Analytical and Numerical Solutions for Nonlinear Equations
©Available online at https://ansne.du.ac.ir/
Online ISSN: 3060–785X
2025, Volume 10, Issue 2, pp. 200–218

Research article

Existence and Qualitative Properties of Radial Solutions to a Nonlinear Equation on Lifshitz Black Hole Backgrounds

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Received: 12/10/2025 **Accepted:** 15/12/2025 **Published:** 30/12/2025 doi 10.22128/ansne.2025.3097.1166

Abstract

We study a nonlinear scalar equation on a $(d+2)$–dimensional Lifshitz black hole background with dynamical critical exponent $z>1$. For static, radially symmetric configurations the Klein–Gordon equation with power-type self-interaction reduces to a non-autonomous second-order ordinary differential equation on the half-line. We formulate the problem in divergence form, identify a natural weighted energy, and prove existence of bounded solutions decaying at infinity via a compactness argument and a monotone iteration scheme built upon the positive Green operator of the linearized problem. We also derive sharp near-horizon and asymptotic expansions, and we discuss uniqueness under a smallness condition on the nonlinearity. An explicit family of backgrounds $f(r)=1-(r_h/r)^{d+z}$ is used to illustrate the boundary behavior and to present a numerically robust shooting strategy consistent with the analysis.

Keywords: Lifshitz black hole, Power-type self-interaction, Radially symmetric solutions

Mathematics Subject Classification (2020): 35J60, 34B40, 35B40, 83C57

1 Introduction

The discovery that certain strongly coupled quantum systems can be modeled by higher-dimensional gravitational theories has reshaped both high-energy and condensed-matter theory over the past quarter century. In the relativistic setting, the AdS/CFT correspondence posits a precise duality between conformal field theories and gravity (or string theory) on asymptotically antide Sitter spaces, furnishing a geometric avatar for renormalization-group flows and a dictionary between operator data and bulk fields [1–3]. A natural next step—crucial for applications to quantum critical matter—is to depart from relativistic scale invariance and admit anisotropic scalings of the form

$$(t,\mathbf{x}) \mapsto (\lambda^z t, \lambda \mathbf{x}), \qquad z>1,$$

characteristic of Lifshitz points in statistical mechanics and of various non-Fermi liquid phases. On the gravity side, this leads to the Lifshitz spacetimes and their black hole deformations introduced in [4] and now reviewed in depth in [5]. In their simplest incarnation, these geometries realize the above nonrelativistic scaling as an isometry and, when endowed with horizons, provide toy models for finite-temperature, dissipative states of Lifshitz-invariant field theories.

While the high-energy perspective emphasizes holography, from a mathematical viewpoint Lifshitz black hole backgrounds give rise to a family of weighted boundary value problems on the half-line whose coefficients are singular at the horizon and asymptote to constants at infinity. In fact, for static, radially symmetric scalar configurations, the KleinGordon equation on a $(d+2)$-dimensional Lifshitz black hole reduces to a nonautonomous second-order ODE in divergence form. This reduction inherits two structural features that drive the analysis below: (i) a first-order degeneration at the horizon, reflecting the vanishing of the blackening factor and enforcing a one-parameter family of regular boundary data, and (ii) an e^{2y} weight after the logarithmic change of variables $y = \log(r/r_h)$, which acts as a confining potential and fixes the decay of solutions at spatial infinity. These features are robust across concrete families of Lifshitz black holes that appear in the physics literature, including solutions and branes in Einstein(dilaton)Maxwell(-Proca) or higher-curvature theories [6–10].

Several motivations converge on the nonlinear, static problem we study. On the holographic side, scalar profiles encode deformations or expectation values of operators and, consequently, govern the thermodynamic and transport responses of the dual state. Nonlinear self-interactions are not merely technical embellishments: they capture backreaction within probe sectors, provide a controlled setting for symmetry breaking, and often model effective Landau potentials. On the purely analytic side, the Lifshitz background supplies a fertile testbed for semilinear ODE/PDE with mixed character: the horizon creates a regular singular point; the asymptotic region is effectively weakly inhomogeneous; and the whole equation admits an energy identity with horizon boundary terms whose sign and size can be read off from the model parameters. As we explain, these ingredients align naturally with maximum principles, positivity of Green operators, and monotone iteration schemes in ordered Banach spaces [11–15].

The aim of this paper is to treat, in a unified and rigorous manner, a prototypical semilinear equation on a broad class of $(d+2)$-dimensional Lifshitz black hole backgrounds. We consider static, radial solutions to the KleinGordon equation with defocusing power-type self-interaction and impose the physically standard boundary conditions of boundedness/regularity at the horizon together with vanishing at infinity. In logarithmic radius, the problem assumes a divergence form with an explicitly coercive weight; we identify the correct function space, establish positivity and compactness of the associated Green operator, and construct global solutions by monotone iteration between ordered barriers. The construction yields *a priori* bounds, monotonicity of profiles, and sharp two-term asymptotics at the horizon and at infinity. For a canonical family with

$$f(r) = 1 - \left(r_h/r\right)^{d+z},$$

we also describe a numerically stable, single-parameter shooting scheme that dovetails with the analytical picture and provides a transparent route to the decaying branch.

It is useful to place our results against familiar landmarks. In asymptotically AdS settings, the interplay between mass terms, boundary conditions, and stability is organized by the BreitenlohnerFreedman bound and by the convexity properties of the energy [16, 17]. For Lifshitz asymptotics, the situation is subtler: although there is no exact analog of the BF bound, the weighted e^{2y} structure acts as a stabilizing surrogate at the level of the static problem, and the existence theory below can be viewed as a nonlinear counterpart of positivity in appropriate weighted spaces. This perspective motivates our emphasis on divergence form and on the horizon boundary term in the energy identity. Technically, the proof rests on a classical toolkit: maximum principles and positivity of Green functions for second-order operators in divergence form [11, 14], fixed-point and compactness arguments in cones [12], and the method of upper and lower solutions together with monotone iterative techniques [13]. Where weights enter in an essential way, we draw on standard results in weighted Sobolev theory and Hardy-type inequalities [18, 19] to control the degeneracy at the horizon and the decay at infinity.

The rest of the paper develops this program in detail. After deriving the reduced equation and the associated function space, we prove existence of bounded, decaying solutions for all defocusing powers $p > 1$ under mild structural assumptions on the blackening factor. We then extract precise near-horizon and far-field expansions and give a uniqueness criterion in a small L^∞-ball by combining positivity of the Green operator with a contraction argument. Finally, to bridge analysis and computation, we discuss an implementation of a shooting/continuation scheme that is globally well-conditioned in the logarithmic coordinate. Throughout we highlight how the hypotheses accommodate standard Lifshitz black hole families [6–10] while remaining flexible enough for further generalizations.

2 Set-Up and the Reduced Equation

We begin by fixing the geometric background and the class of equations under consideration and by recording several equivalent formulations that will be useful later. Throughout, $d \geq 1$ denotes the number of planar spatial directions and $z > 1$ is the dynamical critical exponent that implements the anisotropic scaling $(t, \mathbf{x}) \mapsto (\lambda^z t, \lambda \mathbf{x})$ characteristic of Lifshitz holography [4, 5]. On the $(d+2)$–dimensional manifold

$\mathscr{M} = (r_h, \infty) \times \mathbb{R}_t \times \mathbb{R}^d_{\mathbf{x}}$ we adopt the standard static, planar Lifshitz black hole ansatz

$$ds^2 = -r^{2z} f(r)\, dt^2 + r^2 d\mathbf{x}^2 + \frac{dr^2}{r^2 f(r)}, \tag{1}$$

where $r \in (r_h, \infty)$ is the areal radius, $r_h > 0$ is the (outermost) horizon radius, and the blackening factor f satisfies

$$f \in C^1\big([r_h, \infty)\big), \qquad f(r_h) = 0, \quad f(r) > 0, \text{ for } r > r_h, \quad \lim_{r\to\infty} f(r) = 1. \tag{2}$$

The hypotheses (2) capture the families of solutions most commonly used in the physics literature, including Einstein(dilaton)Maxwell(Proca) and higher-curvature examples [6–10]. We will at times use the stronger local condition that f has a simple zero at r_h, namely $f'(r_h) > 0$; this is automatic in the canonical model $f(r) = 1 - (r_h/r)^{d+z}$.

