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Research article

Nonlinear Eigenvalue Methods for Quantifying Quantum Entanglement

Abrar Ahmed Naqash[1,*], Fardeen Ahmad Sofi[2], Mohammad Haris Khan[2], Sundus Abdi[3]

[1] Department of Physics, National Institute of Technology Srinagar, Jammu and Kashmir, 190006, India
[2] Department of Physics, University of Kashmir, Srinagar 190006, India
[3] Department of Physics, University of Toronto, Toronto, Ontario M5S 1A7, Canada

* **Corresponding author(s):** abrarnaqash_phy@nitsri.ac.in

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Abstract

We present a hybrid analytical numerical method to evaluate the geometric measure of entanglement for pure multipartite states by formulating the closest separable state problem as a coupled nonlinear eigenvalue condition. We develop a hybrid analytical numerical framework in which a formal perturbative linearization around a reference product state is combined with an iterative fixed-point scheme. The approach combines a Gauss-Seidel block fixed-point iteration with a controlled first order linearization about a stationary reference product state. The perturbative analysis provides local structural insight and initialization guidance, while the iterative method yields accurate numerical estimates of the geometric measure of entanglement. We make explicit and prove an equal multiplier stationarity identity showing that, at an optimum, all block Lagrange multipliers coincide and are fixed by the optimal fidelity to the target state. A normalization preserving linearization is obtained by projecting onto local tangent spaces, which yields an explicit first order correction and a corresponding scalar shift in the effective eigenvalue. We further establish a monotonic block ascent property: the overlap with the target state increases at every update, remains bounded, and converges to a stationary value. For standard three qubit benchmarks, the hybrid solver converges smoothly and reproduces the known exact optima for the GHZ and W states.

Keywords: Quantum information theory, Geometric measure of entanglement, Nonlinear eigenvalue problem, Fixed-point iteration, Perturbation theory, GaussSeidel method

Mathematics Subject Classification (2020): 81P40, 81-08, 65H10, 65H17

1 Introduction

Entanglement is the operational core of quantum information processing, enabling quantum teleportation, superdense coding, cryptographic security, and quantum computation [1–4]. This centrality has motivated a large body of work on quantifying entanglement in ways that are both physically meaningful and computationally tractable [5–7]. Although widely used measures such as concurrence, negativity, entanglement of formation, and the relative entropy of entanglement capture important operational and axiomatic aspects [8–11], their evaluation is often difficult for generic states, and the complexity typically increases sharply in multipartite scenarios. As a consequence, there remains a persistent gap between measures that are conceptually well founded and those that can be computed reliably at scale.

The geometric measure of entanglement (GME) provides an appealing compromise. Introduced in [12] and developed systematically in [13], it quantifies multipartite entanglement by the distance of a pure state to the set of fully separable states, equivalently through the maximal overlap with product states. This geometric formulation yields a clear operational intuition and links naturally to resource-theoretic viewpoints and variational methods [14]. However, the same formulation reveals the principal obstruction: computing the closest product state requires solving a nonconvex optimization over a nonlinear manifold. The associated optimality conditions form a coupled nonlinear eigenvalue problem whose variables are the local product factors and an effective eigenvalue encoding the optimal overlap. Unlike linear eigenvalue problems, the nonlinear coupling produces a self-consistency loop: the best local factor depends on reduced operators determined by the full product state, which in turn depends on those local factors. This nonlinear feedback is the source of both analytical intractability and numerical sensitivity.

Nonlinear operator equations of this type are not unique to entanglement quantification. They arise in quantum information problems involving capacities and additivity questions [15], in nonlinear modifications of quantum dynamics [16], and in variational many-body methods where fixed-point conditions characterize optimal tensor network or mean-field states [17]. In the GME setting, existing approaches include direct variational optimization [18], convex and witness-based toolboxes [19], and semidefinite-programming relaxations [20]. These methods can be effective in special cases, but they often trade interpretability for computation, or accuracy for scalability, and they typically do not expose the underlying nonlinear eigenstructure in a form that yields both analytic control and guaranteed numerical progress.

Here we develop a hybrid analytical numerical framework that is designed around the nonlinear eigenvalue structure of the GME problem. Our first contribution is structural: we prove an "equal-multiplier" stationarity identity showing that, at an optimum, all block Lagrange multipliers coincide and are fixed by the optimal fidelity between the target state and its closest product approximation. This identity elevates what is often treated as a purely numerical fixed-point condition into a sharp analytic constraint that ties local optimality to a single global scalar quantity. Our second contribution is a controlled local expansion: we derive a normalization-preserving first-order linearization around a stationary reference product state by projecting the dynamics onto the corresponding local tangent spaces. This produces an analytically tractable baseline estimate together with an explicit first-order correction to the product state and a corresponding scalar shift in the effective eigenvalue, providing a principled perturbative approximation rather than an ad hoc initialization.

The third contribution is algorithmic and provides a rigorous convergence guarantee aligned with the variational objective. We combine the analytic linearization with a Gauss-Seidel block fixed-point iteration and prove a monotonic block-ascent property: the squared overlap with the target state increases at every update, remains bounded, and converges to a stationary value. This monotonicity furnishes a simple progress certificate and a natural stopping criterion, and it clarifies how the iteration fits within the geometry of the separable manifold. Conceptually, the method bridges two strands that are usually separated in practice: perturbative insight (which yields interpretability and error control near stationary points) and iterative fixed-point solving (which yields global refinement and practical performance). It is also consonant with broader developments in nonlinear eigenvalue computations [21], but specialized here to the constrained, multipartite product-structure setting relevant to quantum entanglement.

In benchmark bipartite and multipartite examples, the hybrid solver exhibits smooth monotonic convergence to known optima, illustrating that the combination of an analytically controlled local model with a provably ascending iteration can deliver both reliability and transparency. More broadly, our results suggest that treating entanglement quantification explicitly as a nonlinear eigenvalue problem is not merely a reformulation: it provides a route to solver designs with built-in structure, certificates, and systematic approximations. This perspective is expected to be useful beyond the GME, for related nonconvex optimization tasks that appear in quantum technologies and many-body quantum information.

2 Nonlinear Eigenvalue Equation

To set the stage, we recall that the geometric measure of entanglement (GME) is for a pure state $|\psi\rangle \in \mathscr{H}_{A_1} \otimes \mathscr{H}_{A_2} \otimes \cdots \otimes \mathscr{H}_{A_N}$ is defined as [12, 13, 22]

$$E_G(|\psi\rangle) = 1 - \Lambda_{\max}^2, \qquad \Lambda_{\max} = \max_{|\phi\rangle \in \mathscr{S}} \left| \langle \phi | \psi \rangle \right|, \tag{1}$$

where $\mathscr{S}$ denotes the set of fully separable pure states. Throughout this work, the target state $|\psi\rangle$ and all product states $|\phi\rangle \in \mathscr{S}$ are assumed to be normalized, i.e., $\|\psi\| = 1$ and $\|\phi\| = 1$, so that $0 \leq \Lambda_{\max} \leq 1$. Now for a bipartite system ($N = 2$), this set reduces to product states

$|\phi_A\rangle \otimes |\phi_B\rangle$. Determining $\Lambda_{\max}$ therefore corresponds to solving the constrained optimization problem.

$$\max_{|\phi_A\rangle, |\phi_B\rangle} \left| \langle \phi_A \otimes \phi_B | \psi \rangle \right|. \tag{2}$$

Since the overlap is nonnegative, maximizing the modulus $\left| \langle \phi_A \otimes \phi_B | \psi \rangle \right|$ is equivalent to maximizing its square $\left| \langle \phi_A \otimes \phi_B | \psi \rangle \right|^2$. We therefore work with the squared modulus in the Lagrangian formulation below, which leads to the same maximizers and simplifies the variational derivatives.

