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Research article

Connected 2-Dominating Sets and Connected 2-Dominating Polynomials in Friendship Graphs

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Abstract

Let $G=(V,E)$ be a simple graph. A subset $D\subseteq V$ is called a connected 2-dominating set of G if every vertex in $V\setminus D$ is adjacent to at least two vertices of D, and the induced subgraph $G[D]$ is connected. The minimum size of such a set is referred to as the connected 2-domination number of G, denoted by $\gamma_2^c(G)$. In this work, we investigate the enumeration of connected 2-dominating sets in graphs. For this purpose, we define a generating polynomial, called the connected 2-domination polynomial, which encodes the number of such sets of different cardinalities. Furthermore, several fundamental properties of this polynomial are studied, and explicit forms are derived for certain graph families, in particular the friendship graphs F_n.

Keywords: Connected 2-domination, Connected 2-domination polynomials, Friendship graphs

Mathematics Subject Classification (2020): 05C69, 05C31

1 Introduction

Graph theory, a prominent and widely applicable branch of discrete mathematics, focuses on analyzing the structure and interconnections among different entities. A graph consists of two fundamental components: vertices, representing objects, and edges, representing the relationships among them. Over time, numerous concepts have been introduced within this field to yield deeper insights into these structures. Such concepts have found broad applications in various areas, including computer and communication networks, social network analysis, biological systems, engineering, and economic modeling [1].

Dominating sets constitute a cornerstone of many studies in graph theory. By selecting a subset of vertices that collectively reach every other vertex (via adjacency), one captures essential control or coverage properties relevant to optimization and network design. Beyond the classical definition, researchers have introduced stronger variants to model stricter requirements. One such refinement is the connected 2-dominating set. In this concept, the dominating subset must satisfy two conditions simultaneously. First, it must 2-dominate the graph, meaning that each vertex not in the subset is adjacent to at least two members of it; second, the subgraph induced by the subset must be connected, ensuring internal cohesion and structural integrity [2, 12].

The connected 2-domination polynomial provides a systematic tool for representing and analyzing this specific type of domination in graphs. It not only captures key structural characteristics, but also offers a framework for examining how graphs respond to various modifications. The study of such polynomials is relevant both for advancing theoretical understanding and for supporting the modeling and design of complex systems [3–6, 9, 10, 13, 14].

Formally, for a graph $G = (V,E)$, a subset $D \subseteq V$ is called a connected 2-dominating set if every vertex in $V - D$ is adjacent to at least two vertices of D, while the induced subgraph $G[D]$ is connected. The minimum cardinality of such a set is termed the connected 2-domination number, denoted by $\gamma_2^c(G)$.

The corresponding generating polynomial is defined as

$$D_2^c(G,x) = \sum_{i=\gamma_2^c(G)}^{|V(G)|} d_2^c(G,i)x^i,$$

where $d_2^c(G,i)$ denotes the number of connected 2-dominating sets of cardinality i. Several fundamental properties of this polynomial have been introduced and investigated in previous studies [7, 8, 11, 13, 15].

In this study, we further examine fundamental properties of the connected 2-domination polynomial for specific graph families. In particular, our attention is focused on friendship graphs F_n, which are constructed by augmenting a cycle with a central vertex and joining this vertex to every vertex of the cycle. Owing to their distinctive configuration and broad applications, friendship graphs have been extensively studied in graph theory [6, 16].

We derive the connected 2-domination polynomial for these graphs and analyze its relationship with other well-known graph parameters. The outcomes reveal notable structural regularities in friendship graphs, providing potential directions for future research. Moreover, the insights gained from this study may enhance the understanding of complex graph properties with implications for practical applications.
A connected 2-dominating set in a graph is defined as a vertex subset that meets two essential criteria:

1. Every vertex outside the subset is adjacent to at least two vertices within it.
2. The subgraph induced by the subset is connected, ensuring that all vertices of the subset are reachable through internal paths.

Thus, this notion simultaneously incorporates the requirement of 2-domination along with connectivity, guaranteeing that the dominating vertices form a cohesive and uninterrupted structure. Such a combination is particularly valuable in practical contexts, including communication networks and resource management, where both full coverage and reliable connectivity are of paramount importance.

2 Connected 2-Dominating Sets of the Friendship Graph F_n

In this section, we examine the structure of connected 2-dominating sets in friendship graphs. To build intuition, we first study the cases F_1, F_2, and F_3, and subsequently extend the observations to the general graph F_n.

