma-K,0)\exp\left(r\left(\frac{t^\rho}{\rho}\right)\right)-(\varphi\beta_1+\varsigma\beta_2)\left[\exp\left(r\left(\frac{t^\rho}{\rho}\right)\right)-1\right]. \tag{84}$$

Based on the data presented in Table 1, we will generate several 3D plots corresponding to equation (83), as illustrated in Figures 1–6. The desired Black-Scholes equation has been solved considering the proposed method, and its 2D graphs have been plotted based on various derivative orders $\rho=1,\gamma=0.5,0.7,0.9,1,\rho=0.5,\gamma=0.5,0.7,0.9,1$ and $\gamma=0.5,\rho=0.5,0.7,0.9,1$. Additionally, 3D graphs have been created considering the derivative orders $\gamma=\rho=1,0.5,0.7,0.9$.

Table 1. Values of parameters in Black-Scholes equations

Parameters	value
Strike price (dollars), K	70
Risk-free interest rate (per year), r	0.05
The volatility of φ and ς	0.5
Risk parameters β_1	2
Risk parameters β_2	1

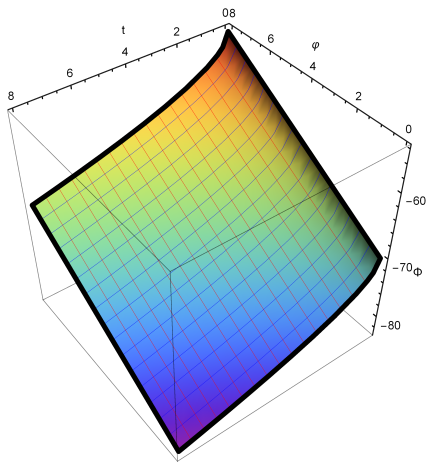

Figure 1. Values of the options for $\rho = 1$ and $\gamma = 0.5$.

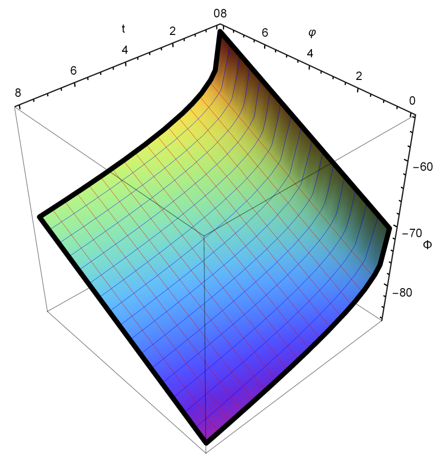

Figure 2. Values of the options for $\rho = 0.5$ and $\gamma = 0.5$.

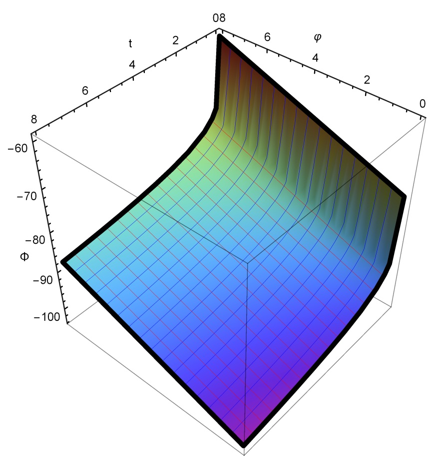

Figure 3. Values of the options for $\rho = 0.2$ and $\gamma = 0.5$.

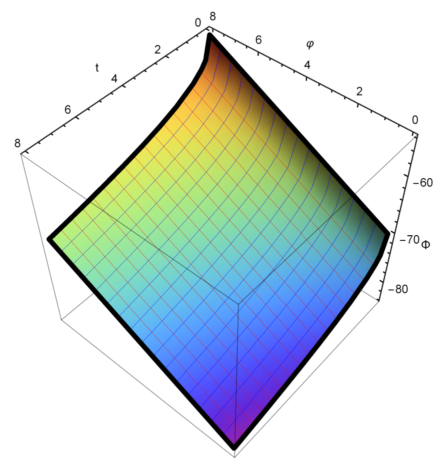

Figure 4. Values of the options for $\rho = 0.7$ and $\gamma = 0.5$.

Example 2. Consider the following two dimensional Naiver-Stokes time fractional differential equations

$$\begin{aligned} {}^{KC}\mathbf{D}_t^{\gamma,\rho}\Phi(\varphi,\varsigma,t) + \Phi\frac{\partial \Phi}{\partial \varphi} + \Psi\frac{\partial \Phi}{\partial \varsigma} &= \rho\left(\frac{\partial^2 \Phi}{\partial \varphi^2} + \frac{\partial^2 \Phi}{\partial \varsigma^2}\right) + \hbar, \quad \varphi,\varsigma,t > 0, \\ {}^{KC}\mathbf{D}_t^{\gamma,\rho}\Psi(\varphi,\varsigma,t) + \Phi\frac{\partial \Psi}{\partial \varphi} + \Psi\frac{\partial \Psi}{\partial \varsigma} &= \rho\left(\frac{\partial^2 \Psi}{\partial \varphi^2} + \frac{\partial^2 \Psi}{\partial \varsigma^2}\right) - \hbar, \quad \varphi,\varsigma,t > 0, \end{aligned} \tag{85}$$

with subject to the initial conditions

$$\begin{aligned} \Phi(\varphi,\varsigma,0) &= -\sin(\varphi+\varsigma), \\ \Psi(\varphi,\varsigma,0) &= \sin(\varphi+\varsigma). \end{aligned} \tag{86}$$

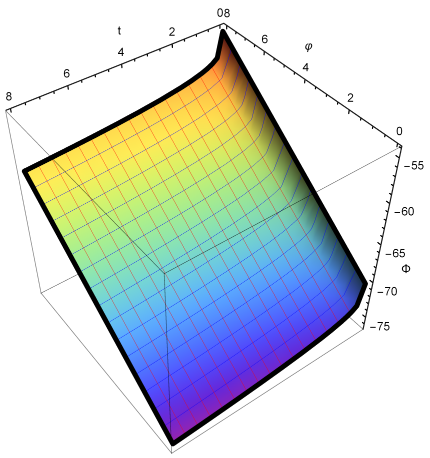

Figure 5. Values of the options for $\rho = 1$ and $\gamma = 0.2$.

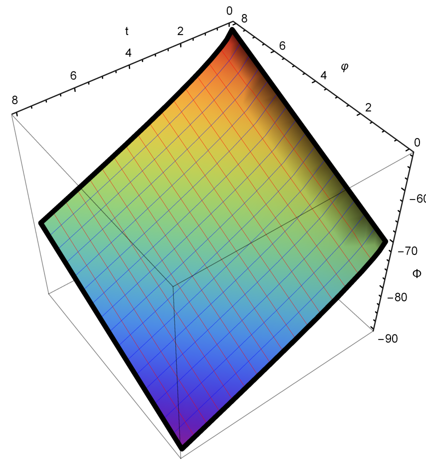

Figure 6. Values of the options for $\rho = 1$ and $\gamma = 0.7$.

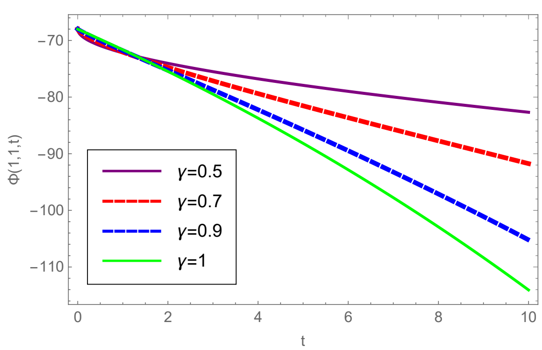

Figure 7. 2D plot of example (1) for $\rho = 1$.

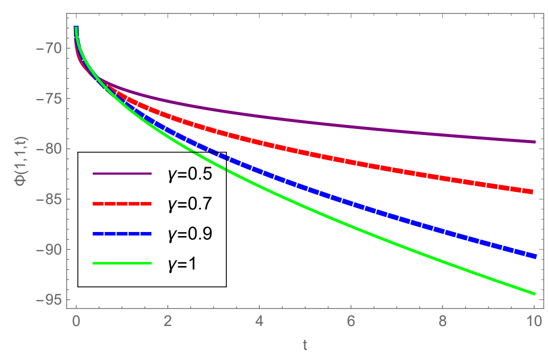

Figure 8. 2D plot of example (1) for $\rho = 0.5$.