We employ the Lagrange multiplier method to enforce the normalization constraints $\langle \phi_A | \phi_A \rangle = 1$ and $\langle \phi_B | \phi_B \rangle = 1$. Define the Lagrangian functional

$$\mathscr{L}[|\phi_A\rangle, |\phi_B\rangle] = \left| \langle \phi_A \otimes \phi_B | \psi \rangle \right|^2 - \lambda_A (\langle \phi_A | \phi_A \rangle - 1) - \lambda_B (\langle \phi_B | \phi_B \rangle - 1). \tag{3}$$

Here the Lagrange multipliers λ_A and λ_B are taken to be real, reflecting the real-valued normalization constraints. Variations are performed in the standard complex variational framework by treating $|\phi\rangle$ and $\langle \phi|$ as independent variables (equivalently, using Wirtinger calculus), so that stationarity with respect to $\langle \phi_A|$ and $\langle \phi_B|$ yields the corresponding Euler-Lagrange equations. To make the variational step explicit, define the scalar overlap

$$c := \langle \phi_A \otimes \phi_B | \psi \rangle = \langle \phi_A | v_A \rangle, \qquad |v_A\rangle := (\mathbb{I}_A \otimes \langle \phi_B|) |\psi\rangle.$$

Then the objective can be written as $|c|^2 = \langle \phi_A | v_A \rangle \langle v_A | \phi_A \rangle = \langle \phi_A| \, |v_A\rangle\langle v_A| \, |\phi_A\rangle$. Treating $\langle \phi_A|$ and $|\phi_A\rangle$ as independent variables, the variation with respect to $\langle \phi_A|$ yields $\frac{\partial}{\partial \langle \phi_A|} |c|^2 = |v_A\rangle\langle v_A| \, |\phi_A\rangle$. The stationarity of $\mathscr{L}$ with respect to the variations in $\langle \phi_A|$ gives

$$\frac{\partial \mathscr{L}}{\partial \langle \phi_A|} = \langle \phi_B | \psi \rangle \langle \psi | \phi_B \rangle |\phi_A\rangle - \lambda_A |\phi_A\rangle = 0. \tag{4}$$

Introducing the (positive semidefinite) operator

$$M_A(|\phi_B\rangle) = \langle \phi_B | \psi \rangle \langle \psi | \phi_B \rangle, \tag{5}$$

Here $\langle \phi_B | \psi \rangle$ is a vector in $\mathscr{H}_A$, which we denote by $|v_A\rangle := (\mathbb{I}_A \otimes \langle \phi_B|) |\psi\rangle$. Accordingly, the operator $M_A(|\phi_B\rangle)$ is a positive semidefinite rank-one operator of the form $M_A = |v_A\rangle\langle v_A|$. Its action on a local state is therefore $M_A |\phi_A\rangle = |v_A\rangle \langle v_A | \phi_A \rangle$, so the associated eigenvalue problem is a projection onto the direction $|v_A\rangle$ rather than a generic linear eigenproblem. we can now rewrite this condition as a nonlinear eigenvalue equation,

$$M_A(|\phi_B\rangle) |\phi_A\rangle = \lambda_A |\phi_A\rangle. \tag{6}$$

Similarly, the variation with respect to $\langle \phi_B|$ produces the following result.

$$M_B(|\phi_A\rangle) |\phi_B\rangle = \lambda_B |\phi_B\rangle, \tag{7}$$

where $M_B(|\phi_A\rangle) = \langle \phi_A | \psi \rangle \langle \psi | \phi_A \rangle$. Multiplying equation (6) on the left by $\langle \phi_A|$ and using the normalization $\|\phi_A\| = 1$, we obtain

$$\lambda_A = \langle \phi_A | M_A(|\phi_B\rangle) | \phi_A \rangle = \left| \langle \phi_A \otimes \phi_B | \psi \rangle \right|^2.$$

An identical argument applied to equation (7) yields $\lambda_B = \langle \phi_B | M_B(|\phi_A\rangle) | \phi_B \rangle = \left| \langle \phi_A \otimes \phi_B | \psi \rangle \right|^2$. Hence, at any stationary point, $\lambda_A = \lambda_B = \Lambda^2$, establishing the equal-multiplier identity for the bipartite case.

Equations (6) and (7) exhibit a crucial feature: the eigenvalue problem for one subsystem depends nonlinearly on the state of the other. To make this structure explicit, note that

$$M_A(|\phi_B\rangle) = \mathrm{Tr}_B \left[|\psi\rangle \langle \psi| \left(\mathbb{I}_A \otimes |\phi_B\rangle \langle \phi_B| \right) \right]. \tag{8}$$

Although equation (8) is written as a partial trace, the presence of the rank-one projector $|\phi_B\rangle\langle \phi_B|$ enforces a single-component contraction. As a result, M_A remains a rank-one positive semidefinite operator and is equivalently given by $M_A = |v_A\rangle\langle v_A|$ with $|v_A\rangle = (\mathbb{I}_A \otimes \langle \phi_B|) |\psi\rangle$. This operator depends quadratically on $|\phi_B\rangle$ and therefore $|\phi_A\rangle$ cannot be determined independently. Together, equation (6) and equation (7) form a system of coupled nonlinear eigenvalue equations. In the special case of a bipartite pure state expressed in its Schmidt decomposition,

$$|\psi\rangle = \sum_{i=1}^{r} \lambda_i |u_i\rangle_A \otimes |v_i\rangle_B, \qquad \lambda_1 \geq \lambda_2 \geq \cdots, \tag{9}$$

Here the Schmidt coefficients $\{\lambda_i\}$ are taken to be real and nonnegative and are ordered as $\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$, so that the maximal overlap is $\Lambda_{\max} = \lambda_1$. with the closest product state $|u_1\rangle_A \otimes |v_1\rangle_B$. Here, the nonlinear problem reduces to a linear eigenvalue equation since the Schmidt decomposition provides an orthogonal factorization. For generic multipartite states, however, no such simplification exists, and one must genuinely solve the nonlinear system.

For a tripartite system, the optimization problem equation (2) is generalized to

$$\max_{|\phi_1\rangle,|\phi_2\rangle,|\phi_3\rangle} \left| \langle \phi_1 \otimes \phi_2 \otimes \phi_3 | \psi \rangle \right|. \tag{10}$$

The variation with respect to $|\phi_i\rangle$ yields the following

$$M_i(\{|\phi_j\rangle\}_{j\neq i})\, |\phi_i\rangle = \lambda_i\, |\phi_i\rangle\,, \qquad i = 1,2,3. \tag{11}$$

We now explicitly derive the equality of the Lagrange multipliers at a stationary point. Left-multiplying equation (11) by $\langle \phi_i |$ and using the normalization $\|\phi_i\| = 1$, we obtain

$$\lambda_i = \langle \phi_i | M_i(\{|\phi_j\rangle\}_{j\neq i}) |\phi_i\rangle\,.$$

By the definition of M_i, this expression evaluates to $\lambda_i = \left| \langle \phi_1 \otimes \cdots \otimes \phi_N | \psi \rangle \right|^2 \equiv \Lambda^2$, which is independent of the subsystem index i. Hence, all Lagrange multipliers coincide at a stationary point, $\lambda_1 = \lambda_2 = \cdots = \lambda_N = \Lambda^2$. Now we can write

$$M_i(\{|\phi_j\rangle\}_{j\neq i}) = \mathrm{Tr}_{\{j\neq i\}} \left[|\psi\rangle\, \langle\psi| \bigotimes_{j\neq i} |\phi_j\rangle\, \langle\phi_j| \right]. \tag{12}$$