Let ϕ be a real scalar field, assumed static and radially symmetric, subject to a defocusing power-law self-interaction. In covariant form we study

$$\Box_g \phi - m^2 \phi - \lambda |\phi|^{p-1} \phi = 0, \qquad m^2 \geq 0,\ \lambda > 0,\ p > 1, \tag{3}$$

where $\Box_g$ is the LaplaceBeltrami operator of (1). For the ansatz (1), the metric density and inverse radial component read

$$\sqrt{-g} = r^{d+z-1}, \qquad g^{rr} = r^2 f(r).$$

Consequently, the radial, static sector reduces to the nonautonomous second-order ODE in *divergence form*

$$\frac{d}{dr}\Big(r^{d+z+1} f(r)\, \phi'(r)\Big) = r^{d+z-1}\Big(m^2 \phi(r) + \lambda |\phi(r)|^{p-1} \phi(r)\Big), \tag{4}$$

posed on the half-line $r \in (r_h, \infty)$. The quantity

$$\mathscr{J}(r) := r^{d+z+1} f(r)\, \phi'(r), \tag{5}$$

is the radial flux associated with the weighted Dirichlet form that will appear in the energy identity below; regularity at the horizon corresponds precisely to finiteness of $\mathscr{J}$ as $r \downarrow r_h$.

We impose the physically canonical boundary conditions

$$\phi \text{ bounded as } r \downarrow r_h, \qquad \lim_{r\to\infty} \phi(r) = 0. \tag{6}$$

The horizon condition enforces smoothness in ingoing EddingtonFinkelstein coordinates and is equivalent to boundedness of ϕ together with finiteness of the flux (5). When $f'(r_h) > 0$, the point $r = r_h$ is a *regular singular point* of (4) in the SturmLiouville sense: writing $p(r) := r^{d+z+1} f(r)$, we have $p(r_h) = 0$ and $p'(r_h) = r_h^{d+z+1} f'(r_h) > 0$, so that p vanishes linearly. A short Frobenius analysis (cf. [14, 15]) reveals that bounded data are parametrized by the single value $\phi(r_h) =: \phi_0$, with finite slope determined by the inhomogeneity. Indeed, expanding $f(r) = f'(r_h)(r - r_h) + O((r - r_h)^2)$ and integrating (4) once yields

$$\lim_{r\downarrow r_h} \phi'(r) = \frac{m^2 \phi_0 + \lambda |\phi_0|^{p-1} \phi_0}{r_h^{d+z+1} f'(r_h)}. \tag{7}$$

Thus the horizon contributes a single regular boundary condition, and the decay at infinity will select a discrete branch (in the linear case) or a monotone solution family (in the nonlinear, defocusing regime studied here).

A convenient change of variables that both compactifies the domain and isolates the degeneracy is the logarithmic radius

$$y := \log\left(\frac{r}{r_h}\right) \in [0, \infty), \qquad r = r_h e^y, \qquad A(y) := f(r_h e^y).$$

In the y-coordinate the equation takes the normalized divergence form

$$\frac{d}{dy}\Big(e^{2y} A(y)\, \phi_y(y)\Big) = m^2 \phi(y) + \lambda |\phi(y)|^{p-1} \phi(y), \tag{8}$$

or, after expansion,

$$A(y)\, \phi_{yy}(y) + \big(A'(y) + 2A(y)\big)\, \phi_y(y) = e^{-2y}\Big(m^2 \phi(y) + \lambda |\phi(y)|^{p-1} \phi(y)\Big). \tag{9}$$

The assumptions (2) translate into $A \in C^1([0,\infty))$ with $A(0) = 0$, $A'(0) = r_h f'(r_h) > 0$, $A(y) > 0$ for $y > 0$, and $A(y) \to 1$ as $y \to \infty$. In this chart the horizon corresponds to $y = 0$ and is again a regular singular point: $e^{2y}A(y) \sim A'(0)\,y$ as $y \downarrow 0$. The finite-slope condition (7) becomes

$$\phi_y(0) \;=\; \frac{m^2\phi(0) + \lambda|\phi(0)|^{p-1}\phi(0)}{A'(0)}. \tag{10}$$

At the opposite end, since $A(y) \to 1$ and the right-hand side of (9) decays like e^{-2y}, solutions that remain small obey the linear asymptotic equation $\phi_{yy} + 2\phi_y \approx 0$ and hence

$$\phi(y) = C_\infty e^{-2y} + o(e^{-2y}) \quad \text{as } y \to \infty, \tag{11}$$

which, in the original radial variable, reads $\phi(r) \sim C_\infty (r_h/r)^2$ as $r \to \infty$. This asymptotic will be sharpened later, but even in this crude form it already explains why the natural energy space features the weight e^{2y}.

To ground the discussion, it is useful to single out the canonical background

$$f(r) = 1 - \left(\frac{r_h}{r}\right)^{d+z}, \qquad \text{equivalently} \qquad A(y) = 1 - e^{-(d+z)y}, \tag{12}$$

for which $A'(0) = d+z$. In this case one may scale out r_h entirely by measuring all lengths in units of the horizon radius; we will often do so tacitly. The equation (8) then couples a uniformly elliptic region ($y \gg 1$) to a weakly degenerate endpoint ($y = 0$). From the point of view of SturmLiouville theory [14, 15], (8) is a second-order equation with a positive, integrable principal weight near $y = 0$,

$$w(y) := e^{2y}A(y) \sim (d+z)\,y,$$

and with asymptotically constant coefficients for large y.

Multiplying (8) by ϕ and integrating by parts over $[0,Y]$ we obtain the balance law

$$\int_0^Y e^{2y}A(y)\,|\phi_y|^2\,dy \;+\; \Big[-\,\phi(y)\,e^{2y}A(y)\,\phi_y(y)\Big]_0^Y \;=\; \int_0^Y \big(m^2\phi^2 + \lambda|\phi|^{p+1}\big)\,dy. \tag{13}$$

Passing to the limit $Y \to \infty$ for solutions satisfying (6) and using the finite-slope horizon condition (10) yields the global identity

$$\int_0^\infty e^{2y}A(y)\,|\phi_y|^2\,dy \;=\; \int_0^\infty \big(m^2\phi^2 + \lambda|\phi|^{p+1}\big)\,dy \;+\; \frac{\phi(0)\,\big(m^2\phi(0) + \lambda|\phi(0)|^{p-1}\phi(0)\big)}{A'(0)}. \tag{14}$$

The left-hand side is a coercive quadratic form in the weighted Sobolev norm $\|e^y\sqrt{A}\,\phi_y\|_{L^2}$, which we will adopt in Section 3 as part of the definition of the natural phase space. The right-hand side is manifestly nonnegative for $\lambda > 0$ and $m^2 \geq 0$, up to the horizon boundary term whose sign is controlled by $\phi(0)$. Identity (14) anticipates both the positivity of the Green operator and the monotonicity of solutions, and it plays a central role in the compactness arguments used later.

For future reference it is convenient to isolate the linear operator

$$(L\phi)(y) := \frac{d}{dy}\Big(e^{2y}A(y)\,\phi_y(y)\Big), \tag{15}$$

so that the full equation is $L\phi = \mathcal{N}(\phi)$ with $\mathcal{N}(\phi) := m^2\phi + \lambda|\phi|^{p-1}\phi$. The boundary conditions (6) and the horizon regularity (10) define a closed realization of L on the weighted space described above; within this framework, L is invertible and its inverse $G := L^{-1}$ is a positive, compact Green operator on bounded subsets of $C([0,\infty))$. These propertiesproved rigorously in the next sectionare the engine behind the monotone iteration scheme and the existence theory. They reflect, in elementary form, the maximum-principle structure of second-order operators in divergence form [11, 14] and will allow us to recast the nonlinear problem as a fixed point in an ordered cone [12, 13].

Equations (4)(11) and the energy law (14) summarize the analytic landscape: a single regular boundary datum at the horizon, a confining e^{2y} weight that enforces decay, and a divergence-form operator with a positive Green kernel. In the next section we formalize the function space dictated by these structures and establish the Green-operator framework that underpins our existence, uniqueness, and asymptotic results.

3 Function Spaces and the Green Operator

The divergence structure unveiled in Sec. 2 suggests a natural weighted energy and, with it, a canonical function space on which the linear part of the equation is well posed. Throughout this section we write

$$p(y) := e^{2y} A(y), \qquad y \in [0,\infty),$$

so that the reduced operator reads

$$(L\phi)(y) = \frac{d}{dy}\big(p(y)\,\phi_y(y)\big).$$

Near the horizon we have $A(y) = A'(0)\,y + O(y^2)$ and hence $p(y) \sim A'(0)\,y$ as $y \downarrow 0$, while in the far region $A(y) \to 1$ and $p(y) \sim e^{2y}$. This nonuniform behavior governs both the choice of space and the mapping properties of L^{-1}.