The graph F_1, the simplest instance of a friendship graph, consists of a single triangle C_3 in which all three vertices are mutually adjacent. We denote its vertex set by $\{v_1, v_2, v_3\}$. The complete adjacency among the vertices provides a useful starting point for understanding the formation of connected 2-dominating sets in larger friendship graphs.

2.1 Definition of a Connected 2-Dominating Set

A connected 2-dominating set of a graph is a vertex subset $D \subseteq V(G)$ such that every vertex in $V(G) \setminus D$ is adjacent to at least two vertices of D, and the induced subgraph $G[D]$ is connected. This definition incorporates both the redundancy requirement of 2-domination and the structural cohesion provided by connectivity.
Case of F_1: Given that all vertices in F_1 are mutually adjacent, the identification of its connected 2-dominating sets is immediate.

- The full vertex set $D = \{v_1, v_2, v_3\}$ is a connected 2-dominating set, since all vertices are included and the induced subgraph is connected.
- Any subset consisting of two vertices, such as $D = \{v_1, v_2\}$, also satisfies the conditions. The remaining vertex (e.g., v_3) is adjacent to both vertices of D, and the induced subgraph remains connected.

Thus, the connected 2-dominating sets of F_1 are:

$$\{v_1, v_2, v_3\}, \quad \{v_1, v_2\}, \quad \{v_1, v_3\}, \quad \{v_2, v_3\}.$$

Case of F_2: The graph F_2 is obtained by joining two triangles C_3 at a common vertex v_5, resulting in a graph with five vertices, each adjacent to at least two others. To determine the connected 2-dominating sets of F_2, we consider the following observations:

- The full vertex set $D = \{v_1, v_2, v_3, v_4, v_5\}$ trivially forms a connected 2-dominating set.
- Any subset of three vertices that includes the central vertex v_5 yields a connected 2-dominating set, provided the remaining vertices are 2-dominated. For instance, the set $D = \{v_1, v_3, v_5\}$ ensures the required domination and induces a connected subgraph.

Consequently, the connected 2-dominating sets of F_2 include:

$$\{v_1, v_2, v_3, v_4, v_5\}, \quad \{v_1, v_4, v_5\}, \quad \{v_2, v_4, v_5\}, \quad \{v_1, v_3, v_5\}.$$

2.2 Counting Connected 2-Dominating Sets for F_3

The friendship graph F_3 is obtained by merging three triangles C_3 at a single common vertex, resulting in a graph with seven vertices. Each vertex in F_3 is adjacent to at least two others, which facilitates the formation of connected 2-dominating sets.

To enumerate all connected 2-dominating sets of F_3, we rely on the following observations:

1. **Inclusion of the central vertex:** The central vertex must be included in every connected 2-dominating set. Excluding it would disconnect the induced subgraph and prevent some vertices from attaining two neighbors in the dominating set.
2. **Selection of vertices from each triangle:** Since F_3 contains three triangles sharing the central vertex, at least one of the two non-central vertices in each triangle must be selected to guarantee 2-domination of the vertices in that triangle. Therefore, each triangle contributes exactly two possible choices to the selection process.
3. **Total number of connected 2-dominating sets:** With three triangles and two valid choices in each, the total number of distinct connected 2-dominating sets is $2 \times 2 \times 2 = 8$.

Hence, every connected 2-dominating set of F_3 consists of the central vertex together with one selected vertex from each of the three triangles, leading to a total of eight distinct sets.

2.3 Connected 2-Dominating Sets in the General Case of F_n

For the general friendship graph F_n, constructed by merging n triangles at a single common vertex v_{2n+1}, the resulting graph contains $2n+1$ vertices and $3n$ edges. Due to this structure, each triangle contributes two non-central vertices, both of which are adjacent to the shared central vertex.

In order to satisfy the requirements of connected 2-domination, one vertex from each triangle must be selected. This guarantees that every non-central vertex is adjacent to at least two vertices in the dominating set, while the inclusion of the common vertex ensures the connectivity of the induced subgraph. Since each triangle offers two possible choices, the total number of connected 2-dominating sets in F_n is 2^n. The reasoning behind this enumeration can be summarized as follows:

1. **Inclusion of the central vertex:** The central vertex must always be part of any connected 2-dominating set; otherwise, the induced subgraph cannot remain connected.
2. **Selection within each triangle:** From each of the n triangles, at least one of the two non-central vertices must be chosen so that every remaining vertex is 2-dominated.
3. **Combinatorial structure:** Since the choices within different triangles are independent, each triangle contributes two possibilities, leading to a total of 2^n valid connected 2-dominating sets.