To make the variational derivation explicit for general N, consider the squared overlap $F(\{|\phi_j\rangle\}) := \left| \langle \phi_1 \otimes \cdots \otimes \phi_N | \psi \rangle \right|^2$. Fix all factors except $|\phi_i\rangle$ and define the partial contraction

$$|v_i\rangle := \left(\mathbb{I}_{A_i} \otimes \langle \phi_{-i} | \right) |\psi\rangle\,, \qquad |\phi_{-i}\rangle := \bigotimes_{j\neq i} |\phi_j\rangle\,.$$

Then F can be written as

$$F = \langle \phi_i | v_i \rangle \, \langle v_i | \phi_i \rangle = \langle \phi_i | \, |v_i\rangle\langle v_i| \, |\phi_i\rangle\,.$$

Introducing a Lagrange multiplier λ_i to enforce $\|\phi_i\| = 1$ and treating $\langle \phi_i |$ and $|\phi_i\rangle$ as independent variables, stationarity with respect to $\langle \phi_i |$ yields $|v_i\rangle\langle v_i|\,|\phi_i\rangle = \lambda_i\,|\phi_i\rangle$. Identifying $M_i = |v_i\rangle\langle v_i|$ and expressing it in partial-trace form gives

$$M_i(\{|\phi_j\rangle\}_{j\neq i}) = \mathrm{Tr}_{j\neq i} \left[|\psi\rangle\langle\psi| \bigotimes_{j\neq i} |\phi_j\rangle\langle\phi_j| \right],$$

which yields the multipartite nonlinear eigen-equation $M_i\,|\phi_i\rangle = \lambda_i\,|\phi_i\rangle$ stated above. These equations are mutually dependent: each operator M_i depends on the states of all other subsystems $\{|\phi_j\rangle\}_{j\neq i}$. The system , equation (11) therefore constitutes a set of self-consistent nonlinear eigenvalue equations that must be solved simultaneously. The coupling strength increases rapidly with the number of subsystems, rendering analytic solutions intractable except for highly symmetric states such as the GHZ and W states. A unifying way to view the above equations is through a fixed-point formulation. Define the map

$$\mathscr{F}\left(\{|\phi_j\rangle\}\right) = \left\{ \frac{M_i(\{|\phi_j\rangle\}_{j\neq i})\,|\phi_i\rangle}{\left\|M_i(\{|\phi_j\rangle\}_{j\neq i})\,|\phi_i\rangle\right\|} \right\}_{i=1}^{N}. \tag{13}$$

The closest separable state corresponds to a fixed point of $\mathscr{F}$,

$$\{|\phi_i^*\rangle\} = \mathscr{F}\left(\{|\phi_i^*\rangle\}\right). \tag{14}$$

The map $\mathscr{F}$ is well defined provided that $M_i(\{|\phi_j\rangle\}_{j\neq i})\,|\phi_i\rangle \neq 0$ for all i. Since each M_i is a rank-one operator of the form $M_i = |v_i\rangle\langle v_i|$, the numerator vanishes whenever $\langle v_i|\phi_i\rangle = 0$. In this degenerate case, the objective $f(\{|\phi_j\rangle\}) = \left| \langle \phi_1 \otimes \cdots \otimes \phi_N | \psi \rangle \right|^2$ is zero and any normalized choice of $|\phi_i\rangle$ leaves f unchanged. Accordingly, the iteration can be extended by defining $\mathscr{F}_i(\{|\phi_j\rangle\}) = |\phi_i\rangle$ whenever $M_i\,|\phi_i\rangle = 0$, which renders the map well defined on the full domain. This convention is consistent with the "safe fallback" used in the numerical algorithm below. This perspective highlights both the difficulty and the potential of the problem, while the structure is nonlinear, it naturally suggests iterative algorithms in which candidate product states are updated recursively until self consistency is achieved. The mathematical properties of $\mathscr{F}$ such as the contractivity and stability of fixed points govern the convergence behavior of such schemes. The nonlinear eigenvalue structure has several important consequences. First, multiple stationary solutions may exist, corresponding to distinct local maxima of the overlap function. Identifying the global maximum $\Lambda_{\max}$ is therefore a nontrivial global optimization task. Second, the equations are not polynomial in the unknowns,the normalization constraints couple to the eigenvalue structure in a transcendental manner. Finally, numerical solvers must balance precision and stability, since naive fixed-point iterations can diverge or converge to suboptimal solutions. These challenges motivate the hybrid perturbativeiterative method introduced in the following section. By starting with a controlled linearization around a reference state and iteratively refining the solution, one can tame the nonlinearities while retaining clear physical interpretability. This strategy draws on insights from variational quantum mechanics [17] and from numerical analysis of nonlinear eigenvalue problems [21], and provides a principled framework tailored to quantum information applications.

Comment 1. (Linearization around the stationary point) To analyze the stability and convergence of the nonlinear eigenvalue equations (6)–(7), we consider a first-order expansion around a stationary solution $\{|\phi_i^{(0)}\rangle, \lambda^{(0)}\}$ satisfying $M_i^{(0)}|\phi_i^{(0)}\rangle = \lambda^{(0)}|\phi_i^{(0)}\rangle$.
Let $|\phi_i\rangle = |\phi_i^{(0)}\rangle + |\delta\phi_i\rangle$ and $\lambda = \lambda^{(0)} + \Delta\lambda$, where the variations $|\delta\phi_i\rangle$ are constrained by the normalization condition $\langle\phi_i^{(0)}|\delta\phi_i\rangle = 0$. Projecting the perturbed equation onto the local tangent space defined by $P_i = \mathbb{I} - |\phi_i^{(0)}\rangle\langle\phi_i^{(0)}|$ yields the normalization-preserving linearized relation

$$P_i\,(M_i^{(0)} - \lambda^{(0)}\mathbb{I})\,P_i\,|\delta\phi_i\rangle = -P_i\,\Delta M_i\,|\phi_i^{(0)}\rangle\,, \tag{15}$$

where ΔM_i denotes the first-order variation of M_i induced by perturbations on the complementary subsystems. Equation (20) forms the basis of the tangent-space correction scheme employed in the hybrid perturbative-iterative algorithm.

3 Solution Method

3.1 Perturbative Formulation

The nonlinear eigenvalue equations derived in the previous section encapsulate the optimization underlying the geometric measure of entanglement, yet their direct solution is generally intractable. We therefore develop a hybrid perturbative-iterative method that combines analytical perturbation theory with fixed-point iteration. The guiding philosophy is twofold: perturbative analysis yields approximate closed-form expressions that elucidate the local structure of the solution space, while iterative refinement exploits the self-consistency of the nonlinear problem to achieve numerical convergence. Together, these techniques provide a practical and conceptually transparent route to the optimal separable approximation. The perturbative expansion should be viewed as a formal linearization that clarifies the local structure of the stationary equations and the role of normalization-preserving tangent directions. While explicit expressions for ΔM_i depend on the detailed parametrization of the complementary subsystems and are not pursued here, the linearized equations provide useful diagnostic information and motivate the subsequent iterative refinement. We begin by assuming access to a candidate separable reference state $|\phi^{(0)}\rangle = \bigotimes_{i=1}^{N}|\phi_i^{(0)}\rangle$, which serves as an expansion point. For highly symmetric states such as GHZ or W states [13,23], a natural reference is the equal superposition of computational basis states, whereas in bipartite systems a product state aligned with the dominant Schmidt vectors often provides a suitable choice. We then parameterize the true stationary solution $|\phi\rangle$ in terms of small deviations around the reference:

$$|\phi_i\rangle = |\phi_i^{(0)}\rangle + \varepsilon\,|\delta\phi_i\rangle\,, \tag{16}$$

where ε is a small bookkeeping parameter controlling the perturbative order. The normalization constraint implies $\langle\phi_i^{(0)}|\delta\phi_i\rangle = 0$, ensuring that $|\delta\phi_i\rangle$ lies in the local tangent space of unit-norm states. Expanding the normalization condition $\langle\phi_i|\phi_i\rangle = 1$ to first order yields $\Re\langle\phi_i^{(0)}|\delta\phi_i\rangle = 0$. The imaginary part of $\langle\phi_i^{(0)}|\delta\phi_i\rangle$ corresponds to an infinitesimal local phase rotation. Throughout this perturbative analysis, we fix this gauge freedom by imposing the phase convention $\langle\phi_i^{(0)}|\delta\phi_i\rangle = 0$, which removes the unphysical phase direction and restricts variations to a gauge-fixed tangent subspace.