We work with the weighted Sobolev class

$$\mathscr{H} \;=\; \Big\{\phi \in C^0([0,\infty)) \cap W^{1,1}_{\mathrm{loc}}([0,\infty)) :\; p^{1/2}\phi_y \in L^2(0,\infty),\; \lim_{y\to\infty}\phi(y) = 0\Big\}, \tag{16}$$

endowed with the seminorm $\|\phi\|_{\mathscr{H}} := \|p^{1/2}\phi_y\|_{L^2(0,\infty)}$ and with the trace $\phi(0)$ understood in the usual sense. The degeneracy $p(y) \sim y$ at 0 ensures that ϕ admits a finite one-sided derivative there if and only if the flux $p\phi_y$ has a finite limit; this is the regularity notion encoded by the horizon boundary condition. A first benefit of (16) is the integration by parts formula: for $\phi, \psi \in \mathscr{H}$ with ψ compactly supported in $[0,\infty)$,

$$\int_0^\infty p\,\phi_y \psi_y\,dy \;=\; -\int_0^\infty (L\phi)\,\psi\,dy - \Big[\psi\,p\,\phi_y\Big]_{y=0}, \tag{17}$$

in which the boundary term is well defined precisely because $p\phi_y$ has a finite trace at 0. The bilinear form

$$\mathfrak{a}[\phi,\psi] \;:=\; \int_0^\infty p(y)\,\phi_y(y)\psi_y(y)\,dy, \tag{18}$$

is nonnegative, symmetric, and closed on $\mathscr{H}$, and the nonlinearity in our problem will only enter through lower-order (unweighted) integrals, to which $\mathscr{H}$ is adapted via Hardy-type controls.

Two elementary inequalities capture the coercive content of the weight p. First, since $p(y) \gtrsim y$ for small y and $p(y) \gtrsim e^{2y}$ for large y, the weighted Hardy inequality (see, e.g., [18,19]) yields, for all $\phi \in \mathscr{H}$,

$$\int_0^1 y\,|\phi_y|^2\,dy \;\gtrsim\; |\phi(0)|^2, \qquad \int_1^\infty e^{2y}\,|\phi_y|^2\,dy \;\gtrsim\; \int_1^\infty |\phi(y)|^2\,dy, \tag{19}$$

so that $\phi \in L^2(1,\infty)$ and the horizon trace is automatically controlled by the energy. Second, combining (19) with CauchySchwarz on $(0,1)$ shows that ϕ is globally bounded and obeys

$$\|\phi\|_{L^\infty(0,\infty)} \;\lesssim\; |\phi(0)| + \|p^{1/2}\phi_y\|_{L^2(0,\infty)}. \tag{20}$$

These estimates are the quantitative backbone for the compactness arguments that will appear below.

Within this framework we specify the realization of L by declaring its domain as

$$\mathscr{D}(L) \;=\; \Big\{\phi \in \mathscr{H} :\; p\,\phi_y \in W^{1,1}_{\mathrm{loc}}([0,\infty)),\; L\phi \in L^1_{\mathrm{loc}}([0,\infty)),\; \lim_{y\to\infty}\phi(y) = 0\Big\}, \tag{21}$$

with horizon regularity encoded by the requirement that $p\phi_y$ have a finite trace at 0. For $\psi \in C_c^\infty([0,\infty))$ the distribution $L\phi$ then acts by (17) and no boundary condition is imposed at $y = \infty$ beyond decay of ϕ. In particular, for any $\psi \in C_c^\infty((0,\infty))$ the operator is symmetric with respect to the unweighted L^2 pairing:

$$\int_0^\infty (L\phi)\,\psi\,dy \;=\; \int_0^\infty \phi\,(L\psi)\,dy.$$

There is no lower-order term in L, and thus L is (formally) a SturmLiouville operator with coefficient p and $q \equiv 0$ in the classical notation [14,15].

The key structural statement we need is that L is invertible on the class of bounded data once the boundary conditions at 0 and at ∞ are enforced, and that L^{-1} is not only bounded but positive and compact on bounded sets. We record this in a single proposition that subsumes the Green-function construction, the maximum principle, and the kernel bounds that will be used by the monotone iteration scheme.

Proposition 1 (Green operator: existence, positivity, compactness)**.** *For every $\psi \in C([0,\infty))$ there exists a unique $\phi \in \mathscr{H} \cap C^1([0,\infty))$ solving*

$$L\phi = \psi \quad \text{on } (0,\infty), \qquad \phi \text{ bounded at } 0, \quad \lim_{y\to\infty} \phi(y) = 0.$$

The solution depends continuously on ψ and can be written as

$$\phi(y) = (G\psi)(y) = \int_0^\infty K(y,s)\,\psi(s)\,ds, \tag{22}$$

where the Green kernel K is continuous on $(0,\infty)\times(0,\infty)$, strictly positive, and satisfies

$$0 < K(y,s) \lesssim \begin{cases} y e^{-2s}, & 0 < y \le s, \\ s e^{-2y}, & 0 < s < y, \end{cases} \qquad \text{and} \qquad \partial_y K(y,s) \lesssim e^{-2\max\{y,s\}}. \tag{23}$$

In particular, $G : C([0,\infty)) \to C^1([0,\infty))$ is bounded and maps bounded sets into equicontinuous families; hence G is compact on bounded subsets of $C([0,\infty))$. Moreover, if $\psi \ge 0$ then $G\psi \ge 0$, and if $\psi \not\equiv 0$ then $G\psi > 0$ on $(0,\infty)$.

Sketch of proof. Let u_- be the unique (up to scaling) solution of $Lu = 0$ that is bounded at $y = 0$ and normalized by $u_-(0) = 1$, and let u_+ be the unique solution of $Lu = 0$ that decays at infinity, normalized by $\lim_{y\to\infty} e^{2y} u_+(y) = 1$. A single integration of $Lu = 0$ shows that $p u_\pm'$ are constant on $[0,\infty)$, whence u_- is strictly increasing and u_+ strictly decreasing; near the endpoints the asymptotics are $u_-(y) = 1 + O(y)$ and $u_+(y) = O(e^{-2y})$. The Wronskian

$$W := p(y)\big(u_-(y)u_+'(y) - u_-'(y)u_+(y)\big),$$

is constant and strictly positive because u_- and u_+ are linearly independent and $p > 0$ on $(0,\infty)$. The standard variation-of-parameters formula (cf. [14, Ch. 3]) then yields

$$K(y,s) = \frac{1}{W} \times \begin{cases} u_-(y)\,u_+(s), & 0 < y \le s, \\ u_-(s)\,u_+(y), & 0 < s < y, \end{cases}$$

which is manifestly continuous and strictly positive on $(0,\infty)^2$. The bounds (23) follow from the endpoint asymptotics for $u_\pm$ and the fact that $W \sim A'(0)$, while the estimate on $\partial_y K$ follows from differentiating the above expression and using that $p u_\pm'$ are constant. The representation (22) is obtained by solving the inhomogeneous equation piecewise and matching at $y = s$; uniqueness comes from the maximum principle (see below) applied to the difference of two solutions with homogeneous right-hand side and the prescribed boundary behavior. The operator bounds and compactness are immediate from (23) and ArzelàAscoli.

For positivity, note that if $\psi \ge 0$ and $\phi = G\psi$ had a negative minimum at some interior point, then at that point we would have $\phi_y = 0$ and $(p\phi_y)' = L\phi = \psi \ge 0$, contradicting the fact that a negative minimum forces $(p\phi_y)' \le 0$. A similar argument excludes strictly negative values whenever $\psi \not\equiv 0$, proving $G\psi > 0$. □

Two consequences are worth recording explicitly because they are repeatedly used later without comment. First, the kernel bounds imply Schur-type estimates: there exists $C > 0$ such that

$$\sup_{y\ge 0} \int_0^\infty K(y,s)\,ds \le C, \qquad \sup_{s\ge 0} \int_0^\infty K(y,s)\,dy \le C, \tag{24}$$

and therefore G acts boundedly on L^∞ and on L^1, with $\|G\|_{L^\infty\to L^\infty} \le C$ and $\|G\|_{L^1\to L^1} \le C$. By interpolation, G extends boundedly to L^p for every $1 \le p \le \infty$. Second, differentiating under the integral sign and using the bound on $\partial_y K$ shows that G maps bounded sets in $C([0,\infty))$ into bounded sets in the weighted Lipschitz class

$$\Big\{\phi \in C^1([0,\infty)) :\ e^{2y}\phi_y \in L^\infty(0,\infty)\Big\},$$

a refinement that simplifies the passage to the limit in monotone iterations.

The maximum principle underlying Proposition 1 admits a clean energy proof that also clarifies how the horizon enters. Suppose $\phi \in \mathscr{H}$ satisfies $L\phi \ge 0$ on $(0,\infty)$, ϕ bounded at 0, and $\lim_{y\to\infty} \phi = 0$. Testing $L\phi$ against $\phi^- := \max\{-\phi, 0\}$ and integrating by parts yields

$$\int_0^\infty p\,|\partial_y \phi^-|^2\,dy \le -\Big[\phi^-\,p\,\phi_y\Big]_0^\infty.$$

The term at ∞ vanishes because $\phi^-(\infty) = 0$, and the term at 0 is nonpositive since $p(0) = 0$ and $\phi^-(0) \geq 0$. Therefore $\phi^- \equiv 0$ and $\phi \geq 0$ everywhere. This argument, a weighted version of the standard one for divergence-form operators [11], is robust under perturbations and will be reused for the monotonicity of nonlinear solutions.

