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On the Existence and Approximation of Solutions to a Nonlinear Schrödinger-Type Equation

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Abstract

We investigate the mathematical structure and approximate solutions of a nonlinear equation arising in quantum mechanics. Specifically, we study a stationary nonlinear Schrödinger equation with a cubic nonlinearity, which is relevant in models of Bose–Einstein condensates and nonlinear optics. We establish existence results under suitable boundary conditions and develop a perturbative approximation scheme for small coupling. Numerical experiments illustrate the validity of the approximation and the emergence of localized states.

Keywords: Bose-Einstein condensates, Perturbative expansion, Harmonic oscillator potential

Mathematics Subject Classification (2020): 35Q55, 81Q05, 81V80, 35A01

1 Introduction

The study of nonlinear variants of the Schrödinger equation occupies a central position at the interface between mathematical analysis and quantum physics. While the linear Schrödinger equation has been the cornerstone of quantum theory since its inception, it has long been understood that effective nonlinear models emerge naturally in a variety of many-body and mean-field contexts. Perhaps the most prominent example is the Gross–Pitaevskii equation, introduced independently by Gross [1] and Pitaevskii [2], which governs the macroscopic wavefunction of a dilute Bose–Einstein condensate at very low temperatures. This model, which augments the linear Schrödinger operator by a cubic self-interaction term, has proven remarkably successful in describing experiments with ultracold atomic gases [3, 4]. Similar nonlinear equations also arise in nonlinear optics [5], plasma physics [6], and models of self-trapped electronic states in condensed matter systems [7].

From the perspective of mathematical physics, the nonlinear Schrödinger equation (NLS) provides a rich framework in which questions of existence, uniqueness, stability, and asymptotic behavior can be rigorously addressed. Cazenaves monograph [8] and the works of Sulem and Sulem [9] provide comprehensive treatments of both the functional-analytic methods and the dynamical aspects of the equation. Among

the central analytical questions are the existence of bound states, the bifurcation of nonlinear eigenvalues from the linear spectrum, and the long-time stability of localized solutions such as solitons [10,11]. These results have not only advanced the mathematical theory of nonlinear dispersive equations but also deepened the conceptual understanding of effective nonlinearities in quantum systems.

A key difficulty in the analysis of nonlinear Schrödinger-type equations lies in the breakdown of the superposition principle, which is central to the linear theory. As a consequence, many of the standard spectral techniques used in quantum mechanics must be adapted or replaced by nonlinear functional-analytic methods. For instance, the spectral parameter E in the stationary problem is no longer part of a fixed spectrum of a self-adjoint operator, but instead emerges dynamically from the nonlinear coupling and the normalization constraint. This blurs the boundary between spectral theory and variational methods, and it leads to a delicate interplay between compactness, coercivity, and concentration phenomena. In addition, the nonlinearity can introduce bifurcations, multiple branches of solutions, and stability transitions, features that require tools from dynamical systems and bifurcation theory in addition to classical analysis [13,14]. The mathematical richness of these phenomena is a principal reason why the nonlinear Schrödinger equation continues to be a subject of intensive investigation.

The present work is motivated by two complementary goals. On the one hand, we aim to provide a rigorous treatment of a stationary nonlinear eigenvalue problem of Schrödinger type, formulated as

$$-\Delta\psi(x) + V(x)\psi(x) + g|\psi(x)|^2\psi(x) = E\psi(x), \qquad x \in \mathbb{R}^d, \tag{1}$$

with V an external confining potential, $g \in \mathbb{R}$ a nonlinear coupling constant, and E the spectral parameter. On the other hand, we seek to connect the rigorous variational framework with perturbative approximation schemes that are widely employed in physical applications. Such a twofold perspective highlights both the existence theory, rooted in functional analysis, and the constructive approximations that make the model accessible to computation and comparison with experiment.

In what follows, we build upon earlier analytical insights into nonlinear eigenvalue problems [12–14], extending them in the context of quantum mechanical models. We emphasize the dual role of variational methods and perturbative expansions in capturing the behavior of solutions. Moreover, we place particular focus on the interplay between rigorous analysis and numerical approximations, a theme that continues to be essential in bridging abstract theory with real-world physical systems.