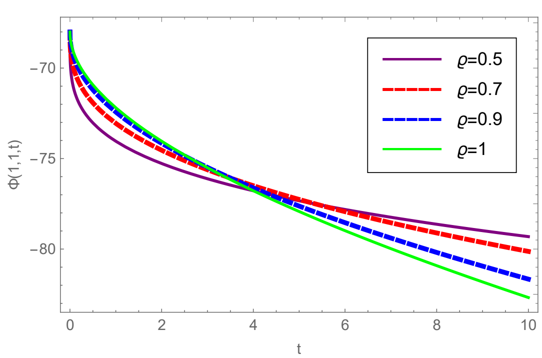

Figure 9. 2D plot of example (1) for $\gamma = 0.5$.

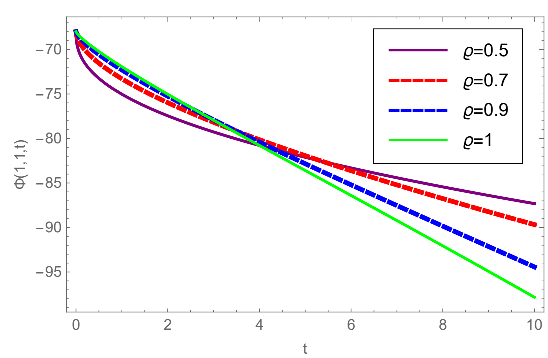

Figure 10. 2D plot of example (1) for $\gamma = 0.8$.

We suppose that the following series be solutions of equations (85), around 0,

$$\begin{aligned} \Phi(\varphi,\varsigma,t) &= \sum_{k=0}^{\infty} \frac{\phi_k}{\Gamma(1+k\gamma)} \left(\frac{t^{\rho}}{\rho}\right)^{k\gamma}, \\ \Psi(\varphi,\varsigma,t) &= \sum_{k=0}^{\infty} \frac{\psi_k}{\Gamma(1+k\gamma)} \left(\frac{t^{\rho}}{\rho}\right)^{k\gamma}, \end{aligned} \tag{87}$$

where $\phi_k = \phi_k(\varphi,\varsigma)$ and $\psi_k = \psi_k(\varphi,\varsigma)$ is coefficients functions with respect to φ and ς,

1. By applying new $\frac{t^\rho}{\rho}$-general transform on both sides of equation (85), we have

$$\begin{aligned}
\mathbb{T}_{\frac{t^\rho}{\rho}}\left\{{}^{KC}\mathbf{D}_t^{\gamma,\rho}\Phi(\varphi,\varsigma,t)\right\} &= \mathbb{T}_{\frac{t^\rho}{\rho}}\left\{\rho\left(\frac{\partial^2\Phi}{\partial\varphi^2}+\frac{\partial^2\Phi}{\partial\varsigma^2}\right)-\Phi\frac{\partial\Phi}{\partial\varphi}-\Psi\frac{\partial\Phi}{\partial\varsigma}+\hbar\right\},\\
\mathbb{T}_{\frac{t^\rho}{\rho}}\left\{{}^{KC}\mathbf{D}_t^{\gamma,\rho}\Psi(\varphi,\varsigma,t)\right\} &= \mathbb{T}_{\frac{t^\rho}{\rho}}\left\{\rho\left(\frac{\partial^2\Psi}{\partial\varphi^2}+\frac{\partial^2\Psi}{\partial\varsigma^2}\right)-\Phi\frac{\partial\Psi}{\partial\varphi}-\Psi\frac{\partial\Psi}{\partial\varsigma}-\hbar\right\},
\end{aligned} \tag{88}$$

or

$$\begin{aligned}
&\Theta(\varphi,\varsigma,\mu(s),\nu(s))+\sin(\varphi+\varsigma)\frac{\mu(s)}{\nu(s)}-\frac{\mu(s)}{\nu^\gamma(s)}\left\{\rho\left(\frac{\partial^2\Theta}{\partial\varphi^2}+\frac{\partial^2\Theta}{\partial\varsigma^2}\right)-\Theta\frac{\partial\Theta}{\partial\varphi}-\Xi\frac{\partial\Theta}{\partial\varsigma}+\hbar\right\}=0,\\
&\Xi(\varphi,\varsigma,\mu(s),\nu(s))-\sin(\varphi+\varsigma)\frac{\mu(s)}{\nu(s)}-\frac{\mu(s)}{\nu^\gamma(s)}\left\{\rho\left(\frac{\partial^2\Xi}{\partial\varphi^2}+\frac{\partial^2\Xi}{\partial\varsigma^2}\right)-\Theta\frac{\partial\Xi}{\partial\varphi}-\Xi\frac{\partial\Xi}{\partial\varsigma}-\hbar\right\}=0,
\end{aligned} \tag{89}$$

where

$$\Theta(\varphi,\varsigma,\mu(s),\nu(s))=\mathbb{T}_{\frac{t^\rho}{\rho}}\left\{{}^{KC}\mathbf{D}_t^{\gamma,\rho}\Phi(\varphi,\varsigma,t)\right\}(\varphi,\varsigma,\mu(s),\nu(s)),$$

and

$$\Xi(\varphi,\varsigma,\mu(s),\nu(s))=\mathbb{T}_{\frac{t^\rho}{\rho}}\left\{{}^{KC}\mathbf{D}_t^{\gamma,\rho}\Psi(\varphi,\varsigma,t)\right\}(\varphi,\varsigma,\mu(s),\nu(s)),\quad \mu(s)>0,\ \nu(s)>0.$$

2. Suppose the solution of equations (89) be the following form

$$\begin{aligned}
\Theta(\varphi,\varsigma,\mu(s),\nu(s)) &= \sum_{k=0}^{\infty}\frac{\phi_k\mu(s)}{\nu(s)^{1+k\gamma}},\\
\Xi(\varphi,\varsigma,\mu(s),\nu(s)) &= \sum_{k=0}^{\infty}\frac{\xi_k\mu(s)}{\nu(s)^{1+k\gamma}}.
\end{aligned} \tag{90}$$

Now, the first n-th terms of series (90) will be written as follow

$$\begin{aligned}
\Theta_n(\varphi,\varsigma,\mu(s),\nu(s)) &= -\sin(\varphi+\varsigma)\frac{\mu(s)}{\nu(s)}+\sum_{k=1}^{n}\frac{\phi_k\mu(s)}{\nu(s)^{1+k\gamma}},\\
\Xi_n(\varphi,\varsigma,\mu(s),\nu(s)) &= \sin(\varphi+\varsigma)\frac{\mu(s)}{\nu(s)}+\sum_{k=1}^{n}\frac{\xi_k\mu(s)}{\nu(s)^{1+k\gamma}}.
\end{aligned} \tag{91}$$

Thus, according to (54), we have

$$\begin{aligned}
\mathbb{T}_{\frac{t^\rho}{\rho}}Res_\Theta(\varphi,\varsigma,\mu(s),\nu(s)) &= \Theta(\varphi,\varsigma,\mu(s),\nu(s))+\sin(\varphi+\varsigma)\frac{\mu(s)}{\nu(s)}-\frac{\mu(s)}{\nu(s)^\gamma}\left\{\rho\left(\frac{\partial^2\Theta}{\partial\varphi^2}+\frac{\partial^2\Theta}{\partial\varsigma^2}\right)-\Theta\frac{\partial\Theta}{\partial\varphi}-\Xi\frac{\partial\Theta}{\partial\varsigma}+\hbar\right\},\\
\mathbb{T}_{\frac{t^\rho}{\rho}}Res_\Xi(\varphi,\varsigma,\mu(s),\nu(s)) &= \Xi(\varphi,\varsigma,\mu(s),\nu(s))-\sin(\varphi+\varsigma)\frac{\mu(s)}{\nu(s)}-\frac{\mu(s)}{\nu(s)^\gamma}\left\{\rho\left(\frac{\partial^2\Xi}{\partial\varphi^2}+\frac{\partial^2\Xi}{\partial\varsigma^2}\right)-\Theta\frac{\partial\Xi}{\partial\varphi}-\Xi\frac{\partial\Xi}{\partial\varsigma}-\hbar\right\},
\end{aligned} \tag{92}$$