Finally, we emphasize the role of the horizon boundary term in the energy identity. For $\phi \in \mathscr{D}(L)$ with $L\phi = \psi \in C([0,\infty))$, multiplying by ϕ and integrating by parts we recover the balance (14) with the right-hand side replaced by $\int_0^\infty \psi\,\phi\,dy$ plus the horizon contribution. In the linear theory this identity implies the resolvent estimate

$$\|p^{1/2}\phi_y\|^2_{L^2(0,\infty)} \;\leq\; \|\psi\|_{L^2(0,\infty)}\,\|\phi\|_{L^2(0,\infty)} \;+\; \frac{|\phi(0)|\,|p\phi_y(0)|}{A'(0)}, \tag{25}$$

which, when coupled with (19)–(20) and the explicit trace formula $p\phi_y(0) = m^2\phi(0)$ in the linear case, gives quantitative control of the linear solution in terms of the data. In the nonlinear setting the same structure yields the *a priori* bounds that anchor the compactness step in the existence proof.

Summarizing, on the weighted space $\mathscr{H}$ the divergence-form operator L has a positive, continuous Green kernel and a compact inverse on bounded sets. The degeneracy of the weight at the horizon is mild and, far from being an obstacle, supplies a built-in trace that replaces an independent boundary condition. These properties are precisely those needed to recast the semilinear problem as a fixed point in an ordered cone, to which the monotone iteration and compactness tools of [12, 13] apply without further complications.

4 Existence Via Monotone Iteration

With the operator framework of Sec. 3 in place, we turn to the nonlinear problem

$$L\phi \;=\; \mathscr{N}(\phi) := m^2\phi + \lambda|\phi|^{p-1}\phi, \tag{26}$$

subject to the horizon regularity and vanishing at infinity encoded in (6) and (10). The strategy is classical: construct an ordered pair of barriers, use the positivity and compactness of the Green operator $G = L^{-1}$ to build a monotone sequence by iteration, and pass to the limit with ArzelàAscoli to obtain a fixed point of $T := G\circ\mathscr{N}$. We emphasize two features that are specific to our setting and simplify the analysis. First, thanks to the defocusing sign $\lambda > 0$ and to the positivity of G (Proposition 1), the operator T is order-preserving on the positive cone. Second, the endpoint structure of $p(y) = e^{2y}A(y)$ supplies both a canonical trace at the horizon and a confining weight at infinity; as a result, the order interval bounded by the barriers is compact for the topology induced by $C([0,\infty))\cap\mathscr{H}$.

To make this precise, let $\mathscr{C} := \{\,\phi \in C([0,\infty))\cap\mathscr{H} :\ \phi \geq 0\,\}$ be the closed cone of nonnegative profiles, and fix $M > 0$. We call a pair $(\underline{\phi}, \overline{\phi}) \in \mathscr{C}^2$ an ordered sub-/supersolution pair if $0 \leq \underline{\phi} \leq \overline{\phi}$ pointwise and

$$L\underline{\phi} \;\geq\; \mathscr{N}(\underline{\phi}), \qquad L\overline{\phi} \;\leq\; \mathscr{N}(\overline{\phi}),$$

in the sense of distributions on $(0,\infty)$, with both functions bounded at 0 and vanishing at ∞. We now show that such a pair exists with $\|\overline{\phi}\|_{L^\infty} \leq M$, for a suitable choice of M determined by the parameters (m^2, λ, p) and by $A'(0)$.

The construction is explicit. For a small constant $c > 0$ consider $\underline{\phi}_c(y) := c\,e^{-2y}$. A direct computation using (8) gives $L\underline{\phi}_c = 2cA'(0) + O(cy)$ near 0 and $L\underline{\phi}_c = O(ce^{-2y})$ for large y, whereas $\mathscr{N}(\underline{\phi}_c) = m^2c\,e^{-2y} + \lambda c^p e^{-2py}$. Hence $L\underline{\phi}_c - \mathscr{N}(\underline{\phi}_c) \geq 0$ on $[0,\infty)$ provided c is chosen sufficiently small; this uses only that $A'(0) > 0$ and $p > 1$. For a supersolution we linearize at a putative L^∞-bound M and solve

$$L\overline{\phi}_M \;=\; (m^2 + \lambda M^{p-1})\,\overline{\phi}_M, \qquad \overline{\phi}_M \text{ bounded at } 0, \quad \overline{\phi}_M(\infty) = 0.$$

By Proposition 1 there is a unique nonnegative solution, and it is strictly decreasing by the maximum principle applied to $(\overline{\phi}_M)_y$. Choosing M large enough that $\|\overline{\phi}_M\|_{L^\infty} \leq M$ (which is possible because the Green norm of the linear operator is finite and the right-hand side is proportional to $\overline{\phi}_M$), we obtain an ordered pair $0 < \underline{\phi}_c \leq \overline{\phi}_M$ with

$$L\overline{\phi}_M \;=\; (m^2 + \lambda M^{p-1})\,\overline{\phi}_M \;\leq\; m^2\overline{\phi}_M + \lambda\overline{\phi}_M^{\,p} \;=\; \mathscr{N}(\overline{\phi}_M),$$

since $\overline{\phi}_M \leq M$. This provides the sought barriers.

Having secured barriers, define $T = G \circ \mathscr{N}$ and initialize the monotone iteration at the lower solution, $\phi_0 := \underline{\phi}_c$, by

$$\phi_{n+1} \; := \; T(\phi_n) \; = \; G\big(m^2\phi_n + \lambda\phi_n^p\big), \qquad n \geq 0.$$

Positivity and order-preservation of T yield $\underline{\phi}_c = \phi_0 \leq \phi_1 \leq \cdots \leq \overline{\phi}_M$. Moreover, the bounds (23) imply that the sequence is uniformly bounded in $C^1([0,\infty))$ on compact subsets and that the derivatives satisfy a global weighted bound of the form $\sup_n \|e^{2y}(\phi_n)_y\|_{L^\infty} < \infty$. ArzelàAscoli then furnishes a subsequence converging uniformly on compacts and weakly in the weighted Sobolev norm to a function $\phi \in \mathscr{C}$ with $\underline{\phi}_c \leq \phi \leq \overline{\phi}_M$. In fact, monotonicity upgrades the subsequence convergence to pointwise monotone convergence, hence to uniform convergence on compacts by Dinis theorem. Passing to the limit in the integral formulation $\phi_{n+1} = G\mathscr{N}(\phi_n)$, which is legitimate by dominated convergence thanks to (24), we obtain $\phi = T(\phi)$ and therefore a solution of (26) with the prescribed boundary behavior.

This procedure yields more than existence. Because the sequence is increasing and bounded above by $\overline{\phi}_M$, the limit ϕ is the *minimal* fixed point of T in the order interval $[\underline{\phi}_c, \overline{\phi}_M]$. Any other nonnegative solution lying between the barriers dominates ϕ. Likewise, starting the iteration from the upper solution and iterating downward with $\psi_{n+1} = T(\psi_n)$ produces a decreasing sequence converging to the *maximal* fixed point in the same interval. In our defocusing setting and for the specific barriers built above, the two limits coincide, and we obtain a unique solution between the barriers; this reflects the strict order-preservation of T and the convexity of $\mathscr{N}$ on $[0,\infty)$, and can be formalized with the standard arguments in ordered Banach spaces [12, 13].

The qualitative features announced in the introduction now follow by simple maximum-principle arguments. The limit ϕ is strictly positive on $(0,\infty)$ because $\phi = T(\phi)$ and T maps nontrivial nonnegative functions to strictly positive ones (Proposition 1). Differentiating (8) shows that $w := \phi_y$ solves a linear equation in divergence form with a nonnegative zeroth-order coefficient:

$$\frac{d}{dy}\big(p\,w_y\big) \; = \; m^2 w + \lambda\, p^{-1}\frac{d}{dy}\big(p\,\phi^p\big) \; = \; m^2 w + \lambda\, p^{-1}\Big(p'\phi^p + p\,p\,\phi^{p-1}w\Big) \; \geq \; \lambda\,\phi^{p-1}w,$$

where we used $p' > 0$, $\phi \geq 0$, and $m^2 \geq 0$. The function w thus obeys a weak maximum principle with homogeneous boundary dataindeed w is bounded at 0 by (10) and $w(\infty) = 0$ by (11)which forces $w \leq 0$ on $(0,\infty)$. Strict negativity ($w < 0$) follows from the strong maximum principle [11]: if w vanished somewhere, then by unique continuation and the horizon asymptotics the solution would be constant, contradicting $\phi(\infty) = 0$ and $\phi(0) > 0$. Hence every solution constructed by the scheme is strictly decreasing.

We record the result in theorem form for future reference.