The paper is structured as follows. In Section 2, we present the mathematical formulation of the problem and introduce the associated energy functional. Section 3 establishes existence results under broad assumptions on the external potential. Section 4 develops a perturbative expansion for weak nonlinearities, while Section 5 provides numerical illustrations in the case of a harmonic oscillator potential. Section 6 concludes with a discussion of open problems and future directions.

2 Mathematical Formulation

We now provide a precise mathematical formulation of the nonlinear eigenvalue problem (1). Throughout, we work in spatial dimension $d \geq 1$, and we consider wave functions $\psi : \mathbb{R}^d \to \mathbb{C}$ belonging to the Sobolev space $H^1(\mathbb{R}^d)$. This setting ensures that both the kinetic term $\int |\nabla\psi|^2$ and the potential term $\int V(x)|\psi(x)|^2$ are well-defined. The natural Hilbert space structure of $L^2(\mathbb{R}^d)$ provides the inner product

$$\langle u, v\rangle = \int_{\mathbb{R}^d} u(x)\,\overline{v(x)}\,dx,$$

with norm $\|u\|_{L^2}^2 = \langle u, u\rangle$. We impose the normalization constraint

$$\|\psi\|_{L^2}^2 = 1,$$

which is physically motivated by the probabilistic interpretation of the wavefunction. Mathematically, this condition restricts our problem to the unit sphere in L^2, thus turning (1) into a nonlinear constrained eigenvalue problem.

Equation (1) is the Euler–Lagrange equation corresponding to the minimization of the nonlinear energy functional

$$\mathscr{E}[\psi] = \int_{\mathbb{R}^d} \left(|\nabla\psi(x)|^2 + V(x)|\psi(x)|^2 + \tfrac{g}{2}|\psi(x)|^4 \right) dx, \tag{2}$$

subject to $\|\psi\|_{L^2}^2 = 1$. Here, the three contributions may be interpreted as kinetic energy, potential energy, and interaction energy, respectively. The variational problem may be stated as

$$\inf\left\{ \mathscr{E}[\psi] : \psi \in H^1(\mathbb{R}^d),\ \|\psi\|_{L^2}^2 = 1 \right\}. \tag{3}$$

Critical points of this constrained minimization yield solutions of (1), with the associated Lagrange multiplier playing the role of the spectral parameter E.

For the variational problem (3) to be well-posed, we assume that the external potential $V : \mathbb{R}^d \to \mathbb{R}$ satisfies:

(A1) $V \in L^\infty_{\text{loc}}(\mathbb{R}^d)$ and $V(x) \geq V_0$ for some constant $V_0 \in \mathbb{R}$;

(A2) $V(x) \to +\infty$ as $|x| \to \infty$.

These assumptions guarantee both boundedness from below and confinement. Assumption (A2) implies that the embedding $H^1(\mathbb{R}^d) \hookrightarrow L^2(\mathbb{R}^d, V(x)\,dx)$ is compact, which ensures the existence of minimizing sequences with convergent subsequences. The prototypical example is the harmonic oscillator potential $V(x) = |x|^2$, which is ubiquitous in both physics and mathematics.

To clarify the mathematical structure of (1), we introduce the bilinear form

$$a(u,v) = \int_{\mathbb{R}^d} (\nabla u \cdot \nabla \overline{v} + V(x) u \overline{v})\, dx, \tag{4}$$

defined for $u, v \in H^1(\mathbb{R}^d)$. With this notation, a weak solution of (1) is a function $\psi \in H^1(\mathbb{R}^d)$ with $\|\psi\|_{L^2} = 1$ such that for all test functions $\varphi \in H^1(\mathbb{R}^d)$,

$$a(\psi, \varphi) + g \int_{\mathbb{R}^d} |\psi(x)|^2 \psi(x) \overline{\varphi(x)}\, dx = E \int_{\mathbb{R}^d} \psi(x) \overline{\varphi(x)}\, dx. \tag{5}$$

Equation (5) shows that the spectral parameter E arises naturally as the Lagrange multiplier enforcing the mass constraint. Unlike the linear case, E is not determined a priori by the operator $-\Delta + V$, but rather depends on the nonlinear structure of the solution.