3.2 First-Order Linearization

We linearize around a reference product state $\{|\phi_i^{(0)}\rangle\}_{i=1}^N$ that is (at least approximately) stationary under the block update, i.e.,

$$M_i^{(0)}\,|\phi_i^{(0)}\rangle = \lambda^{(0)}\,|\phi_i^{(0)}\rangle\,,$$

for all i, where $M_i^{(0)} := M_i(\{|\phi_j^{(0)}\rangle\}_{j\neq i})$ and the common multiplier is $\lambda^{(0)} = \Lambda_0^2$. We write

$$|\phi_i\rangle = |\phi_i^{(0)}\rangle + \varepsilon\,|\delta\phi_i\rangle\,, \tag{17}$$

subject to the first-order normalization constraint $\Re\langle\phi_i^{(0)}|\delta\phi_i\rangle = 0$. Let $Q_i := |\phi_i^{(0)}\rangle\langle\phi_i^{(0)}|$ and $P_i := \mathbb{I} - Q_i$ denote parallel and tangent projectors, respectively. We define the projectors $Q_i := |\phi_i^{(0)}\rangle\langle\phi_i^{(0)}|, P_i := \mathbb{I} - Q_i$. With the above phase convention, P_i acts as the orthogonal projector onto the gauge-fixed tangent subspace at $|\phi_i^{(0)}\rangle$. While the full tangent space to the unit sphere includes the additional phase direction $i|\phi_i^{(0)}\rangle$, this direction has been removed by the chosen gauge fixing. At zeroth order, the perturbative expansion is performed about a reference product state $\{|\phi_i^{(0)}\rangle\}$ that is (exactly or approximately) stationary under the block updates, in the sense that $M_i^{(0)}\,|\phi_i^{(0)}\rangle = \lambda^{(0)}\,|\phi_i^{(0)}\rangle$ for all i, with a common multiplier $\lambda^{(0)} = \Lambda_0^2$. This assumption is satisfied, for example, by symmetric reference states or by product states obtained from a preliminary fixed-point iteration. For a generic nonstationary reference state, one would obtain block-dependent multipliers $\lambda_i^{(0)}$, and the present first-order equation would have to be modified accordingly. Expanding $M_i = M_i^{(0)} + \varepsilon\,\Delta M_i$ and $\lambda = \lambda^{(0)} + \varepsilon\,\Delta\lambda$, the first-order eigenvalue equation becomes

$$\big(M_i^{(0)} - \lambda^{(0)}\mathbb{I}\big)\,|\delta\phi_i\rangle + \big(\Delta M_i - \Delta\lambda\,\mathbb{I}\big)\,|\phi_i^{(0)}\rangle = 0. \tag{18}$$

To make the linearization explicit, we write the first-order variation of M_i induced by perturbations on the complementary subsystems. Recalling $M_i = \mathrm{Tr}_{j\neq i}\big[|\psi\rangle\langle\psi|\bigotimes_{j\neq i}|\phi_j\rangle\langle\phi_j|\big]$, the first-order correction is

$$\Delta M_i = \sum_{k\neq i}\mathrm{Tr}_{j\neq i}\left[|\psi\rangle\langle\psi|\bigotimes_{\substack{j\neq i\\ j\neq k}}|\phi_j^{(0)}\rangle\langle\phi_j^{(0)}|\otimes\big(|\delta\phi_k\rangle\langle\phi_k^{(0)}| + |\phi_k^{(0)}\rangle\langle\delta\phi_k|\big)\right]. \tag{19}$$

Projecting onto the tangent space removes the gauge freedom parallel to $|\phi_i^{(0)}\rangle$ and yields a well-posed linear system

$$P_i\big(M_i^{(0)} - \lambda^{(0)}\mathbb{I}\big)P_i\,|\delta\phi_i\rangle = -\,P_i\,\Delta M_i\,|\phi_i^{(0)}\rangle\,. \tag{20}$$

In practice, this projected linear system provides a diagnostic first-order correction around a stationary reference state, while the fully nonlinear fixed-point iteration is used for numerical refinement. Projecting equation (18) with Q_i fixes the scalar shift, which is consistent across all i on a stationary background:

$$\Delta\lambda = \langle\phi_i^{(0)}|\Delta M_i|\phi_i^{(0)}\rangle\,, \qquad \text{for each } i. \tag{21}$$

Equations (20)(21) together provide the unique first-order correction $|\delta\phi_i\rangle$ (within the tangent space) and the common shift in eigenvalue $\Delta\lambda$, while maintaining normalization to $\mathscr{O}(\varepsilon)$. On a stationary background with a common zeroth-order multiplier $\lambda^{(0)} = \Lambda_0^2$, the first-order scalar shift $\Delta\lambda$ is independent of the block index i. This follows because all blocks share the same objective $\Lambda^2 = \big|\langle\phi_1\otimes\cdots\otimes\phi_N|\psi\rangle\big|^2$ and the perturbation is taken around a product state that satisfies the equal-multiplier condition at zeroth order. Accordingly, equation (19) yields the same value of $\Delta\lambda$ for every i, and there is a single global eigenvalue shift at first order.

At zeroth order, the eigenvalue is simply

$$\lambda_i^{(0)} = \langle\phi_i^{(0)}|M_i(\{|\phi_j^{(0)}\rangle\}_{j\neq i})|\phi_i^{(0)}\rangle\,, \tag{22}$$

which yields a a baseline overlap to the maximal overlap. At zeroth order, the eigenvalue reduces to $\lambda^{(0)} = \big|\langle\phi^{(0)}|\psi\rangle\big|^2$, which is simply the squared overlap between the chosen reference product state $|\phi^{(0)}\rangle$ and the target state $|\psi\rangle$. This quantity serves as a baseline estimate of the objective value, rather than an approximation to the global maximum $\Lambda_{\max}^2$ unless additional assumptions on the proximity of $|\phi^{(0)}\rangle$ to the optimal product state are imposed. This quantity corresponds to the squared overlap of the chosen reference product state with the target state and serves as a baseline estimate for the subsequent iterative optimization, rather than an approximation to the global optimum in general. The First-order corrections $\Delta\lambda$ are then obtained from equation (20), in direct analogy with the Rayleigh-Schrödinger perturbation theory in quantum mechanics. This procedure provides not only approximate estimates of $\Lambda_{\max}$ but also diagnostic insight into the stability of candidate product states: large negative corrections signal that the chosen reference state is far from the true maximizer. Although perturbation theory elucidates the local structure of the solution space, it cannot capture the full nonlinearity of the problem. To obtain accurate results, we employ an iterative refinement scheme based on the fixed-point structure of stationary equations. Recall the update map

$$|\phi_i^{(k+1)}\rangle = \frac{M_i(\{|\phi_j^{(k)}\rangle\}_{j\neq i})\,|\phi_i^{(k)}\rangle}{\left\|M_i(\{|\phi_j^{(k)}\rangle\}_{j\neq i})\,|\phi_i^{(k)}\rangle\right\|}. \tag{23}$$