Theorem 1 (Existence, minimality, and monotonicity)**.** *Assume $A \in C^1([0,\infty))$ satisfies $A(0) = 0$, $A'(0) > 0$, $A(y) > 0$ for $y > 0$, and $A(y) \to 1$ as $y \to \infty$. Let $m^2 \geq 0$, $\lambda > 0$, $p > 1$. Then there exists $\phi \in \mathscr{H} \cap C^2((0,\infty)) \cap C^1([0,\infty))$ solving (26) with ϕ bounded at 0 and $\phi(\infty) = 0$. The solution is strictly positive and strictly decreasing on $(0,\infty)$, admits the horizon slope (10), and satisfies the asymptotic decay (11). Moreover, there exist barriers $\underline{\phi} \leq \overline{\phi}$ with $0 < \underline{\phi} \leq \phi \leq \overline{\phi}$, and ϕ is the minimal fixed point of $T = G \circ \mathscr{N}$ in the order interval $[\underline{\phi}, \overline{\phi}]$.*

Two further properties will be used later and follow from the very same scheme. First, the solution depends monotonically on the parameters: if $(m_1^2, \lambda_1) \leq (m_2^2, \lambda_2)$ componentwise, then the corresponding minimal solutions satisfy $\phi_{m_1,\lambda_1} \leq \phi_{m_2,\lambda_2}$. This is immediate because $T_{m^2,\lambda}$ is order-preserving and larger parameters yield larger right-hand sides in (26). Second, the map $(m^2, \lambda) \mapsto \phi_{m^2,\lambda}$ is continuous from compact subsets of $[0,\infty) \times (0,\infty)$ into $C([0,\infty))$, by dominated convergence in the fixed-point equation together with the compactness of G (cf. [12]). In particular, the family is uniformly controlled in $\mathscr{H}$ on compact parameter sets, a fact that will be useful in the uniqueness and asymptotic refinements below.

We close with a comment on alternatives and robustness. One could replace the explicit barrier construction by an application of Schaefers fixed-point theorem or of the LeraySchauder continuation principle to the compact map T on $\mathscr{C}$; the key a priori bound is then furnished by the energy identity (14), which precludes blow-up along the homotopy in the defocusing case. We prefer monotone iteration because it produces extremal solutions and immediately imports maximum-principle information to the nonlinear setting (see [12, 13] and references therein). The argument is stable under mild perturbations of the nonlinearitye.g. adding lower-order positive terms or replacing the pure power by a locally Lipschitz, monotone function with superlinear growthand under smooth deformations of the background A, provided the basic endpoint structure persists.

5 Local Behavior at the Horizon and at Infinity

We now sharpen the endpoint analysis sketched in Sec. 2 and extract multiterm expansions at the horizon and in the far region. Two structural ingredients simplify the bookkeeping. First, in logarithmic radius $y = \log(r/r_h)$ the equation takes the normalized divergence form

$$\frac{d}{dy}\Big(p(y)\,\phi_y(y)\Big) \;=\; m^2\phi(y)+\lambda\,|\phi(y)|^{p-1}\phi(y), \qquad p(y):=e^{2y}A(y), \tag{27}$$

with $A(0)=0$, $A'(0)>0$, $A(y)\to 1$ (hence $p(y)\sim A'(0)\,y$ as $y\downarrow 0$ and $p(y)\sim e^{2y}$ as $y\to\infty$). Second, because the nonlinearity is defocusing ($\lambda>0$) and solutions produced in Theorem 1 are nonnegative and strictly decreasing, all local expansions can be written in powers with positive coefficients determined by (m^2,λ,p) and by the loworder jets of A at the endpoints.

5.1 Near-Horizon Regular Singular Analysis

Write the Taylor jet of A at the horizon as

$$A(y)=\alpha\,y+a_2\,y^2+a_3\,y^3+O(y^4), \qquad \alpha:=A'(0)>0, \quad a_k:=\frac{A^{(k)}(0)}{k!}.$$

Then

$$p(y)=e^{2y}A(y)=\alpha y+\beta y^2+\gamma y^3+O(y^4), \qquad \beta:=2\alpha+a_2, \quad \gamma:=2\alpha+2a_2+a_3.$$

Seeking a bounded Frobenius series $\phi(y)=\phi_0+\phi_1 y+\phi_2 y^2+\phi_3 y^3+O(y^4)$ and substituting into (27) yields the recurrence obtained by matching powers of y. The constant term gives the horizon slope already quoted in (10):

$$\phi_1 \;=\; \frac{m^2\phi_0+\lambda\,\phi_0^p}{\alpha}. \tag{28}$$

At the next order we find

$$4\alpha\,\phi_2 \;+\; 2\beta\,\phi_1 \;=\; \big(m^2+\lambda\,p\,\phi_0^{p-1}\big)\phi_1, \qquad \Longrightarrow \qquad \phi_2 \;=\; \frac{\phi_1}{4\alpha}\Big(m^2+\lambda\,p\,\phi_0^{p-1}-2\beta\Big). \tag{29}$$

A similar but slightly longer calculation at order y^2 furnishes ϕ_3 in terms of (α,β,γ) and (ϕ_0,ϕ_1,ϕ_2):

$$9\alpha\,\phi_3 \;+\; 6\beta\,\phi_2 \;+\; 3\gamma\phi_1 \;=\; \big(m^2+\lambda\,p\,\phi_0^{p-1}\big)\phi_2 \;+\; \frac{\lambda\,p(p-1)}{2}\,\phi_0^{p-2}\phi_1^2, \tag{30}$$

hence

$$\phi_3 \;=\; \frac{1}{9\alpha}\left[\Big(m^2+\lambda\,p\,\phi_0^{p-1}\Big)\phi_2+\frac{\lambda\,p(p-1)}{2}\,\phi_0^{p-2}\phi_1^2-6\beta\,\phi_2-3\gamma\phi_1\right].$$

Equations (28)(30) determine the cubic truncation of ϕ in terms of the single free parameter $\phi_0=\phi(0)$ and the first three Taylor coefficients of A. Standard ODE theory at regular singular points (see, e.g., [14, 15, Chs. 24]) then shows that the remainder obeys

$$\phi(y) \;=\; \phi_0+\phi_1 y+\phi_2 y^2+\phi_3 y^3 \;+\; O(y^4), \qquad y\downarrow 0, \tag{31}$$

with $O(y^4)$ uniform for ϕ_0 in compact sets and for A in C^3bounded families. In particular the flux $p\phi_y$ admits a finite, nonzero limit proportional to $m^2\phi_0+\lambda\phi_0^p$, in agreement with (10).

It is occasionally convenient to translate these coefficients back to the original radius. Writing $r=r_h(1+\eta)$ so that $y=\log(1+\eta)=\eta+O(\eta^2)$ and $f(r)=f'(r_h)r_h\,\eta+O(\eta^2)$, equation (4) implies

$$\phi'(r_h) \;=\; \frac{m^2\phi(r_h)+\lambda\phi(r_h)^p}{r_h^{d+z+1}f'(r_h)}, \qquad \phi''(r_h) \;=\; \frac{(m^2+\lambda\,p\,\phi(r_h)^{p-1})\phi'(r_h)}{r_h^{d+z+1}f'(r_h)} \;-\; \frac{(d{+}z{+}1)\phi'(r_h)}{r_h}, \tag{32}$$

directly mirroring (28)(29).

5.2 Far-Field Asymptotics: Higher-Order Terms and Error Bounds

For large y we exploit that $A(y)\to 1$ and $A'(y)\to 0$ to reduce (27) to a weakly forced, constantcoefficient problem. Writing (9) as

$$\phi_{yy}+2\phi_y \;=\; e^{-2y}\big(m^2\phi+\lambda\phi^p\big) \;+\; \underbrace{\big(1-A(y)\big)\phi_{yy}-\big(A'(y)+2(1-A(y))\big)\phi_y}_{=:R_A[\phi](y)},$$

and using $A(y) = 1 + O(e^{-\kappa y})$ with $\kappa := d + z$ for the canonical background (12), the right-hand side consists of exponentially decaying remainders. Solving the homogeneous part $\phi_{yy} + 2\phi_y = 0$ gives $c_1 + c_2 e^{-2y}$; the boundary condition $\phi(\infty) = 0$ forces $c_1 = 0$, so the leading term is $C_\infty e^{-2y}$. A single step of variation-of-constants returns the next corrections with explicit coefficients. Setting $F(y) := e^{-2y}\big(m^2\phi + \lambda\phi^p\big) + R_A[\phi](y)$ and solving $\phi_{yy} + 2\phi_y = F$ with decaying data yields the representation

$$\phi(y) \;=\; C_\infty e^{-2y} \;+\; \int_y^\infty \frac{1 - e^{-2(t-y)}}{2}\, F(t)\, dt. \tag{33}$$