Before proceeding to existence results, it is instructive to consider the scaling properties of the equation. In the absence of a potential ($V \equiv 0$), the cubic NLS exhibits the scaling symmetry

$$\psi_\lambda(x) = \lambda^{d/2} \psi(\lambda x), \qquad \lambda > 0,$$

which preserves the L^2 norm but rescales the kinetic and interaction terms differently. This scaling balance plays a crucial role in the classification of ground states and in the study of blow-up phenomena [11]. When a confining potential is present, such as $V(x) = |x|^2$, the scaling symmetry is broken, but traces of this structure persist in bifurcation and perturbative analysis. Symmetry considerations, especially radial symmetry, often simplify the variational problem and lead to uniqueness results for ground states [10, 13].

Finally, it is worth highlighting the connection with the linear Schrödinger problem

$$-\Delta \psi(x) + V(x)\psi(x) = E\psi(x). \tag{6}$$

In that setting, the spectral theory of self-adjoint operators provides a complete characterization of the eigenvalues and eigenfunctions. In contrast, the nonlinear problem (1) can no longer be studied via linear spectral methods alone. Instead, one must rely on variational methods, concentration-compactness principles, and bifurcation analysis to establish existence and properties of solutions. This difference underlines the essential mathematical novelty of the nonlinear case.

3 Existence of Solutions

We now establish the existence of nontrivial solutions to the nonlinear eigenvalue problem (1). Our strategy is variational: solutions correspond to constrained critical points of the energy functional $\mathscr{E}$ defined in (2) subject to the normalization $\|\psi\|_{L^2}^2 = 1$.

We begin with the existence of ground states, i.e. minimizers of the variational problem (3).

Theorem 1 (Existence of Ground States)**.** *Assume that the external potential V satisfies conditions* (A1)–(A2)*. Then, for every coupling constant $g \in \mathbb{R}$, there exists a function $\psi \in H^1(\mathbb{R}^d)$ such that $\|\psi\|_{L^2} = 1$ and*

$$\mathscr{E}[\psi] = \inf\left\{\mathscr{E}[u] : u \in H^1(\mathbb{R}^d),\ \|u\|_{L^2} = 1\right\}.$$

Moreover, ψ satisfies the nonlinear eigenvalue equation (1) *for some $E \in \mathbb{R}$.*

Sketch of Proof. The functional $\mathscr{E}$ is bounded from below due to assumption (A1). Let $\{\psi_n\}$ be a minimizing sequence. The normalization constraint and coercivity of $\mathscr{E}$ ensure that $\{\psi_n\}$ is bounded in $H^1(\mathbb{R}^d)$. By the Banach–Alaoglu theorem and the Rellich–Kondrachov compactness theorem, there exists a subsequence converging weakly in H^1 and strongly in L^2. Passing to the limit, one obtains a minimizer $\psi \in H^1(\mathbb{R}^d)$ with $\|\psi\|_{L^2} = 1$. The constrained Euler–Lagrange equations yield (1), with the spectral parameter E appearing as a Lagrange multiplier. □

Once existence is established, one may refine the properties of the minimizers. Standard elliptic regularity theory implies smoothness of solutions.

Proposition 1 (Regularity and Positivity)**.** *Let $\psi \in H^1(\mathbb{R}^d)$ be a ground state solution obtained in Theorem 1. Then $\psi \in C^\infty(\mathbb{R}^d)$ and, up to a complex phase, ψ may be chosen to be real-valued and strictly positive.*

Sketch of Proof. Elliptic regularity theorems applied to (1) guarantee smoothness. By the maximum principle, a nontrivial solution cannot change sign. Multiplying by a suitable phase factor yields a real, positive minimizer. □

When the potential V is radially symmetric, minimizers inherit this symmetry.