and the n-th of equations (92) will be obtained as follows

$$\begin{aligned}
\mathbb{T}_{\frac{t^\rho}{\rho}}Res_{\Theta_n}(\varphi,\varsigma,\mu(s),\nu(s)) &= \Theta_n(\varphi,\varsigma,\mu(s),\nu(s))+\sin(\varphi+\varsigma)\frac{\mu(s)}{\nu(s)}-\frac{\mu(s)}{\nu(s)^\gamma}\left\{\rho\left(\frac{\partial^2\Theta_n}{\partial\varphi^2}+\frac{\partial^2\Theta_n}{\partial\varsigma^2}\right)-\Theta_n\frac{\partial\Theta_n}{\partial\varphi}-\Xi_n\frac{\partial\Theta_n}{\partial\varsigma}+\hbar\right\},\\
\mathbb{T}_{\frac{t^\rho}{\rho}}Res_{\Xi_n}(\varphi,\varsigma,\mu(s),\nu(s)) &= \Xi_n(\varphi,\varsigma,\mu(s),\nu(s))-\sin(\varphi+\varsigma)\frac{\mu(s)}{\nu(s)}-\frac{\mu(s)}{\nu(s)^\gamma}\left\{\rho\left(\frac{\partial^2\Xi_n}{\partial\varphi^2}+\frac{\partial^2\Xi_n}{\partial\varsigma^2}\right)-\Theta_n\frac{\partial\Xi_n}{\partial\varphi}-\Xi_n\frac{\partial\Xi_n}{\partial\varsigma}-\hbar\right\}.
\end{aligned} \tag{93}$$

3. for $n=1$, we have

$$\begin{aligned}
\Theta_1(\varphi,\varsigma,\mu(s),\nu(s)) &= -\sin(\varphi+\varsigma)\frac{\mu(s)}{\nu(s)}+\frac{\phi_1\mu(s)}{\nu(s)^{1+\gamma}}\\
\Xi_1(\varphi,\varsigma,\mu(s),\nu(s)) &= \sin(\varphi+\varsigma)\frac{\mu(s)}{\nu(s)}+\frac{\xi_1\mu(s)}{\nu(s)^{1+\gamma}}
\end{aligned} \tag{94}$$

Therefore, we have

$$
\begin{aligned}
&\mathbb{T}_{\frac{t^\rho}{\rho}} Res_{\Theta_1}(\varphi,\varsigma,\mu(s),\nu(s)) = \Theta_1(\varphi,\varsigma,\mu(s),\nu(s)) + \sin(\varphi+\varsigma)\frac{\mu(s)}{\nu(s)} \\
&+ \frac{\mu(s)}{\nu(s)^\gamma}\left\{\rho\left(\frac{\partial^2\Theta_1}{\partial\varphi^2} + \frac{\partial^2\Theta_1}{\partial\varsigma^2}\right) + \Theta_1\frac{\partial\Theta_1}{\partial\varphi} + \Xi_1\frac{\partial\Theta_1}{\partial\varsigma} - \hbar\right\} \\
=&\frac{\phi_1\mu(s)}{\nu(s)^{1+\gamma}} - \frac{\mu(s)}{\nu(s)^\gamma}\left\{\rho\left(2\sin(\varphi+\varsigma)\frac{\mu(s)}{\nu(s)} + \frac{\partial^2\phi_1}{\partial\varphi^2}\frac{\mu(s)}{\nu(s)^{1+\gamma}} + \frac{\partial^2\phi_1}{\partial\varsigma^2}\frac{\mu(s)}{\nu(s)^{1+\gamma}}\right)\right\} \\
&+ \frac{\mu(s)}{\nu(s)^\gamma}\left(-\sin(\varphi+\varsigma)\frac{\mu(s)}{\nu(s)} + \frac{\phi_1\mu(s)}{\nu(s)^{1+\gamma}}\right)\left(-\cos(\varphi+\varsigma)\frac{\mu(s)}{\nu(s)} + \frac{\partial\phi_1}{\partial\varphi}\frac{\mu(s)}{\nu(s)^{1+\gamma}}\right) \\
&+ \frac{\mu(s)}{\nu(s)^\gamma}\left(\sin(\varphi+\varsigma)\frac{\mu(s)}{\nu(s)} + \frac{\xi_1\mu(s)}{\nu(s)^{1+\gamma}}\right)\left(-\cos(\varphi+\varsigma)\frac{\mu(s)}{\nu(s)} + \frac{\partial\phi_1}{\partial\varphi}\frac{\mu(s)}{\nu(s)^{1+\gamma}}\right) - \hbar\frac{\mu(s)}{\nu(s)^{1+\gamma}}, \\
&\mathbb{T}_{\frac{t^\rho}{\rho}} Res_{\Xi_1}(\varphi,\varsigma,\mu(s),\nu(s)) = \Xi_1(\varphi,\varsigma,\mu(s),\nu(s)) - \sin(\varphi+\varsigma)\frac{\mu(s)}{\nu(s)} \\
&- \frac{\mu(s)}{\nu(s)^\gamma}\left(\rho\left(\frac{\partial^2\Xi_1}{\partial\varphi^2} + \frac{\partial^2\Xi_1}{\partial\varsigma^2}\right) + \Theta_1\frac{\partial\Xi_1}{\partial\varphi} + \Xi_1\frac{\partial\Xi_1}{\partial\varsigma} + \hbar\right) \\
=&\frac{\xi_1\mu(s)}{\nu(s)^{1+\gamma}} - \frac{\mu(s)}{\nu(s)^\gamma}\left\{\rho\left(-2\sin(\varphi+\varsigma)\frac{\mu(s)}{\nu(s)} + \frac{\partial^2\xi_1}{\partial\varphi^2}\frac{\mu(s)}{\nu(s)^{1+\gamma}} + \frac{\partial^2\xi_1}{\partial\varsigma^2}\frac{\mu(s)}{\nu(s)^{1+\gamma}}\right)\right\} \\
&+ \frac{\mu(s)}{\nu(s)^\gamma}\left(-\sin(\varphi+\varsigma)\frac{\mu(s)}{\nu(s)}\frac{\phi_1\mu(s)}{\nu(s)^{1+\gamma}}\right)\left(\cos(\varphi+\varsigma)\frac{\mu(s)}{\nu(s)} + \frac{\partial\xi_1}{\partial\varphi}\frac{\mu(s)}{\nu(s)^{1+\gamma}}\right) \\
&+ \frac{\mu(s)}{\nu(s)^\gamma}\left(\sin(\varphi+\varsigma)\frac{\mu(s)}{\nu(s)} + \frac{\xi_1\mu(s)}{\nu(s)^{1+\gamma}}\right)\left(\cos(\varphi+\varsigma)\frac{\mu(s)}{\nu(s)} + \frac{\partial\xi_1}{\partial\varsigma}\frac{\mu(s)}{\nu(s)^{1+\gamma}}\right) + \hbar\frac{\mu(s)}{\nu(s)^{1+\gamma}}.
\end{aligned}
\tag{95}
$$

After multiplying $\nu(s)^{1+\gamma}$ on both sides of equation (95) and taking $_{\nu(s)\to\infty}$, we have

$$
\begin{aligned}
&\lim_{\nu(s)\to\infty}\nu(s)^{1+\gamma}\mathbb{T}_{\frac{t^\rho}{\rho}} Res_{\Xi_1}(\varphi,\varsigma,\mu(s),\nu(s)) = \mu(s)\left(\phi_1 - 2\rho\sin(\varphi+\varsigma) - \hbar\right) = 0, \\
&\phi_1 = 2\rho\sin(\varphi+\varsigma) + \hbar. \\
&\lim_{\nu(s)\to\infty}\nu(s)^{1+\gamma}\mathbb{T}_{\frac{t^\rho}{\rho}} Res_{\Xi_1}(\varphi,\varsigma,\mu(s),\nu(s)) = \mu(s)\left(\xi_1 + 2\rho\sin(\varphi+\varsigma) + \hbar\right), \\
&\xi_1 = -2\rho\sin(\varphi+\varsigma) - \hbar.
\end{aligned}
\tag{96}
$$