Thus, for any friendship graph F_n, the number of connected 2-dominating sets is precisely 2^n.

3 The Connected 2-Domination Number of the Friendship Graph F_n

In this section, we determine the connected 2-domination number of the friendship graph F_n. The result is stated in the following theorem.

Theorem 1. *For every integer $n \geq 1$, the connected 2-domination number of the friendship graph F_n is*

$$\gamma_2^c(F_n) = n+1.$$

Proof. The graph F_n is formed by merging n triangles that share a single common vertex v_{2n+1}. Each triangle thus consists of the central vertex together with two peripheral vertices.

Step 1: Construction of a connected 2-dominating set. Label the vertices of F_n as $\{v_1, v_2, \ldots, v_{2n+1}\}$. Consider the set

$$D = \{v_1, v_3, \ldots, v_{2n-1}, v_{2n+1}\}.$$

This set contains exactly one peripheral vertex from each triangle together with the central vertex, and the induced subgraph is clearly connected. Furthermore, every vertex outside D is adjacent to at least two vertices of D. Hence,

$$\gamma_2^c(F_n) \leq n+1.$$

Step 2: Establishing a lower bound. Suppose, to the contrary, that there exists a connected 2-dominating set D with $|D| \leq n$. We examine two cases.

Case 1: If the central vertex $v_{2n+1} \in D$, then with at most $n-1$ additional vertices available, there exists at least one triangle for which neither peripheral vertex is included in D. In such a triangle, at least one vertex fails to be 2-dominated, contradicting the assumption.

Case 2: If the central vertex $v_{2n+1} \notin D$, then no peripheral vertex can be adjacent to two vertices of D, because all triangles intersect only at the central vertex (see Figure 1). □

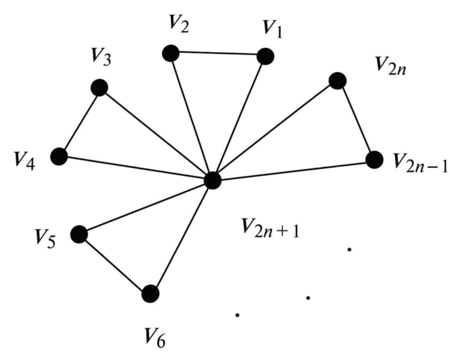

Figure 1. The labeled friendship graph F_n

Thus, the 2-domination requirement cannot be satisfied in this case. In both scenarios, a connected 2-dominating set of size at most n cannot exist. Therefore,

$$|D| \geq n+1.$$

Combining both bounds yields the desired conclusion:

$$\gamma_2^c(F_n) = n+1.$$

4 The Connected 2-Domination Polynomials of the Friendship Graph F_n

In this section, we first determine the connected 2-domination polynomial of the basic friendship graph F_1 and subsequently extend the analysis to the general case of F_n.

Theorem 2. *For the friendship graph F_1, the connected 2-domination polynomial is*

$$D_2^c(F_1,x) = x^3 + 3x^2. \tag{1}$$

Proof. The graph F_1 is isomorphic to the 3-cycle C_3, in which each vertex is adjacent to both of the others. Since the connected 2-domination number of this graph is $\gamma_2^c(F_1) = 2$, the numbers of connected 2-dominating sets of cardinalities 2 and 3 are

$$d_2^c(F_1,2) = 3, \qquad d_2^c(F_1,3) = 1.$$

Applying the definition of the connected 2-domination polynomial yields $D_2^c(F_1,x) = x^3 + 3x^2$. □

The following theorem, which will be invoked later in the proof of Theorem 4, provides a closed-form expression for the 2-domination polynomial of the cycle C_n.

Theorem 3. *For every integer $n \geq 3$, the 2-domination polynomial of the cycle C_n is given by*

$$D_2(C_n,x) = \sum_{i=0}^{n} \left[\binom{n-i}{i} + \binom{n-i-1}{i-1} \right] x^{n-i}. \tag{2}$$

Proof. Let the vertices of C_n be labeled as $v_1, v_2, \ldots, v_n$. Removing the edge $v_1 v_n$ produces the path P_n on the same vertex set. Let D be a connected 2-dominating set of P_n. Such sets can be constructed in two distinct ways:

Case 1: Both endpoints v_1 and v_n belong to D. The remaining vertices $v_2, \ldots, v_{n-1}$ may then be selected in any admissible configuration that preserves connected 2-domination. In this case, the number of sets is

$$d_2(C_n,i) = \binom{n-i-1}{i-1}, \qquad 0 \leq i \leq n.$$

Case 2: Introduce an auxiliary vertex v_{n+1} connected to v_n, thereby forming the path P_{n+1}. In this situation, the definition requires that both v_1 and v_{n+1} are contained in D. Applying arguments analogous to those in Case 1 but on P_{n-1}, we obtain

$$d_2(C_n,i) = \binom{n-i}{i}, \qquad 0 \leq i \leq n.$$

Combining the contributions from both cases and substituting them into the definition of the 2-domination polynomial yields the stated expression. See [8, 10, 11] for related discussions. □

4.1 The Connected 2-Domination Polynomial of the Friendship Graph F_n

To obtain the explicit form of the connected 2-domination polynomial of the friendship graph F_n, it is essential to ensure that every 2-dominating set also induces a connected subgraph. Thus, any connected 2-dominating set must simultaneously satisfy both the 2-domination requirement and the connectivity constraint.

Theorem 4. *For every integer $n \geq 2$, the connected 2-domination polynomial of the friendship graph F_n is*

$$D_2^c(F_n,x) = x^{2n+1} + (2n+1)x^{2n} + \sum_{i=0}^{n-2} \binom{n}{i} 2^{n-i} x^{n+1+i}. \tag{3}$$

Proof. The formulation follows from the structural properties of F_n, which consists of n triangles sharing a common central vertex.

1. **Inclusion of the central vertex.** The central vertex v_{2n+1} must appear in every connected 2-dominating set. Since it is adjacent to all other vertices, its presence guarantees that the induced subgraph remains connected and serves as the hub linking the triangular components.

2. **Selection of vertices in each triangle.** To satisfy 2-domination, at least one of the two non-central vertices from each triangle must be included. This requirement also maintains connectivity, since each chosen vertex is linked directly to the central vertex.

3. **Counting valid configurations.** For a given value of i, the number of connected 2-dominating sets that include the central vertex and maintain domination across the triangles is
$$\binom{n}{i}2^{n-i},$$
which accounts for all permissible ways of selecting non-central vertices from the n triangular components.

4. **Boundary cases.** When $i = 2n$, the connected 2-dominating sets correspond to choosing all but one vertex, yielding exactly $2n+1$ such sets. When $i = 2n+1$, the only possible set is the entire vertex set $V(F_n)$.

Collecting all contributions leads to the stated expression:
$$D_2^c(F_n,x) = x^{2n+1} + (2n+1)x^{2n} + \sum_{i=0}^{n-2}\binom{n}{i}2^{n-i}x^{n+1+i}. \tag{3}$$

This completes the proof. □

5 Conclusion

The expression corresponding to relation (3) in Theorem 4 can be reformulated in the following adjusted form.

Result 1. For every integer $n \geq 2$, the connected 2-domination polynomial of the friendship graph F_n is given by
$$D_2^c(F_n,x) = x^{2n+1} + \sum_{i=1}^{n-1}\left[\binom{2n-i}{i} + \binom{2n-i-1}{i-1} + \binom{n}{i-1}\right]x^{2n+1-i} + 2^n x^{n+1}. \tag{4}$$

Proof. 1. **Inclusion of the central vertex.** Given the structural configuration of F_n, the central vertex v_{2n+1} must be contained in every connected 2-dominating set. Its presence guarantees that all selected vertices remain part of a connected induced subgraph.

2. **Enumeration by cardinality.** When $i = n+1$, precisely 2^n connected 2-dominating sets exist, each corresponding to the choice of one non-central vertex from every triangle. For $i = 2n+1$, the only connected 2-dominating set is the full vertex set $V(F_n)$.

3. **Intermediate ranges.** For values $n+2 \leq i \leq 2n-1$, preserving connectivity requires viewing F_n as a modified cycle C_{2n} with additional edges incident to the central vertex.

Using the counting technique from Theorem 3 and applying induction on n, the number of connected 2-dominating sets in this range can be determined. In conclusion, by combining the boundary cases with the enumerated intermediate cases, we obtain the complete expression for the connected 2-domination polynomial of F_n, as stated in equation (4). □

Authors' Contributions

All authors have the same contribution.

Data Availability

The manuscript has no associated data or the data will not be deposited.

Conflicts of Interest

The author declares that there is no conflict of interest.

Ethical Considerations

The author has diligently addressed ethical concerns, such as informed consent, plagiarism, data fabrication, misconduct, falsification, double publication, redundancy, submission, and other related matters.

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