For the operators M_i arising from contraction of the pure state $|\psi\rangle$ with projectors on the complementary subsystems, we have the rank-one form $M_i(\{|\phi_j\rangle\}_{j\neq i}) = |v_i\rangle\langle v_i|, \qquad |v_i\rangle := (\mathbb{I}_{A_i}\otimes\langle\phi_{-i}|)\,|\psi\rangle$. Consequently, $M_i\,|\phi_i\rangle = |v_i\rangle\,\langle v_i|\phi_i\rangle$. Provided $\langle v_i|\phi_i\rangle \neq 0$, normalization removes the scalar factor $\langle v_i|\phi_i\rangle$, and the update rule equation (21) reduces to $|\phi_i^{(k+1)}\rangle = \frac{|v_i^{(k)}\rangle}{\||v_i^{(k)}\rangle\|}$, which is exactly the block-wise maximization step used in the monotone-ascent lemma below. Thus, the operator-based map, the fixed-point formulation, and the algorithmic update are equivalent descriptions of the same iteration. Starting from the perturbative reference $\{|\phi_i^{(0)}\rangle\}$, the scheme updates each local factor by applying its corresponding reduced operator and re-normalizing. This normalization is enforced $\langle\phi_i^{(k)}|\phi_i^{(k)}\rangle = 1$ in every iteration. In practice, the update equation (23) corresponds to a Gauss-Seidel block-coordinate ascent on the squared overlap $\Lambda^2 = |\langle\Phi|\psi\rangle|^2$, ensuring monotone convergence under broad conditions discussed below.

3.3 Convergence Considerations.

The convergence of the iterative update equation (23) is not guaranteed a priori, as the map may possess multiple fixed points corresponding to distinct local maxima of the overlap function. However, if the true solution $\{|\phi_i^*\rangle\}$ is isolated and the update map $\mathscr{F}$ is contractive in its neighborhood, then the Banach fixed-point theorem ensures local convergence:

$$\left\|\mathscr{F}(\{|\phi_i\rangle\}) - \mathscr{F}(\{|\phi_i^*\rangle\})\right\| \leq \kappa\left\|\{|\phi_i\rangle\} - \{|\phi_i^*\rangle\}\right\|, \qquad 0 \leq \kappa < 1. \tag{24}$$

Formally, one may view the iteration as a fixed-point map $\mathscr{F}$ on the manifold of normalized product states. Heuristically, if a fixed point $\{|\phi_i^*\rangle\}$ is isolated and $\mathscr{F}$ acts as a contraction in a sufficiently small neighborhood where all updates are well defined, one expects local convergence, in the spirit of the Banach fixed-point theorem. We emphasize, however, that we do not claim a rigorous contractivity result: the norm on the product-state manifold is not specified, and the map $\mathscr{F}$ is only locally defined due to possible vanishing denominators. Accordingly, equation (22) should be interpreted as a qualitative guideline rather than a theorem-level statement. In practice, verifying strict contractivity is challenging, since the map $\mathscr{F}$ depends nonlinearly on all subsystems. Nevertheless, empirical results presented in Sec.4 demonstrate robust convergence for a wide range of initializations. The inclusion of the perturbative initialization step significantly enhances convergence reliability by placing the starting point within the basin of attraction of the global optimum. For completeness, we clarify below the precise relation between the block update used in the convergence proof and the fixed-point map $\mathscr{F}$, as well as the treatment of degenerate updates.

Lemma 1 (Monotone block ascent and convergence to stationarity)**.** *Let* $f(\{|\phi_i\rangle\}) := \left|\langle\phi_1\otimes\cdots\otimes\phi_N|\psi\rangle\right|^2$ *with* $\|\phi_i\| = 1$ *for all i and finite dimensional Hilbert spaces. Given an iterate* $\{|\phi_j^{(k)}\rangle\}$ *define the partial contraction*

$$|v_i^{(k)}\rangle := (\langle\phi_{-i}^{(k)}|)\,|\psi\rangle, \qquad |\phi_{-i}^{(k)}\rangle := \bigotimes_{j\neq i}|\phi_j^{(k)}\rangle,$$

and perform a cyclic (Gauss-Seidel) block update

$$|\phi_i^{(k+1)}\rangle = \frac{|v_i^{(k)}\rangle}{\|\,|v_i^{(k)}\rangle\,\|}, \qquad i = 1,\ldots,N.$$

Then the objective values $f(\{|\phi_i^{(k)}\rangle\})$ *form a non decreasing sequence bounded above by* 1*, and hence converge. Moreover, every limit point of the sequence of iterates is a stationary point of f, i.e.* $M_i(\{|\phi_j\rangle\}_{j\neq i})\,|\phi_i\rangle = \Lambda^2\,|\phi_i\rangle$*, for all i, with the common multiplier*

$$\Lambda^2 = \left|\langle\phi_1\otimes\cdots\otimes\phi_N|\psi\rangle\right|^2.$$

Proof. Fix the complementary blocks $\{|\phi_j\rangle\}_{j\neq i}$ and write $|v_i\rangle = (\langle\phi_{-i}|)\,|\psi\rangle$. Then

$$f(\{|\phi_j\rangle\}) = \big|\,\langle\phi_i|v_i\rangle\,\big|^2 \leq \|\,|\phi_i\rangle\,\|^2\,\|\,|v_i\rangle\,\|^2 = \|\,|v_i\rangle\,\|^2,$$

by the Cauchy-Schwarz inequality [24] , with equality if and only if $|\phi_i\rangle$ is collinear with $|v_i\rangle$. Hence, for fixed other blocks the choice $|\phi_i\rangle \leftarrow |v_i\rangle\,/\|\,|v_i\rangle\,\|$ maximizes f over block i, so each block update cannot decrease the objective. A full cyclic sweep (updating $i = 1,\ldots,N$) therefore produces a non decreasing sequence of objective values. Because $f \leq 1$ for normalized states, the monotone bounded sequence $f(\{|\phi_i^{(k)}\rangle\})$ converges. Let $\{|\phi_i^{(k_m)}\rangle\}$ be a convergent subsequence of iterates with limit $\{|\phi_i^*\rangle\}$. The total improvement of f over one sweep equals the sum of the nonnegative per-block improvements; since the sequence of total improvements converges to zero, each per-block improvement must vanish along the subsequence. By continuity of f and the fact that each block update is the exact maximizer for that block, it follows that for every i the limit $|\phi_i^*\rangle$ attains the maximum of f with the other blocks fixed. Therefore $|\phi_i^*\rangle$ is co-linear with $|v_i^*\rangle = (\langle\phi_{-i}^*|)\,|\psi\rangle$, which is equivalent to $M_i(\{|\phi_j^*\rangle\}_{j\neq i})\,|\phi_i^*\rangle = \Lambda^2\,|\phi_i^*\rangle$ with $\Lambda^2 = \big|\,\langle\phi_1^*\otimes\cdots\otimes\phi_N^*|\psi\rangle\,\big|^2$. Recalling that for the present problem $M_i(\{|\phi_j\rangle\}_{j\neq i}) = |v_i\rangle\langle v_i|$, the condition that $|\phi_i\rangle$ is colinear with $|v_i\rangle$ is equivalent to $M_i\,|\phi_i\rangle = \|\,|v_i\rangle\,\|^2\,|\phi_i\rangle = \Lambda^2\,|\phi_i\rangle$, establishing the equivalence with the stationary eigen-equation. Hence every limit point is a stationary point of f. The statement assumes $|v_i\rangle \neq 0$ for all block updates; if $|v_i\rangle = 0$, the objective value vanishes and any normalized choice of $|\phi_i\rangle$ leaves f unchanged, consistent with the fallback rule used in Algorithm 1. □

The block update appearing in Lemma 3.1 can be written equivalently in terms of the fixed-point map $\mathscr{F}$. Indeed, with $|v_i\rangle = (\langle\phi_{-i}|)\,|\psi\rangle$ one has $M_i(\{|\phi_j\rangle\}_{j\neq i})\,|\phi_i\rangle$ collinear with $|v_i\rangle$, so both updates define the same normalized direction. If $|v_i\rangle = 0$, the objective value vanishes and leaving $|\phi_i\rangle$ unchanged yields a well-defined update. The two componentsperturbative initialization and iterative refinement can be combined into a practical algorithm.