Now, the first approximate can be written as follows

$$
\begin{aligned}
\Theta_2(\varphi,\varsigma,\mu(s),\nu(s)) &= -\sin(\varphi+\varsigma)\frac{\mu(s)}{\nu(s)} + (2\rho\sin(\varphi+\varsigma)+\hbar)\frac{\mu(s)}{\nu^{1+\gamma}(s)}, \\
\Xi_2(\varphi,\varsigma,\mu(s),\nu(s)) &= \sin(\varphi+\varsigma)\frac{\mu(s)}{\nu(s)} + (-2\rho\sin(\varphi+\varsigma)-\hbar)\frac{\mu(s)}{\nu^{1+\gamma}(s)},
\end{aligned}
\tag{97}
$$

for $n = 2$, we have

$$
\begin{aligned}
\Theta_2(\varphi,\varsigma,\mu(s),\nu(s)) &= -\sin(\varphi+\varsigma)\frac{\mu(s)}{\nu(s)} + \frac{\phi_1\mu(s)}{\nu(s)^{1+\gamma}} + \frac{\phi_2\mu(s)}{\nu(s)^{1+2\gamma}}, \\
\Xi_2(\varphi,\varsigma,\mu(s),\nu(s)) &= \sin(\varphi+\varsigma)\frac{\mu(s)}{\nu(s)} + \frac{\xi_1\mu(s)}{\nu(s)^{1+\gamma}} + \frac{\xi_2\mu(s)}{\nu(s)^{1+2\gamma}},
\end{aligned}
\tag{98}
$$

Therefore, we have

$$
\begin{aligned}
&\mathbb{T}_{\frac{t^\rho}{\rho}} Res_{\Theta_2}(\varphi,\varsigma,\mu(s),\nu(s)) = \Theta_2(\varphi,\varsigma,\mu(s),\nu(s)) + \sin(\varphi+\varsigma)\frac{\mu(s)}{\nu(s)} \\
&\quad - \frac{\mu(s)}{\nu(s)^\gamma}\left\{\rho\left(\frac{\partial^2\Theta_2}{\partial\varphi^2}+\frac{\partial^2\Theta_2}{\partial\varsigma^2}\right) - \Theta_2\frac{\partial\Theta_2}{\partial\varphi} - \Xi_2\frac{\partial\Theta_2}{\partial\varsigma} + \hbar\right\} = \frac{\phi_1\mu(s)}{\nu(s)^{1+\gamma}} + \frac{\phi_2\mu(s)}{\nu(s)^{1+2\gamma}} \\
&\quad - \frac{\mu(s)}{\nu(s)^\gamma}\left\{\rho\left(2\sin(\varphi+\varsigma)\frac{\mu(s)}{\nu(s)} + \frac{\partial^2\phi_1}{\partial\varphi^2}\frac{\mu(s)}{\nu(s)^{1+\gamma}} + \frac{\partial^2\phi_2}{\partial\varphi^2}\frac{\mu(s)}{\nu(s)^{1+2\gamma}}\right.\right. \\
&\quad \left.\left. + \frac{\partial^2\phi_1}{\partial\varsigma^2}\frac{\mu(s)}{\nu(s)^{1+\gamma}} + \frac{\partial^2\phi_2}{\partial\varsigma^2}\frac{\mu(s)}{\nu(s)^{1+2\gamma}}\right)\right\} \\
&\quad + \frac{\mu(s)}{\nu(s)^\gamma}\left(-\sin(\varphi+\varsigma)\frac{\mu(s)}{\nu(s)} + \frac{\phi_1\mu(s)}{\nu(s)^{1+\gamma}} + \frac{\phi_2\mu(s)}{\nu(s)^{1+2\gamma}}\right) \\
&\quad \times\left(-\cos(\varphi+\varsigma)\frac{\mu(s)}{\nu(s)} + \frac{\partial\phi_1}{\partial\varphi}\frac{\mu(s)}{\nu(s)^{1+\gamma}} + \frac{\partial\phi_2}{\partial\varphi}\frac{\mu(s)}{\nu(s)^{1+2\gamma}}\right) \\
&\quad + \frac{\mu(s)}{\nu(s)^\gamma}\left(\sin(\varphi+\varsigma)\frac{\mu(s)}{\nu(s)} + \frac{\xi_1\mu(s)}{\nu(s)^{1+\gamma}} + \frac{\xi_2\mu(s)}{\nu(s)^{1+2\gamma}}\right) \\
&\quad \times\left(-\cos(\varphi+\varsigma)\frac{\mu(s)}{\nu(s)} + \frac{\partial\phi_1}{\partial\varphi}\frac{\mu(s)}{\nu(s)^{1+\gamma}} + \frac{\partial\phi_2}{\partial\varphi}\frac{\mu(s)}{\nu(s)^{1+2\gamma}}\right) - \hbar\frac{\mu(s)}{\nu(s)^{1+\gamma}}, \\
&\mathbb{T}_{\frac{t^\rho}{\rho}} Res_{\Xi_2}(\varphi,\varsigma,\mu(s),\nu(s)) = \Xi_2(\varphi,\varsigma,\mu(s),\nu(s)) - \sin(\varphi+\varsigma)\frac{\mu(s)}{\nu(s)} \\
&\quad - \frac{\mu(s)}{\nu(s)^\gamma}\left(\rho\left(\frac{\partial^2\Xi_2}{\partial\varphi^2}+\frac{\partial^2\Xi_2}{\partial\varsigma^2}\right) - \Theta_2\frac{\partial\Xi_1}{\partial\varphi} - \Xi_2\frac{\partial\Xi_2}{\partial\varsigma} - \hbar\right) = \frac{\xi_1\mu(s)}{\nu(s)^{1+\gamma}} + \frac{\xi_2\mu(s)}{\nu(s)^{1+2\gamma}} \\
&\quad - \frac{\mu(s)}{\nu(s)^\gamma}\left\{\rho\left(-2\sin(\varphi+\varsigma)\frac{\mu(s)}{\nu(s)} + \frac{\partial^2\xi_1}{\partial\varphi^2}\frac{\mu(s)}{\nu(s)^{1+\gamma}} + \frac{\partial^2\xi_2}{\partial\varphi^2}\frac{\mu(s)}{\nu(s)^{1+2\gamma}}\right.\right. \\
&\quad \left.\left. + \frac{\partial^2\xi_1}{\partial\varsigma^2}\frac{\mu(s)}{\nu(s)^{1+\gamma}} + \frac{\partial^2\xi_2}{\partial\varsigma^2}\frac{\mu(s)}{\nu(s)^{1+2\gamma}}\right)\right\} \\
&\quad + \frac{\mu(s)}{\nu(s)^\gamma}\left(-\sin(\varphi+\varsigma)\frac{\mu(s)}{\nu(s)} + \frac{\phi_1\mu(s)}{\nu(s)^{1+\gamma}} + \frac{\phi_2\mu(s)}{\nu(s)^{1+2\gamma}}\right) \\
&\quad \times\left(\cos(\varphi+\varsigma)\frac{\mu(s)}{\nu(s)} + \frac{\partial\xi_1}{\partial\varphi}\frac{\mu(s)}{\nu(s)^{1+\gamma}} + \frac{\partial\xi_2}{\partial\varphi}\frac{\mu(s)}{\nu(s)^{1+2\gamma}}\right) \\
&\quad + \frac{\mu(s)}{\nu(s)^\gamma}\left(\sin(\varphi+\varsigma)\frac{\mu(s)}{\nu(s)} + \frac{\xi_1\mu(s)}{\nu(s)^{1+\gamma}} + \frac{\xi_2\mu(s)}{\nu(s)^{1+2\gamma}}\right) \\
&\quad \times\left(\cos(\varphi+\varsigma)\frac{\mu(s)}{\nu(s)} + \frac{\partial\xi_1}{\partial\varsigma}\frac{\mu(s)}{\nu(s)^{1+\gamma}} + \frac{\partial\xi_2}{\partial\varsigma}\frac{\mu(s)}{\nu(s)^{1+2\gamma}}\right) + \hbar\frac{\mu(s)}{\nu(s)^{1+\gamma}},
\end{aligned} \tag{99}
$$

After multiplying $\nu(s)^{1+2\gamma}$ on both sides of equation (99) and taking $\lim\limits_{\nu(s)\to\infty}$, we have

$$
\begin{aligned}
&\lim_{\nu(s)\to\infty} \nu(s)^{1+2\gamma}\mathbb{T}_{\frac{t^\rho}{\rho}} Res_{\Theta_2}(\varphi,\varsigma,\mu(s),\nu(s)) = \mu(s)\left(\phi_2 - \rho\frac{\partial^2\phi_1}{\partial\varphi^2} - \rho\frac{\partial^2\phi_1}{\partial\varsigma^2}\right) = 0, \\
&\phi_2 = -4\rho^2\sin(\varphi+\varsigma). \\
&\lim_{\nu(s)\to\infty} \nu(s)^{1+2\gamma}\mathbb{T}_{\frac{t^\rho}{\rho}} Res_{\Xi_2}(\varphi,\varsigma,\mu(s),\nu(s)) = \mu(s)\left(\xi_2 - \rho\frac{\partial^2\xi_1}{\partial\varphi^2} - \rho\frac{\partial^2\xi_1}{\partial\varsigma^2}\right), \\
&\xi_2 = 4\rho^2\sin(\varphi+\varsigma).
\end{aligned} \tag{100}
$$