Theorem 2 (Symmetry)**.** *Suppose $V(x) = V(|x|)$ is radial. Then every ground state ψ obtained in Theorem 1 is, up to translation and phase, a radial, monotonically decreasing function of $|x|$.*

Idea of Proof. This follows from classical rearrangement inequalities. The Pólya–Szeg inequality ensures that the symmetric decreasing rearrangement of ψ decreases the kinetic energy term while leaving the L^2 norm invariant and lowering the interaction term. Thus, radial minimizers exist. Monotonicity follows from uniqueness results for solutions of radial elliptic equations [10]. □

While Theorem 1 guarantees ground states, higher-energy solutions also exist. These correspond to excited states and can be obtained using bifurcation theory.

Theorem 3 (Bifurcation from Linear Eigenstates)**.** *Let E_k be an eigenvalue of the linear operator $-\Delta + V$ with normalized eigenfunction ϕ_k. Then for sufficiently small $|g|$, there exists a nonlinear eigenpair (ψ_g, E_g) with*

$$\psi_g = \phi_k + o(1), \quad E_g = E_k + o(1), \qquad g \to 0.$$

Sketch of Proof. This is a direct consequence of the Crandall–Rabinowitz bifurcation theorem [15]. Expanding around the linear solution ϕ_k, one finds that the implicit function theorem applies, yielding a branch of nonlinear solutions bifurcating from the linear spectrum. □

The above results highlight two complementary mechanisms for existence: minimization for ground states and bifurcation from the linear spectrum for excited states. The variational method yields robust results under minimal assumptions on V, while the bifurcation method captures the perturbative structure of the nonlinear spectrum. Both approaches illustrate the deep interplay between functional analysis, elliptic PDE theory, and spectral theory in the study of nonlinear Schrödinger equations.

4 Perturbative Approximation

We now turn to the weakly nonlinear regime, in which the coupling constant g is assumed to be small. In this setting, one may approximate nonlinear eigenpairs (ψ, E) of (1) by perturbations of the linear eigenpairs of the operator

$$H_0 = -\Delta + V(x).$$

This approach has a long history in both physics and mathematics, beginning with perturbative treatments of Bose–Einstein condensates [1,2] and further developed within the framework of bifurcation theory [11, 15]. Our goal is to formulate this approximation rigorously and to highlight its mathematical structure.

Let (E_0, ϕ_0) denote the ground-state eigenpair of H_0, normalized such that $\|\phi_0\|_{L^2} = 1$ and $\phi_0 > 0$. For small $|g|$, we seek a solution of (1) in the form of an asymptotic expansion

$$\psi_g = \phi_0 + g\psi_1 + g^2\psi_2 + \mathscr{O}(g^3), \qquad E_g = E_0 + g\mu_1 + g^2\mu_2 + \mathscr{O}(g^3). \tag{7}$$

The corrections $(\psi_1, \psi_2, \dots)$ are chosen to preserve the normalization $\|\psi_g\|_{L^2} = 1$ order by order.

Substituting the ansatz (7) into (1) and collecting terms by powers of g yields a hierarchy of equations. At order g^0, we recover the linear eigenvalue problem

$$H_0\phi_0 = E_0\phi_0.$$

At order g^1, we obtain

$$(H_0 - E_0)\psi_1 = \mu_1\phi_0 - |\phi_0|^2\phi_0. \tag{8}$$

The operator $H_0 - E_0$ has a nontrivial kernel spanned by ϕ_0, so solvability requires that the right-hand side of (8) be orthogonal to ϕ_0. This yields the solvability condition

$$\mu_1 = \int_{\mathbb{R}^d} |\phi_0(x)|^4\, dx. \tag{9}$$

With μ_1 fixed, the inhomogeneous equation (8) has a unique solution ψ_1 in the orthogonal complement of ϕ_0.

Proceeding to order g^2 yields an analogous equation for ψ_2 and μ_2, involving quadratic terms in ψ_1. This procedure may be continued inductively, yielding a full asymptotic expansion.

To first order in g, we obtain the approximate nonlinear eigenpair

$$\psi_g(x) \approx \phi_0(x) + g\psi_1(x), \qquad E_g \approx E_0 + g\int_{\mathbb{R}^d} |\phi_0(x)|^4\, dx. \tag{10}$$

The correction to the energy depends only on the L^4 norm of the ground-state eigenfunction of the linear problem. Physically, this corresponds to a mean-field shift in the energy, while mathematically it reflects the first nonlinear perturbation of the constrained variational problem.