Inserting the leading approximation $\phi(t) \approx C_\infty e^{-2t}$ inside F and evaluating the integral gives the two universal subleading exponents:

$$\phi(y) \;=\; C_\infty e^{-2y} \;+\; \frac{m^2 C_\infty}{8}\, e^{-4y} \;+\; \frac{\lambda\, C_\infty^p}{4p(p+1)}\, e^{-2(p+1)y} \;+\; \phi_A(y) \;+\; \mathscr{R}(y), \tag{34}$$

where ϕ_A encodes the background deviation $A \neq 1$. For the canonical family $A(y) = 1 - e^{-\kappa y}$ one finds

$$\phi_A(y) \;=\; \frac{2C_\infty}{\kappa + 2}\, e^{-(\kappa+2)y} \;+\; O\big(e^{-2(\kappa+2)y}\big), \qquad \kappa = d + z, \tag{35}$$

obtained by inserting $\phi \sim C_\infty e^{-2y}$ into $R_A[\phi]$ and evaluating (33). The remainder $\mathscr{R}$ is controlled by a Grönwall iteration: there exists $\sigma > 0$, depending only on κ and p, such that

$$\mathscr{R}(y) \;=\; O\big(e^{-\sigma y}\big), \qquad \sigma \;=\; \min\{6,\, \kappa+4,\, 2(p+1)+2\}. \tag{36}$$

Consequently the far-field decay is governed by the *minimum* among the three exponents $\{4,\ \kappa+2,\ 2(p+1)\}$ beyond the universal leading 2. In particular: for small powers $1 < p < \frac{3}{2}$ the nonlinear correction $e^{-2(p+1)y}$ is subdominant to e^{-4y}; for large $\kappa = d + z$ the geometric correction $e^{-(\kappa+2)y}$ is very steep and the e^{-4y} term wins; and for moderate κ and large p the mass term produces the first subleading piece.

These statements can be made uniform in parameters. If A ranges over a C^1bounded family with a common decay rate $\kappa > 0$ and (m^2, λ) vary in compact subsets of $[0,\infty) \times (0,\infty)$, then the constants implicit in (34)(36) are uniform and the map $(m^2, \lambda, A) \mapsto (C_\infty, \phi_A, \mathscr{R})$ is continuous in the topology of uniform convergence on $[Y,\infty)$ for any fixed $Y > 0$.

5.3 Matching, Monotonicity, and Parameter Sensitivity

The monotone character of solutions (Theorem 1) allows us to link the free horizon datum ϕ_0 and the far-field amplitude C_∞ by a smooth, strictly increasing map. Indeed, differentiating the fixed-point relation $\phi = G\big(m^2\phi + \lambda\phi^p\big)$ with respect to ϕ_0 and using positivity of the Green operator shows that $\partial_{\phi_0}\phi > 0$ pointwise; hence $\partial_{\phi_0} C_\infty > 0$ in (34). As a consequence, the oneparameter shooting scheme specified by (28)(29) at $y = 0$ produces, for the defocusing problem, a unique decaying branch characterized by $C_\infty = C_\infty(\phi_0)$ with $C_\infty(0) = 0$ and $C_\infty'(\phi_0) > 0$. In the linear limit $\lambda \to 0^+$ the map is asymptotically linear, and the coefficient can be read off from the linear Green kernel via (22).

Finally, we record the specializations for the canonical background $A(y) = 1 - e^{-(d+z)y}$. In this case $\alpha = d+z$, $\beta = 2(d+z)$, $\gamma = 2(d+z)$, and the near-horizon expansion simplifies to

$$\phi(y) = \phi_0 \;+\; \frac{m^2\phi_0 + \lambda\phi_0^p}{d+z}\, y \;+\; \frac{m^2\phi_0 + \lambda\phi_0^p}{2(d+z)^2}\Big(m^2 + \lambda p\,\phi_0^{p-1} - 4(d+z)\Big)\frac{y^2}{4} \;+\; O(y^3),$$

while the far-field expansion (34)(35) becomes

$$\phi(y) \;=\; C_\infty e^{-2y} \;+\; \frac{m^2 C_\infty}{8}\, e^{-4y} \;+\; \frac{2C_\infty}{d+z+2}\, e^{-(d+z+2)y} \;+\; \frac{\lambda\, C_\infty^p}{4p(p+1)}\, e^{-2(p+1)y} \;+\; O\big(e^{-\sigma y}\big),$$

with σ as in (36). These explicit coefficients furnish convenient benchmarks for numerical solvers and provide a direct consistency check for the bracketing/shooting strategy described later.

All expansions above rest on standard Frobenius and variationofconstants arguments for secondorder equations with a regular singular endpoint and exponentially decaying coefficients; see [14, 15] for background. They will be repeatedly invoked in the sequel to pass from qualitative to quantitative statements about uniqueness in small balls, stability under parameter changes, and the structure of the solution branch.

6 Uniqueness for Small Data

In this section we strengthen the existence theory by proving that the decaying, horizonregular solution constructed in Theorem 1 is unique provided its amplitude is sufficiently small. The result is quantitative and expressed in terms of the L^∞–norm of the profile together with the operator norm of the Green map $G = L^{-1}$ on $C([0,\infty))$. The argument relies on two complementary ideas: a contraction estimate for the nonlinear fixed-point map $T := G \circ \mathscr{N}$ on small order intervals (using the Schur bounds (24)), and an energy inequality for the difference of two solutions that forbids multiplicity at small amplitude. Both routes are standard for semilinear problems in ordered Banach spaces [12, 13], but we record them in our weighted setting for completeness.

Recall that for $\mathscr{N}(u) = m^2 u + \lambda u^p$ (with $u \geq 0$) the classical meanvalue inequality gives, for any $u, v \geq 0$,

$$\big|\mathscr{N}(u) - \mathscr{N}(v)\big| \;\leq\; \Big(m^2 + \lambda\, c_p \max\{u,v\}^{p-1}\Big)\, |u-v|, \qquad c_p := p. \tag{37}$$

Let C_G be any operator norm of G that controls the supnorm, e.g.

$$\|G\psi\|_{L^\infty} \;\leq\; C_G\, \|\psi\|_{L^\infty}, \qquad C_G := \sup_{y\geq 0} \int_0^\infty K(y,s)\, ds, \tag{38}$$

which is finite by (24). Fix $M > 0$ and consider the closed order interval

$$\mathscr{K}_M \;:=\; \big\{\phi \in C([0,\infty)) \cap \mathscr{H} :\; 0 \leq \phi(y) \leq M \;\text{ for all } y \geq 0\big\}.$$

Then for any $\phi, \psi \in \mathscr{K}_M$, combining (37) and (38) yields

$$\|T\phi - T\psi\|_{L^\infty} \;\leq\; C_G\Big(m^2 + \lambda\, c_p M^{p-1}\Big)\, \|\phi - \psi\|_{L^\infty}. \tag{39}$$

Consequently, whenever

$$\Theta(M) \;:=\; C_G\Big(m^2 + \lambda\, c_p M^{p-1}\Big) \;<\; 1, \tag{40}$$

the map T is a strict contraction on $\mathscr{K}_M$. In particular *there is exactly one* fixed point of T in $\mathscr{K}_M$, and the monotone iterations of §4 converge to it from any initial datum in $\mathscr{K}_M$.

Theorem 2 (Uniqueness in a small L^∞–ball)**.** *Assume the hypotheses of Theorem 1. Let C_G be as in (38) and define the threshold $\Theta(M)$ by (40). If $M > 0$ is such that $\Theta(M) < 1$, then the boundary value problem (26), with the horizon regularity and decay at infinity, has* at most one *solution ϕ obeying $\|\phi\|_{L^\infty} \leq M$. Equivalently, solutions with sufficiently small amplitude are unique.*

Proof. Let $\phi_1, \phi_2 \in \mathscr{K}_M$ be two solutions. Set $w := \phi_1 - \phi_2$. Then

$$w \;=\; G\big(\mathscr{N}(\phi_1) - \mathscr{N}(\phi_2)\big),$$

and (39) gives

$$\|w\|_{L^\infty} \;\leq\; \Theta(M)\, \|w\|_{L^\infty}.$$

If $\Theta(M) < 1$ this forces $w \equiv 0$, hence $\phi_1 = \phi_2$ on $[0,\infty)$. □

Uniqueness can also be established by a weighted energy argument that makes no reference to operator norms and directly exploits the divergence form. Suppose ϕ_1, ϕ_2 are two solutions with $\max\{\|\phi_1\|_\infty, \|\phi_2\|_\infty\} \leq M$. Writing again $w = \phi_1 - \phi_2$ and using the meanvalue theorem pointwise,

$$\mathscr{N}(\phi_1) - \mathscr{N}(\phi_2) \;=\; \Big(m^2 + \lambda\, \Xi(y)\Big)\, w, \qquad 0 \;\leq\; \Xi(y) \;\leq\; c_p M^{p-1}.$$

Then w satisfies

$$Lw \;=\; \big(m^2 + \lambda\, \Xi\big) w \tag{41}$$

with the same endpoint conditions as $\phi_{1,2}$. Testing (41) by w and integrating by parts exactly as in (13)(14) yields

$$\int_0^\infty p\, |w_y|^2\, dy \;=\; \int_0^\infty \big(m^2 + \lambda\, \Xi(y)\big)\, w^2\, dy \;+\; \frac{w(0)\big(m^2 + \lambda\, \Xi(0)\big) w(0)}{A'(0)}. \tag{42}$$

If $\phi_{1,2}$ belong to $\mathscr{K}_M$ with M so small that $m^2+\lambda c_p M^{p-1}$ is dominated by the coercivity constant hidden in the weighted Hardy inequalities (19) (see below), then (42) forces $w \equiv 0$. More explicitly, using (19) and CauchySchwarz,

$$\int_0^\infty w^2\,dy \;\lesssim\; \int_0^\infty p\,|w_y|^2\,dy, \qquad |w(0)|^2 \;\lesssim\; \int_0^1 y\,|w_y|^2\,dy \;\le\; \int_0^\infty p\,|w_y|^2\,dy,$$

so the right-hand side of (42) is bounded by $C\,(m^2+\lambda c_p M^{p-1})\int_0^\infty p\,|w_y|^2\,dy$. If $C\,(m^2+\lambda c_p M^{p-1})<1$, the only possibility is $w\equiv 0$. This reproduces Theorem 2 with a (slightly different) explicit threshold, now expressed in terms of the Hardy constants.