Therefore, the second approximate can be written as follows

$$
\begin{aligned}
\Theta_2(\varphi,\varsigma,\mu(s),\nu(s)) &= -\sin(\varphi+\varsigma)\frac{\mu(s)}{\nu(s)} + (2\rho\sin(\varphi+\varsigma)+\hbar)\frac{\mu(s)}{\nu(s)^{1+\gamma}} - 4\rho^2\sin(\varphi+\varsigma)\frac{\mu(s)}{\nu(s)^{1+2\gamma}}, \\
\Xi_2(\varphi,\varsigma,\mu(s),\nu(s)) &= \sin(\varphi+\varsigma)\frac{\mu(s)}{\nu(s)} + (-2\rho\sin(\varphi+\varsigma)-\hbar)\frac{\mu(s)}{\nu(s)^{1+\gamma}} + 4\rho^2\sin(\varphi+\varsigma))\frac{\mu(s)}{\nu(s)^{1+2\gamma}},
\end{aligned} \tag{101}
$$

Finally, the n-th approximate can be written as follows

$$\begin{aligned}\Theta_n(\varphi,\varsigma,\mu(s),\nu(s)) =& -\sin(\varphi+\varsigma)\frac{\mu(s)}{\nu(s)}+(2\rho\sin(\varphi+\varsigma)+\hbar)\frac{\mu(s)}{\nu(s)^{1+\gamma}}\\ &-4\rho^2\sin(\varphi+\varsigma)\frac{\mu(s)}{\nu(s)^{1+2\gamma}}\\ &+\cdots+(-1)^{n+1}(2\rho)^n\sin(\varphi+\varsigma)\frac{\mu(s)}{\nu(s)^{1+n\gamma}}, \quad n=0,1,2\cdots.\\ \Xi_n(\varphi,\varsigma,\mu(s),\nu(s)) =& \sin(\varphi+\varsigma)\frac{\mu(s)}{\nu(s)}+(-2\rho\sin(\varphi+\varsigma)-\hbar)\frac{\mu(s)}{\nu(s)^{1+\gamma}}\\ &+4\rho^2\sin(\varphi+\varsigma))\frac{\mu(s)}{\nu(s)^{1+2\gamma}}\\ &+\cdots+(-1)^{n}(2\rho)^n\sin(\varphi+\varsigma)\frac{\mu(s)}{\nu(s)^{1+n\gamma}}, \quad n=0,1,2,\cdots.\end{aligned} \tag{102}$$

If $n\to\infty$, we have

$$\begin{aligned}&\Theta(\varphi,\varsigma,\mu(s),\nu(s)) = -\sin(\varphi+\varsigma)\frac{\mu(s)}{\nu(s)}+(2\rho\sin(\varphi+\varsigma)+\hbar)\frac{\mu(s)}{\nu(s)^{1+\gamma}}\\ &\quad -4\rho^2\sin(\varphi+\varsigma)\frac{\mu(s)}{\nu(s)^{1+2\gamma}}\\ &\quad +\cdots+(-1)^{n+1}(2\rho)^n\sin(\varphi+\varsigma)\frac{\mu(s)}{\nu(s)^{1+n\gamma}}+\cdots, \quad n=0,1,2\cdots.\\ &=\frac{\hbar\mu(s)}{\nu(s)^{1+\gamma}}-\sin(\varphi+\varsigma)\frac{\mu(s)}{\nu(s)}+2\rho\sin(\varphi+\varsigma)\frac{\mu(s)}{\nu(s)^{1+\gamma}}-4\rho^2\sin(\varphi+\varsigma)\frac{\mu(s)}{\nu(s)^{1+2\gamma}}+\cdots,\\ &\Xi(\varphi,\varsigma,\mu(s),\nu(s)) = \sin(\varphi+\varsigma)\frac{\mu(s)}{\nu(s)}+(-2\rho\sin(\varphi+\varsigma)-\hbar)\frac{\mu(s)}{\nu(s)^{1+\gamma}}\\ &\quad +4\rho^2\sin(\varphi+\varsigma))\frac{\mu(s)}{\nu(s)^{1+2\gamma}}\\ &\quad +\cdots+(-1)^{n}(2\rho)^n\sin(\varphi+\varsigma)\frac{\mu(s)}{\nu(s)^{1+n\gamma}}+\cdots, \quad n=0,1,2,\cdots.\\ &=-\frac{\hbar\mu(s)}{\nu(s)^{1+\gamma}}+\sin(\varphi+\varsigma)\frac{\mu(s)}{\nu(s)}-2\rho\sin(\varphi+\varsigma)\frac{\mu(s)}{\nu(s)^{1+\gamma}}+4\rho^2\sin(\varphi-\varsigma)\frac{\mu(s)}{\nu(s)^{1+2\gamma}}-\cdots,\end{aligned} \tag{103}$$

By imposing inverse new $\frac{t^\rho}{\rho}$–generalized transform on both sides of above equations, we obtained as follows

$$\begin{aligned}&\Phi(\varphi,\varsigma,t) = \frac{\hbar}{\Gamma(1+\gamma)}\left(\frac{t^\rho}{\rho}\right)^\gamma\\ &\quad -\sin(\varphi+\varsigma)\left\{1-\frac{2\rho}{\Gamma(1+\gamma)}\left(\frac{t^\rho}{\rho}\right)^\gamma+\frac{4\rho^2}{\Gamma(1+2\gamma)}\left(\frac{t^\rho}{\rho}\right)^{2\gamma}-\frac{8\rho^3}{\Gamma(1+3\gamma)}\left(\frac{t^\rho}{\rho}\right)^{3\gamma}+\cdots\right\}\\ &=\frac{\hbar}{\Gamma(1+\gamma)}\left(\frac{t^\rho}{\rho}\right)^\gamma-\sin(\varphi+\varsigma)\sum_{k=0}^{\infty}\frac{(-1)^k(2\rho)^k}{\Gamma(1+k\gamma)}\left(\frac{t^\rho}{\rho}\right)^{k\gamma}\\ &=\frac{\hbar}{\Gamma(1+\gamma)}\left(\frac{t^\rho}{\rho}\right)^\gamma-\sin(\varphi+\varsigma)E_\gamma\left(-2\rho\left(\frac{t^\rho}{\rho}\right)^\gamma\right).\\ &\Psi(\varphi,\varsigma,t) = -\frac{\hbar}{\Gamma(1+\gamma)}\left(\frac{t^\rho}{\rho}\right)^\gamma\\ &\quad +\sin(\varphi+\varsigma)\left\{1-\frac{2\rho}{\Gamma(1+\gamma)}\left(\frac{t^\rho}{\rho}\right)^\gamma+\frac{4\rho^2}{\Gamma(1+2\gamma)}\left(\frac{t^\rho}{\rho}\right)^{2\gamma}-\frac{8\rho^3}{\Gamma(1+3\gamma)}\left(\frac{t^\rho}{\rho}\right)^{3\gamma}+\cdots\right\}\\ &=-\frac{\hbar}{\Gamma(1+\gamma)}\left(\frac{t^\rho}{\rho}\right)^\gamma+\sin(\varphi+\varsigma)\sum_{k=0}^{\infty}\frac{(-1)^k(2\rho)^k}{\Gamma(1+k\gamma)}\left(\frac{t^\rho}{\rho}\right)^{k\gamma}\\ &=-\frac{\hbar}{\Gamma(1+\gamma)}\left(\frac{t^\rho}{\rho}\right)^\gamma+\sin(\varphi+\varsigma)E_\gamma\left(-2\rho\left(\frac{t^\rho}{\rho}\right)^\gamma\right).\end{aligned} \tag{104}$$

For $\gamma = 1$ we have

$$\begin{aligned}\Phi(\varphi,\varsigma,t) =& \hbar\left(\frac{t^{\rho}}{\rho}\right) - \sin(\varphi+\varsigma)\exp\left(-2\rho\left(\frac{t^{\rho}}{\rho}\right)\right),\\ \Psi(\varphi,\varsigma,t) =& -\hbar\left(\frac{t^{\rho}}{\rho}\right) + \sin(\varphi+\varsigma)\exp\left(-2\rho\left(\frac{t^{\rho}}{\rho}\right)\right).\end{aligned} \tag{105}$$

Several 3D plots of the first equation (85) for varying values of $\gamma = 0.25, 0.5, 0.75$, and 1 are displayed in Figures 11–14.