1. Selection of references. Choose a separable reference state $|\phi^{(0)}\rangle$ based on prior knowledge of $|\psi\rangle$ (e.g., Schmidt basis or symmetry considerations).
2. Perturbative expansion; Compute zeroth and first-order corrections to estimate $\Lambda_{\max}$ and refine $|\phi^{(0)}\rangle$ to $|\phi^{(1)}\rangle$.
3. Iterative refinement. Apply the fixed-point update rule (equation 23) until convergence within a chosen tolerance.
4. Selection of solution. If multiple fixed points are found, select the one that yields the largest overlap $\Lambda_{\max}$.

The hybrid strategy combines the interpretability of analytical approximations with the reliability of iterative refinement. Perturbative analysis identifies approximate solutions and clarifies their stability, while the iterative scheme provides high-accuracy numerical results. While each update step involves only local operations on single subsystems, this locality does not by itself remove the exponential scaling associated with a generic multipartite state representation. In particular, computing the partial contractions $|v_i\rangle = (\langle\phi_{-i}|)\,|\psi\rangle$ requires access to all amplitudes of $|\psi\rangle$ and is therefore exponentially costly for dense states. Accordingly, the method does not eliminate exponential overhead in the generic case. Its practical efficiency arises when the target state admits additional structure, such as tensor-network representations, sparsity, or symmetry constraints, where these contractions can be computed efficiently. Limitations arise in strongly entangled systems where no natural reference state lies close to the true maximizer. In such cases, perturbation theory may provide poor initializations, and iteration may converge to suboptimal fixed points. Nevertheless, as demonstrated in the Results section, the method performs remarkably well for a range of canonical bipartite and multipartite states.

3.4 Bipartite Schmidt States

Consider the bipartite pure state

$$|\psi\rangle = \lambda_1|00\rangle + \lambda_2|11\rangle + \lambda_3|22\rangle, \tag{25}$$

with Schmidt coefficients $\{\lambda_i\}$ satisfying $\sum_i \lambda_i^2 = 1$. The geometric measure of entanglement is known to be determined by the largest Schmidt coefficient $\Lambda_{\max} = \max_i \lambda_i$.

We applied the fixed-point iteration equation (23) starting from a random initial product state. Fig.1 shows the convergence of the estimated overlap $\langle\phi_A^{(k)}|\rho_A|\phi_A^{(k)}\rangle$ as a function of the iteration step. Here $\rho_A := \mathrm{Tr}_B\,|\psi\rangle\langle\psi|$ denotes the reduced density operator of subsystem A. The method converges rapidly to the dominant Schmidt coefficient, demonstrating both the stability and accuracy of the iterative refinement scheme.

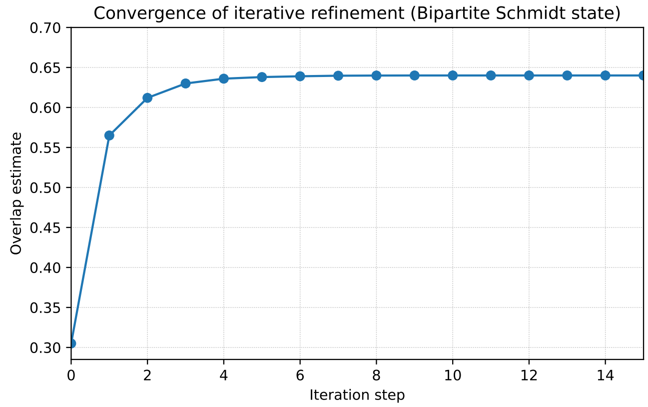

Figure 1. Convergence of the iterative refinement scheme for a bipartite Schmidt state with coefficients $(0.8, 0.6, 0)$. The overlap estimate approaches the largest Schmidt coefficient within a few iterations.

3.5 Multipartite GHZ State and W State

The N-qubit GHZ state is defined as:

$$|\mathrm{GHZ}_N\rangle = \frac{1}{\sqrt{2}}\big(|0\rangle^{\otimes N} + |1\rangle^{\otimes N}\big). \tag{26}$$

For $N = 3$, the maximum overlap with separable states is $\Lambda^2_{\max} = 1/2$, attained by the product states $|0\rangle^{\otimes 3}$ and $|1\rangle^{\otimes 3}$. This value is optimal: any product state has overlap bounded by $1/\sqrt{2}$ with GHZ_3, since the two components $|0\rangle^{\otimes 3}$ and $|1\rangle^{\otimes 3}$ are orthogonal and appear with equal weight [13]. Our algorithm successfully converges to these fixed points, depending on the initialization. This validates that the nonlinear iteration can identify multiple symmetry-related solutions. The three-qubit W state is given by:

$$|W_3\rangle = \frac{1}{\sqrt{3}}\big(|100\rangle + |010\rangle + |001\rangle\big). \tag{27}$$

In contrast to the GHZ state, the maximum overlap with separable states is $\Lambda^2_{\max} = 4/9$ [13], corresponding to product states of the form:

$$|\phi\rangle = \left(\sqrt{\tfrac{2}{3}}|0\rangle + \sqrt{\tfrac{1}{3}}|1\rangle\right)^{\otimes 3}. \tag{28}$$

This example highlights the genuinely nonlinear character of the eigenvalue equations, since the maximizing product state is not aligned with the computational basis. Fig.2 compares the maximum overlaps found numerically for the GHZ and W states using random initializations followed by iterative refinement. For $N = 3$ we have $\Lambda^2_{\max}(\mathrm{GHZ}_3) = \frac{1}{2}$ and $\Lambda^2_{\max}(W_3) = \frac{4}{9}$ [13]. Hence, W is slightly more entangled than GHZ under the geometric measure ($E_G(W_3) = 5/9 > E_G(\mathrm{GHZ}_3) = 1/2$), while GHZ has the highest maximum product overlap. Our iteration converges to symmetry-related maximizers in both cases.

These examples demonstrate that the proposed hybrid method faithfully reproduces known analytical results for canonical states. The iterative scheme converges rapidly to dominant Schmidt components in bipartite cases and distinguishes between qualitatively different types of multipartite entanglement (GHZ vs W). In higher dimensions, where closed-form results are unavailable, the method provides a practical tool for entanglement quantification.

3.6 Numerical Validation

We validate the hybrid Gauss-Seidel block ascent (Algorithm1) on the canonical three-qubit benchmarks GHZ_3 and W_3. In each case, we perform five random initializations with tolerance 10^{-14} and report convergence behavior. We emphasize that the fixed-point map $\mathscr{F}$ and the monotone-ascent lemma are to be understood on the domain where all block updates are well defined.In degenerate cases where $|v_i\rangle = 0$, the objective value is zero and the block update leaves the state unchanged, as implemented explicitly by the safe fallback in algorithm 1.