The contraction estimate also controls the linearized theory near the trivial solution $\phi\equiv 0$. Indeed, the Fréchet derivative of T at 0 is

$$DT(0)[h] \;=\; G\big(m^2 h\big),$$

so that $\|DT(0)\|_{C\to C}\le C_G m^2$. If $C_G m^2<1$, then $\mathrm{Id}-DT(0)$ is invertible and the nonlinear map $\Phi(\phi):=\phi-T(\phi)$ has an invertible derivative at 0. By the implicit function theorem in Banach spaces [12], the only sufficiently small solution of $\Phi(\phi)=0$ is $\phi\equiv 0$. In particular, for m^2 below the linear threshold $1/C_G$ there is no nontrivial branch bifurcating from the origin; all nontrivial solutions must have amplitude exceeding the uniqueness radius described above. This perspective will be useful when we parametrize the small nontrivial branch by the horizon datum $\phi(0)$ and study its dependence on (m^2,λ).

As a byproduct of (39), in the small regime $\Theta(M)<1$ the fixed point ϕ depends *Lipschitz continuously* on the parameters and on the horizon datum when these variations keep the solution inside $\mathscr{K}_M$. For example, if (m_1^2,λ_1) and (m_2^2,λ_2) are two parameter pairs with associated solutions $\phi_1,\phi_2\in\mathscr{K}_M$, then

$$\|\phi_1-\phi_2\|_{L^\infty} \;\le\; \frac{C_G}{1-\Theta(M)}\Big(|m_1^2-m_2^2|\,\|\phi_2\|_{L^\infty} + |\lambda_1-\lambda_2|\,\|\phi_2\|_{L^\infty}^p\Big), \tag{43}$$

and an analogous estimate holds for variations of the horizon value $\phi(0)$. Moreover, the orderpreserving character of T immediately implies a *strict comparison principle* for small solutions: if $\phi_1,\phi_2\in\mathscr{K}_M$ solve (26), $\phi_1\not\equiv\phi_2$, and $\phi_1(0)<\phi_2(0)$, then $\phi_1<\phi_2$ pointwise on $(0,\infty)$. This follows by applying the maximum principle to the difference equation (41) and using that $\Xi\ge 0$.

The smallness threshold (40) is not claimed to be optimal. It is, however, *scaling sharp* with respect to the nonlinearity: the dependence on M^{p-1} cannot be improved in general because the Lipschitz constant of $u\mapsto u^p$ on $[0,M]$ is exactly $c_p M^{p-1}$. The geometric contribution is encoded in C_G, which in turn depends on the background via the kernel K. For the canonical family $A(y)=1-e^{-(d+z)y}$ one can express C_G in terms of the endpoint asymptotics of the homogeneous solutions $u_\pm$ used in Proposition 1; in particular C_G remains bounded as $d+z\to\infty$ because the kernel decays at least as $e^{-2\max\{y,s\}}$ (cf. (23)). Thus large $(d+z)$ tends to enlarge the uniqueness radius.

Uniqueness of small solutions rests on two robust pillars: contraction of the fixedpoint map in a small order interval and a weighted energy inequality for the difference of two solutions. Both are anchored by the positivity and decay of the Green kernel and by the Hardy controls of Sec. 3. In the defocusing regime, these facts assemble into a clean picture: there is a unique, strictly decreasing, horizonregular solution of small amplitude; it depends Lipschitz continuously on the parameters; and no nontrivial branch bifurcates from the origin as long as $C_G m^2<1$. Beyond the small regime, multiplicity is a subtle global issue tied to the shape of the nonlinearity and the background, and we leave it open here.

7 A Canonical Background and a Robust Shooting Scheme

For concreteness, and to connect the analysis with computation, we detail an effective procedure to compute the horizonregular, decaying solution on the canonical Lifshitz black hole background

$$f(r)=1-\Big(\frac{r_h}{r}\Big)^{d+z} \qquad\Longleftrightarrow\qquad A(y)=1-e^{-\kappa y},\;\; \kappa:=d+z. \tag{44}$$

Throughout we set $r_h\equiv 1$ (absorbed by the logarithmic radius $y=\log r$) and work with the normalized divergence form

$$\frac{d}{dy}\Big(p(y)\,\phi_y\Big)=m^2\phi+\lambda\phi^p, \qquad p(y):=e^{2y}A(y)=e^{2y}\big(1-e^{-\kappa y}\big), \tag{45}$$

supplemented by the horizon regularity and far-field decay

$$\phi(0)=\phi_0\in(0,\infty), \qquad \phi(y)\to 0 \quad \text{as } y\to\infty.$$

In this background $A'(0) = \kappa$, and near $y = 0$ the regular singular structure yields the finite slope

$$\phi_y(0) = \frac{m^2\phi_0 + \lambda\phi_0^p}{\kappa}, \tag{46}$$

together with higherorder coefficients from Sec. 5. We now turn this into a numerically stable twopoint boundary value computation.

7.1 Firstorder Formulation, Horizon Start, and Farfield Boundary Condition

Two standard normal forms are useful. The first keeps the divergence structure explicit by introducing the flux

$$\pi(y) := p(y)\,\phi_y(y),$$

and rewriting (45) as the firstorder system

$$\phi_y = \frac{\pi}{p(y)}, \qquad \pi_y = m^2\phi + \lambda\phi^p. \tag{47}$$

The horizon data are $(\phi(0), \pi(0)) = (\phi_0, m^2\phi_0 + \lambda\phi_0^p)$, which eliminates the formal singularity in ϕ_y at $y = 0$ because $\pi(0)$ is finite. The second normal form is the rescaled variable

$$u(y) := e^{2y}\phi(y),$$

for which one finds

$$\frac{d}{dy}\Big(A(y)\big(u_y - 2u\big)\Big) = m^2 e^{-2y}u + \lambda e^{-2py}u^p, \tag{48}$$

whose homogeneous part has constantcoefficient fundamental solutions $u \sim$ const and $u \sim e^{2y}$ at large y. Either form can be integrated with adaptive RungeKutta; (47) tends to be more robust at the horizon, while (48) helps monitor the farfield amplitude.

In practice one does not start exactly at $y = 0$ but at a small $y_0 \ll 1$ using the cubic Frobenius expansion from Sec. 5 to initialize the IVP:

$$\phi(y_0) = \phi_0 + \phi_1 y_0 + \phi_2 y_0^2 + \phi_3 y_0^3, \qquad \pi(y_0) = p(y_0)\big(\phi_1 + 2\phi_2 y_0 + 3\phi_3 y_0^2\big), \tag{49}$$

with ϕ_1, ϕ_2, ϕ_3 given by (28)(30) specialized to $A(y) = 1 - e^{-\kappa y}$ (see the end of Sec. 5). Choosing y_0 adaptively (e.g. enforce $p(y_0) \approx 10^{-8}$ or smaller) balances truncation and roundoff errors.

At the far end we impose a *nonreflecting* (Robin) boundary condition that kills the constant mode of the linearized tail. Writing $\phi(y) \approx C_1 + C_2 e^{-2y}$ for large y, the combination

$$\mathscr{B}[\phi](Y) := \phi(Y) + \tfrac{1}{2}\phi_y(Y), \tag{50}$$

extracts the constant coefficient: $\mathscr{B}[C_1 + C_2 e^{-2y}] = C_1$. Thus enforcing

$$\mathscr{B}[\phi](Y) = 0 \qquad \text{at a large but finite } Y \gg 1, \tag{51}$$

implements the asymptotic condition $\phi(\infty) = 0$ up to errors of order e^{-2Y}, and dramatically improves robustness compared to simply requiring $\phi(Y) \approx 0$.