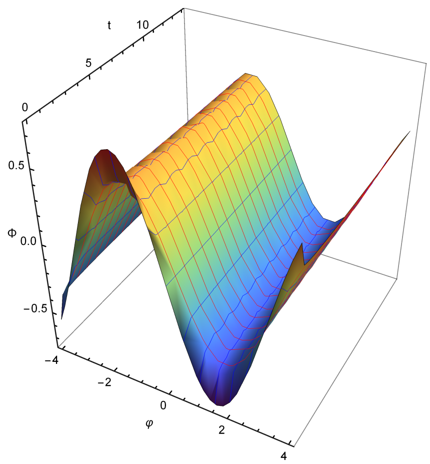

Figure 11. Values of the options for $\rho = \rho = 0.3$, $\gamma = 0.25$, $\hbar = 0$ and $\varsigma = 0$.

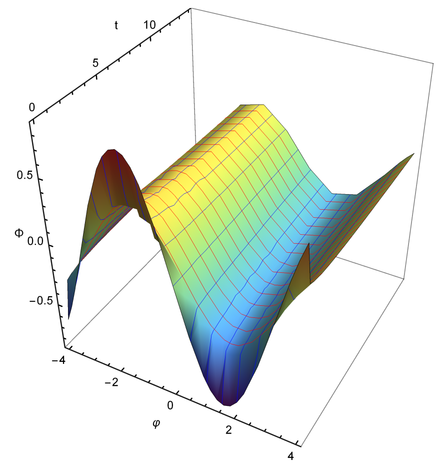

Figure 12. Values of the options for $\rho = \rho = 0.3$, $\gamma = 0.5$, $\hbar = 0$ and $\varsigma = 0$.

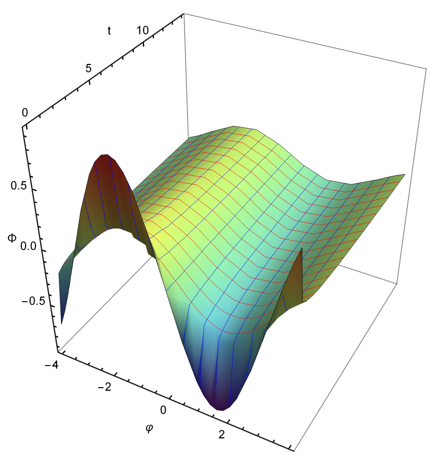

Figure 13. Values of the options for $\rho = \rho = 0.3$, $\gamma = 0.75$, $\hbar = 0$ and $\varsigma = 0$.

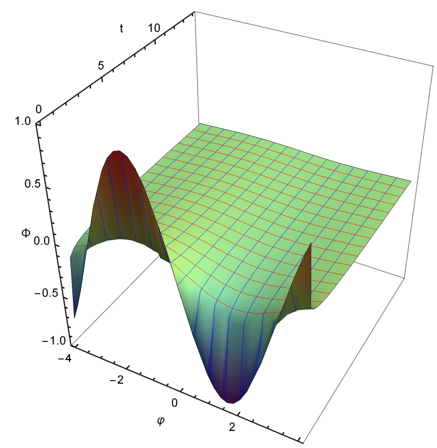

Figure 14. Values of the options for $\rho = \rho = 0.3$, $\gamma = 1$, $\hbar = 0$ and $\varsigma = 0$.

Some 3D-plot of second equation (85) for diverse $\gamma = 0.25, 0.5, 0.75$ and 1 are shown in Figures 15-18.

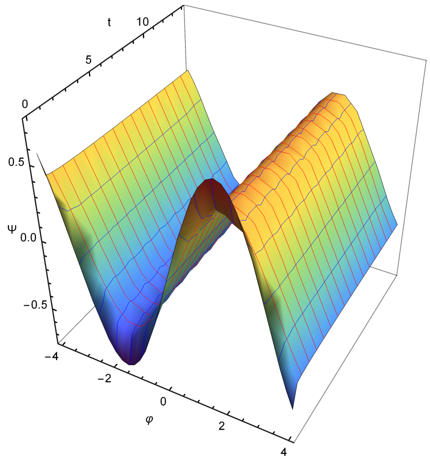

Figure 15. Values of the options for $\rho = \rho = 0.3$, $\gamma = 0.25$, $\hbar = 0$ and $\varsigma = 0$.

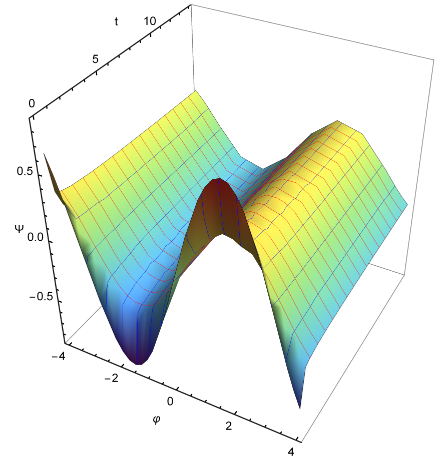

Figure 16. Values of the options for $\rho = \rho = 0.3$, $\gamma = 0.5$, $\hbar = 0$ and $\varsigma = 0$.

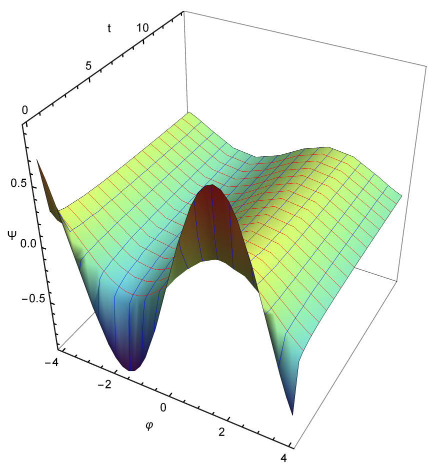

Figure 17. Values of the options for $\rho = \rho = 0.3$, $\gamma = 0.75$, $\hbar = 0$ and $\varsigma = 0$.

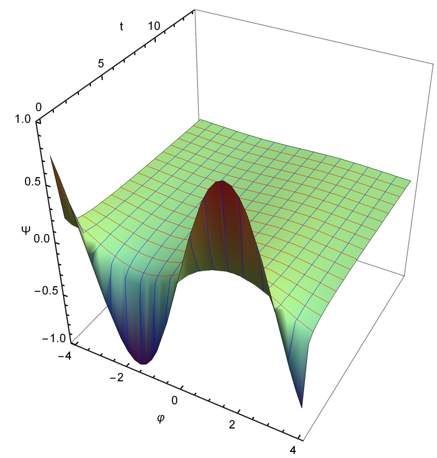

Figure 18. Values of the options for $\rho = \rho = 0.3$, $\gamma = 1$, $\hbar = 0$ and $\varsigma = 0$.

Several 3D-plot of first equation (85) for diverse $\gamma = 0.5$ and 0.8 are shown in Figures 19-20. Several 3D-plot of first equation (85) for diverse $\gamma = 0.5$ and 0.8 are shown in Figures 21-22.

In the aforementioned example, the system of 2D Navier-Stokes equations has been solved using the proposed method. 2D graphs for the first equation have been plotted with parameters $\rho = \rho = 1$, $\hbar = 0$, and $\gamma = 0.4, 0.6, 0.8, 1$; for the second equation, the same parameters have been applied. Additionally, 3D graphs for the first equation have been created with parameters $\rho = \rho = 0.3$, $\hbar = 0$, $\varsigma = 0$, and $\gamma = 0.25, 0.5, 0.75, 1$; and for the second equation with identical parameters. For the first equation under the conditions $\rho = \rho = 0.75$, $\hbar = 0$, $t = 2$, and $\gamma = 0.5, 0.8$; and for the second equation under the same conditions. Comparative tables have also been plotted based on the values $\rho = \rho = 1$, $\hbar = 0$, and $\gamma = 0.25, 0.5, 0.75, 1$. Finally, A comparative analysis of the approximate and exact solutions of $\Phi(\varphi, \varsigma, t)$ and $\Psi(\varphi, \varsigma, t)$ for example 2, together with the absolute error, is performed for $\gamma = 1$, $\rho = 0.3$, $\rho = 1$, and $t = 0.1$, over various values of φ and ς, using the multi–Laplace transform decomposition method (LTDM) and the triple Elzaki transform decomposition method (TETDM) proposed by [38] and [39], respectively.