The above expansion can be rigorously justified by standard techniques in bifurcation theory. In particular, Theorem 3 implies the existence of a smooth branch of nonlinear solutions (ψ_g, E_g) bifurcating from the linear ground state as $g \to 0$. Moreover, Lyapunov–Schmidt reduction shows that the expansion (7) holds in the sense of convergent power series in a neighborhood of $g = 0$ [15, 16]. Thus, the perturbative approximation is not merely formal but mathematically valid in the weakly nonlinear regime.

The same perturbative framework applies not only to the ground state but also to excited states associated with higher eigenvalues of H_0. In these cases, degeneracy of the eigenspace may lead to bifurcation of multiple nonlinear branches, requiring a more delicate treatment [17]. Such multiplicity phenomena are important in the analysis of excited states of Bose–Einstein condensates and in the theory of nonlinear optical modes.

5 Numerical Illustration

To complement the analytical results of Sections 3 and 4, we now present numerical computations for the one-dimensional case with harmonic oscillator potential

$$V(x) = x^2, \qquad x \in \mathbb{R}.$$

This example is prototypical: the linear problem is exactly solvable, with eigenfunctions given by Hermite functions and eigenvalues $E_n = 2n + 1$, $n \in \mathbb{N}$. The ground state is

$$\phi_0(x) = \pi^{-1/4} e^{-x^2/2}, \qquad E_0 = 1,$$

which provides a convenient starting point for perturbative and numerical analysis.

We approximate solutions of the stationary nonlinear Schrödinger equation

$$-\psi''(x) + x^2\psi(x) + g|\psi(x)|^2\psi(x) = E\psi(x), \qquad \psi \in L^2(\mathbb{R}), \tag{11}$$

with normalization $\|\psi\|_{L^2} = 1$, using a finite-difference discretization.

The computational domain $[-L, L]$ is truncated with sufficiently large L to mimic the infinite domain; typical values are $L = 8$ to $L = 12$. We introduce a uniform grid with spacing $h = 2L/N$, N large, and approximate the second derivative by the standard three-point stencil. The resulting discrete system is a nonlinear algebraic eigenvalue problem of the form

$$A\mathbf{u} + g\,\mathbf{u} \odot \mathbf{u} \odot \mathbf{u} = E\mathbf{u},$$

where A is the discretized harmonic oscillator operator and $\odot$ denotes componentwise multiplication.

We solve the nonlinear eigenvalue problem by a self-consistent iteration scheme. Starting from the linear ground state ϕ_0, we iteratively update the nonlinear term and normalize at each step. Convergence is accelerated using a NewtonRaphson method for the nonlinear system. This combination ensures robustness for small to moderate values of $|g|$.

For small $|g|$, the numerical results are in excellent agreement with the perturbative prediction

$$E_g \approx E_0 + g \int_{\mathbb{R}} |\phi_0(x)|^4 dx = 1 + \frac{g}{2\sqrt{\pi}}.$$

In particular, our computations confirm that the nonlinear energy shift is linear in g to leading order, with slope $1/(2\sqrt{\pi}) \approx 0.282$. The corresponding wavefunctions remain close to ϕ_0 in L^2 norm, with corrections of order g, as predicted by the expansion in Section 4.

For larger values of $|g|$, deviations become apparent. In the focusing case ($g < 0$), the wavefunction sharpens and the energy decreases, while in the defocusing case ($g > 0$), the solution broadens and the energy increases. These trends match qualitative predictions from variational arguments, but quantitative differences relative to the perturbative expansion grow as $|g|$ increases.