For GHZ_3 (Fig.3) the method converges in 5 iterations to $\Lambda^2_{\max} = \frac{1}{2}$, with local factors collapsing to computational-basis product states (symmetry-related). For W_3, it converges in 16-18 iterations to $\Lambda^2_{\max} = \frac{4}{9}$, with each single-qubit factor $(\sqrt{2/3}, \sqrt{1/3})$ up to a phase (Fig.4). In all runs the objective $|\langle\Phi^{(k)}|\psi\rangle|^2$ is monotonically non-decreasing, in agreement with the monotone block-ascent lemma.

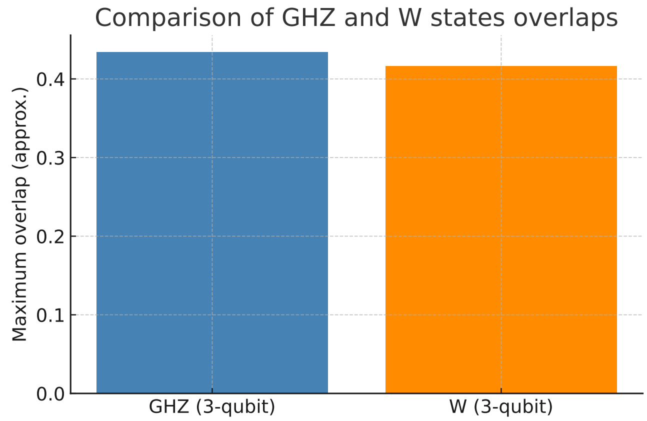

Figure 2. Approximate maximum overlaps with product states for the three-qubit GHZ and W states. The values approach the theoretical predictions $\Lambda^2_{\max} = 1/2$ (GHZ) and $\Lambda^2_{\max} = 4/9$ (W).

Algorithm 1 Hybrid Perturbative-Iterative GME Solver (pure states)

Input: pure state $|\psi\rangle$ on $\mathscr{H}_{A_1} \otimes \cdots \otimes \mathscr{H}_{A_N}$, tolerance $\varepsilon > 0$, (optional) `max_iter`
Initialize product factors $\{|\phi_i^{(0)}\rangle\}$ (symmetry/SVD/perturbative guess), each $\left\| \phi_i^{(0)} \right\| = 1$
$\Lambda_0^2 \leftarrow \left| \langle \phi_1^{(0)} \otimes \cdots \otimes \phi_N^{(0)} | \psi \rangle \right|^2$
$k \leftarrow 0$
repeat
  for $i = 1, \ldots, N$ **do** ▷ Gauss-Seidel block update
    Construct the current product excluding subsystem i:
    $|\phi_{-i}^{(\mathrm{cur})}\rangle \leftarrow \bigotimes_{j<i} |\phi_j^{(k+1)}\rangle \otimes \bigotimes_{j>i} |\phi_j^{(k)}\rangle$
    $|v_i\rangle \leftarrow \left(\langle \phi_{-i}^{(\mathrm{cur})} | \right) |\psi\rangle$
    if $\|v_i\| = 0$ **then**
      $|\phi_i^{(k+1)}\rangle \leftarrow |\phi_i^{(k)}\rangle$ ▷ safe fallback
    else
      $|\phi_i^{(k+1)}\rangle \leftarrow \dfrac{|v_i\rangle}{\|v_i\|}$
    end if
  end for
  $\Lambda_{k+1}^2 \leftarrow \left| \langle \phi_1^{(k+1)} \otimes \cdots \otimes \phi_N^{(k+1)} | \psi \rangle \right|^2$
  $k \leftarrow k+1$
until $\left| \Lambda_k^2 - \Lambda_{k-1}^2 \right| \le \varepsilon$ **or** $k \ge$ `max_iter`
Output: $\Lambda^2_{\max} \approx \Lambda_k^2$, closest product state $\bigotimes_i |\phi_i^{(k)}\rangle$

Table 1. Summary over five random initializations per state (tolerance 10^{-14}). Exact values agree with [13].

State	Seed	Iterations	Final Λ^2
GHZ_3	1-5	5	0.5
W_3	1-5	16-18	0.4444444444

3.7 Equal-Multiplier Stationarity Identity

We now make explicit the identity between the block Lagrange multipliers at a stationary point of the constrained optimization. Consider the stationary conditions $M_i(\{|\phi_j\rangle\}_{j\neq i})\,|\phi_i\rangle = \lambda_i\,|\phi_i\rangle\,, \|\phi_i\| = 1$, arising from the Lagrangian formulation. Multiplying on the left by $\langle\phi_i|$

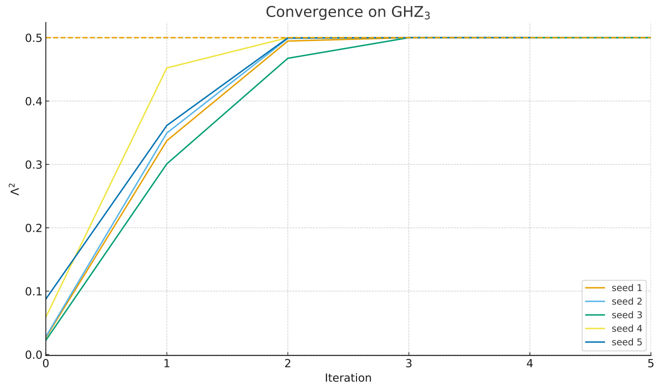

Figure 3. Convergence of Λ^2 for GHZ_3 across five random initializations (iteration index starts at 0). Curves rise monotonically and saturate at the theoretical value $1/2$ (dashed), typically within 2 to 3 iterations; the subsequent plateau reflects numerical tolerance.

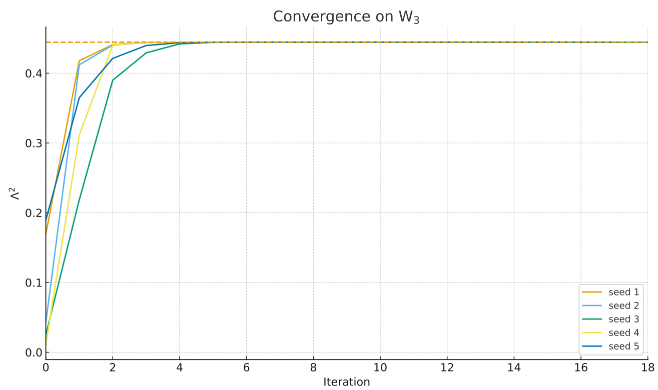

Figure 4. Convergence of Λ^2 for W_3 across five random initializations (iteration index starts at 0). Curves rise monotonically and saturate at the theoretical value $4/9$ (dashed) within a few iterations; the long flat tail indicates convergence to tolerance.

yields $\lambda_i = \langle\phi_i|M_i(\{|\phi_j\rangle\}_{j\neq i})|\phi_i\rangle = \left|\langle\phi_1\otimes\cdots\otimes\phi_N|\psi\rangle\right|^2$, which is independent of the block index i. Hence, at any stationary point of the variational problem, all block multipliers coincide, $\lambda_1 = \cdots = \lambda_N = \Lambda^2$, where Λ^2 is the squared overlap between the optimal product state and the target state. This establishes the equal-multiplier stationarity identity directly from the Lagrange-multiplier conditions. Lemma 3.1 recovers the same identity dynamically for stationary limit points of the block-coordinate ascent algorithm.