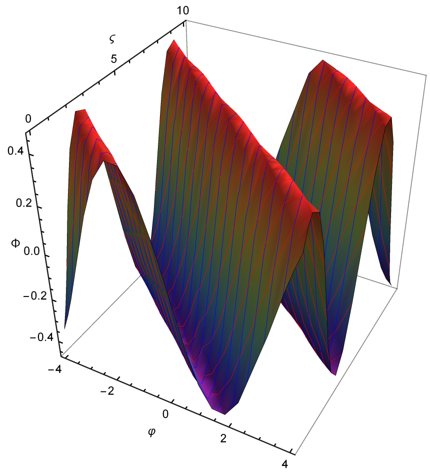

Figure 19. Values of the options for $\rho = \rho = 0.75$, $\gamma = 0.5$, $\hbar = 0$ and $t = 2$.

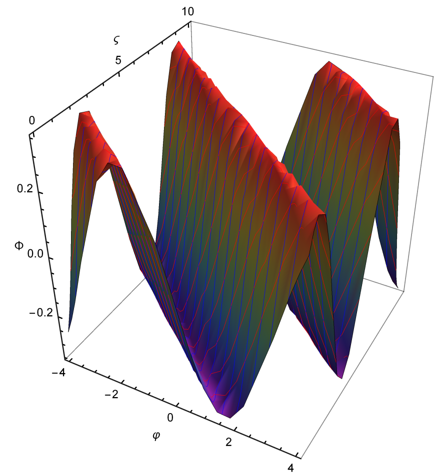

Figure 20. Values of the options for $\rho = \rho = 0.75$, $\gamma = 0.8$, $\hbar = 0$ and $t = 2$.

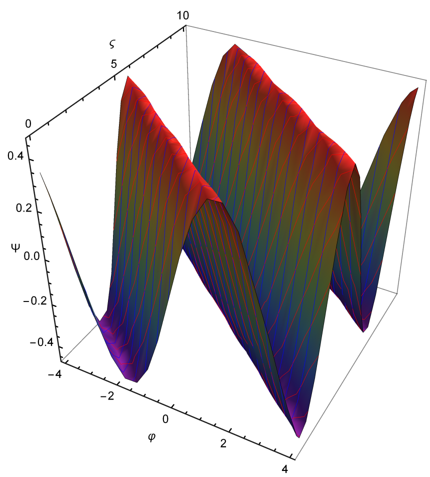

Figure 21. Values of the options for $\rho = \rho = 0.75$, $\gamma = 0.5$, $\hbar = 0$ and $t = 2$.

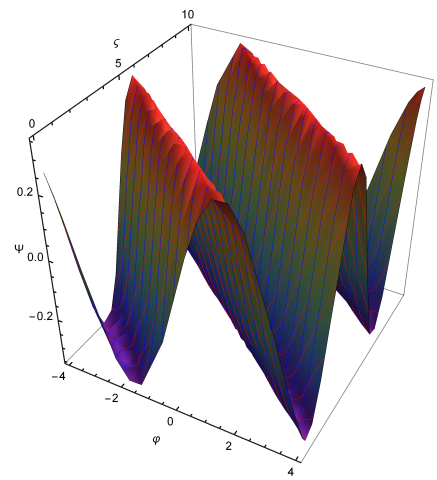

Figure 22. Values of the options for $\rho = \rho = 0.75$, $\gamma = 0.8$, $\rho = 0.3$, $\hbar = 0$ and $t = 2$.

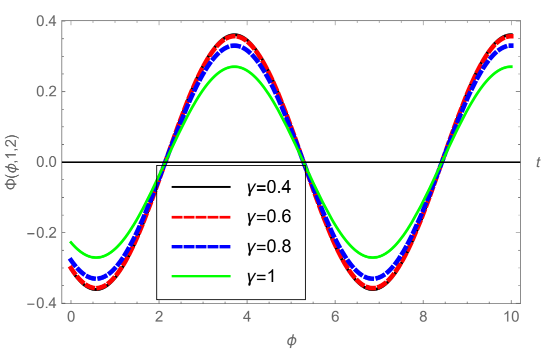

Figure 23. 2D plot of first equation (2) according to $\rho = \rho = 1, \varsigma = 1, t = 2$ and $\hbar = 0$.

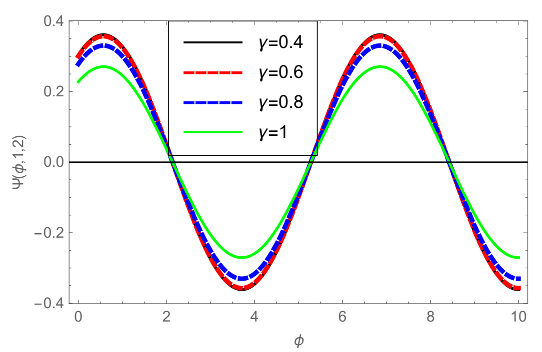

Figure 24. 2D plot of second equation (2) according to $\rho = \rho = 1, \varsigma = 1, t = 2$ and $\hbar = 0$.

5 Conclusion

This study successfully introduced a novel hybrid computational framework, the generalized integral transform residual power series method, specifically designed to address nonlinear two-dimensional time-fractional partial differential equations. By utilizing the Katugampola

Table 2. Solution of example 2 for the first equation according to $\rho = \rho = 1, \hbar = 0$.

φ	ς	t	$\gamma = 0.25$	$\gamma = 0.5$	$\gamma = 0.75$	$\gamma = 1$
1	1	0.00	-0.9092974268	-0.9092974268	-0.9092974268	-0.9092974268
		0.20	-0.3554686590	-0.4166819294	-0.5017853189	-0.6095202930
		0.40	-0.3177800722	-0.3316505416	-0.3557835960	-0.4085736709
		0.60	-0.2966250404	-0.2852617879	-0.2731382519	-0.2738751219
		0.80	-0.2820720498	-0.2545724791	-0.2202091324	-0.1835839843
		1.00	-0.2710837973	-0.2322306313	-0.1837494450	-0.1230600248
1.5	1.5	0.00	-0.1411200081	-0.1411200081	-0.1411200081	-0.1411200081
		0.20	-0.0551675816	-0.0646676824	-0.0778754521	-0.0945955703
		0.40	-0.0493184353	-0.0514710871	-0.0552164588	-0.0634093070
		0.60	-0.0460352431	-0.0442717032	-0.0423901698	-0.0425045296
		0.80	-0.0437766662	-0.0395088221	-0.0341757423	-0.0284916382
		1.00	-0.0420713251	-0.0360414399	-0.0285173172	-0.0190985163
2	2	0.00	0.7568024953	0.7568024953	0.7568024953	0.7568024953
		0.20	0.2958543158	0.3468017335	0.4176327461	0.5072998835
		0.40	0.2644863435	0.2760306474	0.2961164360	0.3400532813
		0.60	0.2468791444	0.2374215812	0.2273312389	0.2279445311
		0.80	0.2347667824	0.2118790637	0.1832786677	0.1527957886
		1.00	0.2256213294	0.1932840851	0.1529335005	0.1024220801

Table 3. Solution of example 2 for the second equation according to $\rho = \rho = 1, \hbar = 0$.