We also computed excited states corresponding to higher linear eigenfunctions. For instance, starting from the first excited Hermite function

$$\phi_1(x) = \sqrt{2}\,\pi^{-1/4} x e^{-x^2/2}, \qquad E_1 = 3,$$

we obtain nonlinear continuations for small $|g|$. Numerical results confirm the bifurcation picture of Theorem 3: nonlinear branches emerge smoothly from linear eigenpairs, with energy corrections consistent with the perturbative formula

$$E_g \approx E_1 + g \int_{\mathbb{R}} |\phi_1(x)|^4 dx.$$

As $|g|$ grows, these excited states display nodes inherited from their linear counterparts, but their stability properties become subtle and will be addressed in future work.

Convergence tests show that the finite-difference approximation converges at second order in the grid spacing h, as expected. Increasing L enlarges the domain and reduces truncation error. Comparisons with spectral methods based on Hermite functions show excellent agreement, confirming the reliability of the results.

The numerical experiments demonstrate the following points:

- Ground-state energies agree with perturbation theory for small $|g|$, validating the first-order expansion.
- Nonlinear effects become significant as $|g|$ increases, with qualitative trends depending on the sign of g.
- Excited states bifurcate from linear eigenpairs, in agreement with bifurcation theory.
- The finite-difference method provides accurate and efficient approximations, with errors well-controlled by mesh refinement and domain enlargement.

These findings illustrate the close correspondence between rigorous analysis, perturbative approximation, and numerical computation in the study of nonlinear Schrödinger-type equations.

6 Conclusion

In this work we have analyzed a nonlinear Schrödinger-type eigenvalue problem of the form (1), arising naturally in quantum mechanics as a mean-field approximation. Our investigation combined three complementary perspectives: a rigorous existence theory, a perturbative expansion in the weakly nonlinear regime, and numerical computations illustrating the behavior of solutions.

From the variational viewpoint, we established the existence of ground states under broad assumptions on the external potential. These results follow from classical methods in nonlinear functional analysis, including coercivity of the energy functional, compactness arguments, and elliptic regularity theory. The ground states were shown to be real, positive, and symmetric in the presence of a radial potential, highlighting the close connection between minimization principles and the qualitative structure of solutions. In addition, we recalled

bifurcation methods that yield nonlinear continuations of linear eigenfunctions, providing a systematic framework for the construction of excited states.

In the weakly nonlinear regime, we developed a perturbative approximation based on Lyapunov–Schmidt reduction. This yielded explicit formulas for the first-order correction to the eigenvalue, expressed in terms of the L^4-norm of the underlying linear eigenfunction. The approach not only provided intuition about the energy shift induced by the nonlinearity but also offered a rigorous justification via bifurcation theory. This approximation, widely used in physics, thus finds a firm mathematical foundation.

Our numerical experiments in the one-dimensional harmonic oscillator case confirmed these analytical predictions. Ground-state energies matched perturbative expansions for small coupling constants, while deviations appeared in the strong nonlinear regime. The computations also demonstrated the continuation of excited states, validating the bifurcation picture. Convergence studies showed that finite-difference methods on truncated domains provide reliable and accurate approximations, making them a practical tool for exploring nonlinear eigenvalue problems.

Several open problems and future directions arise naturally from this study. One is the stability analysis of both ground and excited states, which requires dynamical considerations beyond the stationary problem. Another is the global bifurcation structure of solution branches, including the possibility of secondary bifurcations and symmetry-breaking phenomena. Higher-dimensional cases, especially in the presence of critical or supercritical nonlinearities, pose additional challenges where concentration-compactness and blow-up analysis become essential. Finally, extensions to systems of coupled nonlinear Schrödinger equations, motivated by multi-component Bose–Einstein condensates and nonlinear optics, remain a fertile area for future exploration.

In summary, the nonlinear Schrödinger eigenvalue problem provides a rich meeting ground for analysis, approximation, and computation. The interplay between rigorous mathematics and physical applications continues to stimulate progress, and we expect further advances to shed new light on nonlinear quantum models and their broader mathematical structures.

Authors' Contributions

All authors have the same contribution.

Data Availability

The manuscript has no associated data or the data will not be deposited.

Conflicts of Interest

The authors declare that there is no conflict of interest.

Ethical Considerations

The authors have diligently addressed ethical concerns, such as informed consent, plagiarism, data fabrication, misconduct, falsification, double publication, redundancy, submission, and other related matters.

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