4 Results And Discussion

To illustrate the effectiveness of the hybrid perturbativeiterative method, we present results for three representative classes of states: bipartite Schmidt states, the three-qubit GHZ state, and the three-qubit W state. Each case highlights different features of the nonlinear eigenvalue equation and the performance of our solution scheme. The results presented above underscore the interplay between nonlinear mathematics and quantum information theory. While linear algebra provides the foundation of standard quantum mechanics, nonlinear eigenvalue equations arise naturally in optimization-based definitions of quantum resources, most notably in the geometric measure of entanglement. The hybrid perturbativeiterative method developed here illustrates that such nonlinear problems can be tackled with a balance of analytical approximations and numerical self-consistency schemes. The examples of Schmidt states, GHZ, and W states serve both as benchmarks and as archetypes of the diversity of entanglement structures.

A central conceptual message of our work is that entanglement quantification is not merely a matter of diagonalizing reduced density operators, but instead requires solving fixed-point equations whose solutions may be highly nontrivial. This stands in contrast with bipartite pure states, where the Schmidt decomposition reduces the problem to a simple linear one. Multipartite systems, however, do not admit such simplifications, and the self-consistency between subsystems introduces genuine nonlinearity. By framing the search for closest separable states as a nonlinear eigenvalue problem, we provide a unified mathematical perspective that can be leveraged across different contexts.

Beyond entanglement quantification, the methodology has broader implications. Similar nonlinear optimization problems appear in the determination of quantum channel capacities, in fidelity-based measures of coherence, and in variational approaches to many-body quantum states, such as matrix product states and projected entangled pair states [17]. In all of these cases, the iterative fixed-point structure of the equations bears close resemblance to the self-consistent field methods widely used in computational physics and chemistry. Thus, insights from decades of research in numerical analysis and nonlinear optimization can be fruitfully imported into quantum information theory.

From a practical standpoint, the hybrid method offers a trade-off between interpretability and computational power. Perturbative expansions clarify which product states are stable candidates for maximizing overlap, while iterative refinement ensures that the numerical results are accurate and robust. This combination is particularly valuable for medium-sized multipartite systems, where brute-force optimization over product states is infeasible, yet analytical solutions are out of reach. The rapid convergence observed in our simulations suggests that the method may scale more favorably than general-purpose optimization routines, though a detailed complexity analysis remains an open question.

The discussion also highlights limitations and open challenges. Convergence to the global maximum is not guaranteed in the presence of multiple local fixed points, and the quality of the perturbative initialization strongly influences the success of the iteration. For highly entangled states with no clear separable reference, more sophisticated initialization strategies may be required. One promising direction is to integrate machine learning methods to predict good initial product states, effectively combining data-driven heuristics with mathematically rigorous iterative schemes. Another avenue is to adapt tools from convex optimization and semidefinite programming to provide certificates of optimality, complementing the heuristic but efficient iterative refinement.

On the conceptual side, understanding the geometry of the solution landscape remains an intriguing challenge. The fixed-point nonlinear map $\mathscr{F}$ defines a dynamical system in the manifold of product states, with attractors corresponding to locally optimal separable approximations. Characterizing the basins of attraction and their relation to entanglement classes could offer new geometric insights into the structure of multipartite quantum states. This approach resonates with ongoing efforts to connect the theory of entanglement with differential geometry, algebraic geometry, and information geometry.

In summary, the nonlinear eigenvalue perspective provides both a unifying framework and a practical algorithmic tool for quantum information theory. Bridge conceptual questions about the nature of entanglement with computational strategies capable of addressing realistic systems. We anticipate that further development of this approach possibly integrating perturbative analysis, fixed-point iteration, and modern machine learning will not only deepen our theoretical understanding of entanglement but also accelerate its application in emerging quantum technologies.

5 Conclusion

We have developed a systematic framework for addressing the coupled nonlinear eigenvalue equations that underlie entanglement quantification via the geometric measure of entanglement. By integrating a controlled perturbative expansion about stationary reference product states with an iterative fixed-point refinement, the approach combines analytical transparency with numerical robustness. In particular, the structural identities established at stationarity clarify the self-consistent eigenstructure of the optimization, while the normalization-preserving linearization yields a principled local approximation that can be upgraded to a convergent numerical solution. The resulting hybrid solver reproduces the known optima for standard bipartite and canonical multipartite benchmarks and provides a practical route for treating regimes where closed-form solutions are unavailable and black-box variational search becomes fragile.

Beyond its immediate utility as a computational tool, the framework emphasizes a conceptual point: entanglement quantification through the GME is intrinsically nonlinear. The optimal separable approximation is not determined by a single spectral computation, but by a self-consistent fixed-point condition in which local optimality and global fidelity are coupled. From this viewpoint, the relevant mathematical objects are nonlinear eigenpairs on product manifolds, together with their tangent-space dynamics and stability properties, rather than eigenvectors of a fixed linear operator. Algorithmically, our results show that hybrid strategies-combining perturbation theory, fixed-point analysis, and monotone block updates-can tame this nonlinearity with built-in progress certificates and interpretable intermediate approximations, suggesting a general design principle for quantum-resource optimization problems.

Several open directions follow naturally. A first problem is to characterize when the monotone block-ascent dynamics converges to the global optimum rather than to a nonglobal stationary point, and to relate this basin structure to entanglement landscapes, symmetry sectors, and degeneracies. A second problem is to develop acceleration mechanisms with guarantees-for example, Anderson-type mixing, quasi-Newton updates on product manifolds, or trust-region schemes on tangent spaces-that preserve monotonic ascent while improving convergence rates in high-dimensional or weakly gapped instances. A third problem is to quantify analytic error bounds for the first-order correction, including conditions under which the perturbative baseline controls the distance to the true optimum, and to determine whether higher-order tangent-space expansions can be made stable without sacrificing normalization and feasibility. A fourth direction is to extend the framework to structured notions of separability, such as biseparable or k-producible sets, where the constraint geometry changes and new nonlinear eigenstructures emerge.

On the applications side, it is natural to ask how the method performs for families of states relevant to near-term experiments, including noisy preparations, symmetry-constrained ansätze, and high-party-number states with limited tomography. This raises an additional set of questions about incorporating statistical uncertainty, certification from partial information, and exploiting locality or tensor-network structure to reduce computational cost. Finally, it would be valuable to generalize the hybrid philosophy to other quantum-information optimization tasks that share the same self-consistency motif, including coherence quantifiers, variational bounds on channel capacities, and tensor-network variational states, where nonlinear fixed-point equations encode optimality [15, 17]. In these settings, one may seek analogous stationarity identities, tangent-space linearizations, and monotone ascent principles tailored to the corresponding constraint manifolds.

In a broader perspective, nonlinear analysis may become as central to quantum-resource theory as linear algebra has been to the foundations of quantum mechanics. Many operational questions in quantum technologies reduce to constrained, nonconvex, self-consistent optimizations, and progress requires toolkits that jointly deliver rigor, interpretability, and scalability. The framework developed here provides one such step: it makes the nonlinear eigenstructure explicit, supplies analytically controlled local approximations, and couples them to an iterative procedure with guaranteed ascent. We expect these ingredients to be useful well beyond the GME, as quantum information science increasingly demands algorithms whose theoretical guarantees and practical performance advance in tandem.

Authors' Contributions

All authors have the same contribution.

Data Availability

The manuscript has no associated data or the data will not be deposited.

Conflicts of Interest

The authors declare that there is no conflict of interest.

Ethical Considerations

The authors have diligently addressed ethical concerns, such as informed consent, plagiarism, data fabrication, misconduct, falsification, double publication, redundancy, submission, and other related matters.

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