φ	ς	t	$\gamma = 0.25$	$\gamma = 0.5$	$\gamma = 0.75$	$\gamma = 1$
1	1	0.00	0.9092974268	0.9092974268	0.9092974268	0.9092974268
		0.20	0.3554686590	0.4166819294	0.5017853189	0.6095202930
		0.40	0.3177800722	0.3316505416	0.3557835960	0.4085736709
		0.60	0.2966250404	0.2852617879	0.2731382519	0.2738751219
		0.80	0.2820720498	0.2545724791	0.2202091324	0.1835839843
		1.00	0.2710837973	0.2322306313	0.1837494450	0.1230600248
1.5	1.5	0.00	0.1411200081	0.1411200081	0.1411200081	0.1411200081
		0.20	0.0551675816	0.0646676824	0.0778754521	0.0945955703
		0.40	0.0493184353	0.0514710871	0.0552164588	0.0634093070
		0.60	0.0460352431	0.0442717032	0.0423901698	0.0425045296
		0.80	0.0437766662	0.0395088221	0.0341757423	0.0284916382
		1.00	0.0420713251	0.0360414399	0.0285173172	0.0190985163
2	2	0.00	-0.7568024953	-0.7568024953	-0.7568024953	-0.7568024953
		0.20	-0.2958543158	-0.3468017335	-0.4176327461	-0.5072998835
		0.40	-0.2644863435	-0.2760306474	-0.2961164360	-0.3400532813
		0.60	-0.2468791444	-0.2374215812	-0.2273312389	-0.2279445311
		0.80	-0.2347667824	-0.2118790637	-0.1832786677	-0.1527957886
		1.00	-0.2256213294	-0.1932840851	-0.1529335005	-0.1024220801

fractional derivative and a generalized power function transform, we have bridged the gap between transform-based methods and series solutions. The numerical results indicate that the proposed method provides high-order precision and rapid convergence rates for both the Navier–Stokes and Black–Scholes models. The primary advantage of this approach lies in its ability to handle nonlinearities without the need for linearization, discretization, or large computational fluctuations. However, the method is not without limitations; the symbolic

Table 4. Compression between terms approximate solution 85 for the first equation according to $\rho = \rho = 1, \gamma = 1$ and $\hbar = 0$.

φ	ς	t	20 term	30 term	40 term
1	1	0.00	-0.909297426826	-0.909297426826	-0.909297426826
		0.20	-0.416681908261	-0.416681929378	-0.416681929378
		0.40	-0.331630695081	-0.331650540974	-0.331650541581
		0.60	-0.284185362392	-0.285261535675	-0.285261787847
		0.80	-0.236370782757	-0.254554398661	-0.254572475708
		1.00	-0.069566663907	-0.231735056834	-0.232230346044
1.5	1.5	0.00	-0.141120008060	-0.141120008060	-0.141120008060
		0.20	-0.064667683328	-0.064667682430	-0.064667682430
		0.40	-0.051472282916	-0.051471087180	-0.051471087150
		0.60	-0.044351214182	-0.044271718545	-0.044271703201
		0.80	-0.041062477851	-0.039510093088	-0.039508822342
		1.00	-0.051574478233	-0.036080403114	-0.036041459407

Table 5. Compression between terms approximate solution (85) for the second equation according to $\rho = \rho = 1, \gamma = 1$ and $\hbar = 0$.

φ	ς	t	20 term	30 term	40 term
1	1	0.00	0.909297426826	0.909297426826	0.909297426826
		0.20	0.416681908261	0.416681929378	0.416681929378
		0.40	0.331630695081	0.331650540974	0.331650541581
		0.60	0.284185362392	0.285261535675	0.285261787847
		0.80	0.236370782757	0.254554398661	0.254572475708
		1.00	0.069566663907	0.231735056834	0.232230346044
1.5	1.5	0.00	0.141120008060	0.141120008060	0.141120008060
		0.20	0.064667683328	0.064667682430	0.064667682430
		0.40	0.051472282916	0.051471087180	0.051471087150
		0.60	0.044351214182	0.044271718545	0.044271703201
		0.80	0.041062477851	0.039510093088	0.039508822342
		1.00	0.051574478233	0.036080403114	0.036041459407

Table 6. Comparison of exact and approximate solutions with absolute errors

	Data Points						
Parameter	1	2	3	4	5	6	7
w_1	1.2	1.4	1.6	1.8	2.0	2.2	2.4
w_2	0.2	0.4	0.6	0.8	1.0	1.2	1.4
Exact solution	0.928061605333306	0.917135159876129	0.761413238647699	0.485480908995482	0.132901818569906	-0.240659546761904	-0.576226061283512
Triple Elzaki DM	0.928061605333849	0.917135159876666	0.761413238648144	0.485480908995766	0.132901818569984	0.240659546762045	0.576226061283849
Approximate solution	0.928061605333302	0.917135159876125	0.761413238647695	0.485480908995479	0.132901818569906	-0.240659546761903	-0.576226061283509
Absolute error	4.108×10^{-15}	4.000×10^{-15}	3.331×10^{-15}	2.109×10^{-15}	5.829×10^{-16}	1.055×10^{-15}	2.554×10^{-15}
LTDM [31]	0.928061605333849	0.917135159876666	0.761413238648144	0.485480908995766	0.132901818569984	0.240659546762045	0.576226061283849

derivation of the residual functional can become computationally expensive for systems with high-dimensional spatial variables or high-order nonlinearities, potentially increasing the demand on memory resources during execution. Future research should aim to extend this framework to solve coupled systems of fractional equations and explore its integration with machine learning algorithms for parameter estimation. Furthermore, investigating the method's performance on irregular domains and boundary-layer problems remains a promising direction for further development. This work not only advances the theoretical framework for solving fractional differential equations but also provides a practical computational tool with significant potential for applications across various scientific and engineering domains.

Table 7. A comparative analysis of the approximate and exact solutions of $\Phi(\varphi,\varsigma,t)$ for example 2, together with the absolute error, is performed for $\gamma=1$, $\rho=0.3$, $\rho=1$, and $t=0.1$, over various values of φ and ς, using the multi–Laplace transform decomposition method (LTDM) and the triple Elzaki transform decomposition method (TETDM) proposed by [38] and [39], respectively.

φ	ς	Exact solution	Approximate solution	Absolute error	LTDM [38]	TETDM [39]
1.2	0.2	-0.928061605333306	-0.928061605333302	$4.107825191113079\times10^{-15}$	-0.928061605333849	-0.928061605333849
1.4	0.4	-0.917135159876129	-0.917135159876125	$3.996802888650564\times10^{-15}$	-0.917135159876666	-0.917135159876666
1.6	0.6	-0.761413238647699	-0.761413238647695	$3.330669073875470\times10^{-15}$	-0.761413238648144	-0.761413238648144
1.8	0.8	-0.485480908995482	-0.485480908995479	$2.109423746787797\times10^{-15}$	-0.485480908995766	-0.485480908995766
2.0	1.0	-0.132901818569906	-0.132901818569906	$5.828670879282072\times10^{-16}$	-0.132901818569984	-0.132901818569984
2.2	1.2	0.240659546761904	0.240659546761903	$1.054711873393899\times10^{-15}$	0.240659546762045	0.240659546762045
2.4	1.4	0.576226061283512	0.576226061283509	$2.553512956637860\times10^{-15}$	0.576226061283849	0.576226061283849

Table 8. A comparative analysis of the approximate and exact solutions of $\Psi(\varphi,\varsigma,t)$ for example 2, together with the absolute error, is performed for $\gamma=1$, $\rho=0.3$, $\rho=1$, and $t=0.1$, over various values of φ and ς, using the multi–Laplace transform decomposition method (LTDM) and the triple Elzaki transform decomposition method (TETDM) proposed by [38] and [39], respectively.

φ	ς	Exact solution	Approximate solution	Absolute error	LTDM [38]	TETDM [39]
1.2	0.2	0.928061605333306	0.928061605333302	$4.107825191113079\times10^{-15}$	0.928061605333849	0.928061605333849
1.4	0.4	0.917135159876129	0.917135159876125	$3.996802888650564\times10^{-15}$	0.917135159876666	0.917135159876666
1.6	0.6	0.761413238647699	0.761413238647695	$3.330669073875470\times10^{-15}$	0.761413238648144	0.761413238648144
1.8	0.8	0.485480908995482	0.485480908995479	$2.109423746787797\times10^{-15}$	0.485480908995766	0.485480908995766
2.0	1.0	0.132901818569906	0.132901818569906	$5.828670879282072\times10^{-16}$	0.132901818569984	0.132901818569984
2.2	1.2	-0.240659546761904	-0.240659546761903	$1.054711873393899\times10^{-15}$	-0.240659546762045	-0.240659546762045
2.4	1.4	-0.576226061283512	-0.576226061283509	$2.553512956637860\times10^{-15}$	-0.576226061283849	-0.576226061283849

Authors' Contributions

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data Availability

All data in the paper are available from the corresponding authors upon reasonable request.

Conflicts of Interest

The authors declare that there is no conflict of interest.

Ethical Considerations

The authors have diligently addressed ethical concerns, such as informed consent, plagiarism, data fabrication, misconduct, falsification, double publication, redundancy, submission, and other related matters.

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