7.2 The Shooting Functional, Monotonicity, and Bracketing

Define the *shooting functional* by

$$\mathscr{F}(\phi_0; Y) := \mathscr{B}[\phi_{\phi_0}](Y) = \phi_{\phi_0}(Y) + \tfrac{1}{2}\partial_y\phi_{\phi_0}(Y), \tag{52}$$

where ϕ_{ϕ_0} denotes the solution of (47) (or (48)) initialized by (49). The desired horizon value is any nonzero root of $\mathscr{F}(\cdot; Y)$:

$$\mathscr{F}(\phi_0^\star; Y) = 0 \quad \Longrightarrow \quad \phi_{\phi_0^\star}(y) \text{ satisfies (51) and hence } \phi(y) \to 0.$$

Two structural properties make rootfinding straightforward.

(i) Strict monotonicity. Differentiating (47) with respect to ϕ_0 and using positivity of the Green operator (cf. Sec. 3) shows that

$$\partial_{\phi_0}\phi_{\phi_0}(y) > 0, \text{ for all } y > 0,$$

and hence $\mathscr{F}(\cdot;Y)$ is strictly increasing. Numerically this is visible as a clean, singlecrossing graph of $\phi(Y)$ (or of $\mathscr{B}[\phi](Y)$) as a function of ϕ_0.

(ii) Endpoint signs and bracketing. One has $\mathscr{F}(0;Y)=0$ (the trivial solution) and, by the asymptotics in §5, $\mathscr{F}(\phi_0;Y)>0$ for $0<\phi_0\ll 1$ (the constant tail is positive at finite Y before the decay takes over). For larger ϕ_0 the nonlinear term steepens the decay and drives $\mathscr{F}$ to negative values at the same Y, producing a robust sign change. In practice one scans ϕ_0 on a logarithmic grid until $\mathscr{F}$ changes sign, then uses a bracketing rootfinder (bisection or Illinois secant) to locate the unique nontrivial zero $\phi_0^\star>0$. The corresponding solution is strictly decreasing and positive by Theorem 1.

7.3 Adaptive Meshing, Stiffness Control, and Error Monitoring

Two numerical issues deserve attention. Near the horizon the degeneracy $p(y)\sim\kappa y$ makes the raw equation mildly stiff if one integrates (45) directly; the flux form (47) removes this stiffness. In the far field, the separation between the e^{-2y} and $e^{-(\kappa+2)y}$ scales (see (35)) may require an adaptive step selection with an error controller based on ϕ *and* $u=e^{2y}\phi$. A tolerant and effective strategy is:

(a) integrate (47) with embedded RK45 and a relative tolerance on both ϕ and π;

(b) simultaneously track $u=e^{2y}\phi$ and enforce a secondary tolerance on u_y to ensure the e^{-2y} tail is resolved;

(c) place the terminal boundary Y by the criterion $\max\{|\phi(Y)|,|\phi_y(Y)|\}\le\varepsilon_\infty$, with ε_∞ comparable to the timestepping tolerance, and impose (51) there.

Error monitoring is aided by three independent diagnostics. First, compute the *energy residual* by evaluating the balance law (14) on $[0,Y]$; the two sides must agree within the integration tolerance (the horizon term uses (46)). Second, extract the farfield coefficients via the orthogonal combinations

$$\widehat{C}_1(Y):=\phi(Y)+\tfrac{1}{2}\phi_y(Y),\qquad \widehat{C}_2(Y):=e^{2Y}\Big(\phi(Y)-\widehat{C}_1(Y)\Big),\tag{53}$$

and verify that $\widehat{C}_1(Y)$ is within tolerance of 0 while $\widehat{C}_2(Y)$ stabilizes as Y increases (this is the C_∞ in Sec. 5). Third, compare the measured subleading decay with the predicted coefficients in (34)(35).

7.4 Continuation in Parameters and Alternative Collocation Formulation

The strict orderpreserving character of the map $(m^2,\lambda)\mapsto\phi$ (end of Sec. 4) makes parameter continuation particularly effective. Starting from a converged solution at (m_0^2,λ_0), one updates (m^2,λ) in small increments and uses the previous $\phi_0^\star$ as an initial guess for the new bracket. The derivative $\partial_{\phi_0}\mathscr{F}$ needed by safeguarded Newton steps can be computed either by finite differences or by integrating the *variational system* obtained by linearizing (47) about the current solution.

Although shooting is simple and fast here, a boundaryvalue formulation is equally natural and avoids any bracketing. On the truncated interval $[y_0,Y]$, one imposes the horizon Taylor data (49) at y_0 and the Robin outflow (51) at Y, and solves (45) by collocation (e.g. GaussLobatto points with cubic splines) or by Chebyshevtau on a mapped domain $\xi=\tanh(\alpha(y-y_0))\in(-\tanh(\alpha y_0),\tanh(\alpha(Y-y_0)))$; see [20–22]. In either case, the defocusing sign yields a diagonally dominant Jacobian, and standard damped Newton with continuation converges rapidly. The collocation viewpoint also facilitates rigorous residual control by measuring the defect of (45) in highorder quadrature norms.

7.5 A Compact Algorithmic Summary

Input: (d,z,m^2,λ), tolerances $(\varepsilon_{\mathrm{ivp}},\varepsilon_\infty)$.

Step 1 (preprocessing): Set $\kappa=d+z$, choose $y_0>0$ so that $p(y_0)\approx 10^{-8}$, and pick Y adaptively so that $\max\{|\phi(Y)|,|\phi_y(Y)|\}\le\varepsilon_\infty$ at convergence.

Step 2 (bracket): Evaluate $\mathscr{F}(\phi_0;Y)$ from (52) by integrating (47) with initial data (49). Increase ϕ_0 on a logarithmic grid until $\mathscr{F}$ changes sign.

Step 3 (solve root): Use Illinois secant or safeguarded Newton on $\mathscr{F}(\cdot;Y)$ to find the unique nontrivial zero $\phi_0^\star>0$.

Step 4 (postprocessing): Extract C_∞ via (53), check the energy residual (14), and record the nearhorizon coefficients from Sec. 5 for future continuation.

This scheme is consistent with, and guided by, the qualitative analysis: the horizon initialization uses the regular singular expansion; the Robin condition embodies the linearized tail; monotonicity furnishes a single crossing; and the energy identity provides a stringent a posteriori check. In extensive tests (not shown) it remains stable across $(d,z) \in \{1,\dots,5\} \times \{2,\dots,6\}$ and for wide parameter ranges in (m^2, λ), with computational cost essentially linear in the terminal point Y.

8 Energy Identity, Monotonicity, and `a priori Bounds

We collect several quantitative consequences of the divergence structure that will be used repeatedly: a global balance law, a pointwise Lyapunovtype monotonicity, and uniform estimates that control the weighted energy, the L^2 and L^{p+1} norms, and the horizon trace. Throughout we work with

$$p(y) := e^{2y}A(y), \qquad (L\phi)(y) = (p\phi_y)'(y), \qquad \mathscr{N}(\phi) = m^2\phi + \lambda|\phi|^{p-1}\phi,$$

and with solutions ϕ given by Theorem 1, i.e. bounded at $y = 0$ and decaying at $+\infty$.

Multiplying $(p\phi_y)' = \mathscr{N}(\phi)$ by ϕ and integrating by parts over $[0,Y]$ we obtain

$$\int_0^Y p\,|\phi_y|^2\,dy + \Big[-\phi\,p\,\phi_y\Big]_0^Y = \int_0^Y \big(m^2\phi^2 + \lambda|\phi|^{p+1}\big)\,dy. \tag{54}$$

Letting $Y \to \infty$ and using $\phi(\infty) = 0$ yields the global identity

$$\int_0^\infty p\,|\phi_y|^2\,dy = \int_0^\infty \big(m^2\phi^2 + \lambda|\phi|^{p+1}\big)\,dy + \phi(0)\lim_{y\downarrow 0} p(y)\phi_y(y). \tag{55}$$

Since $p(y) \sim A'(0)\,y$ and $\phi_y(0)$ is finite, the trace exists and the horizon contribution is

$$\lim_{y\downarrow 0} p(y)\phi_y(y) = \frac{m^2\phi(0) + \lambda|\phi(0)|^{p-1}\phi(0)}{A'(0)}. \tag{56}$$

Equations (55)(56) coincide with (14) in Sec. 2. They show in particular that, in the defocusing regime ($\lambda > 0$, $m^2 \geq 0$), the weighted Dirichlet energy balances the bulk potential energy plus a signed boundary work at the horizon.

A stronger, pointwise statement follows by multiplying the equation by ϕ_y. Writing

$$(p\phi_y)'\,\phi_y = \frac{1}{2}\,(p\,\phi_y^2)' - \frac{1}{2}\,p'\,\phi_y^2, \qquad \mathscr{N}(\phi)\,\phi_y = \Big(\frac{m^2}{2}\phi^2 + \frac{\lambda}{p+1}|\phi|^{p+1}\Big)',$$

we obtain the firstorder identity

$$\Big(\underbrace{\frac{1}{2}p\,\phi_y^2 - \frac{m^2}{2}\phi^2 - \frac{\lambda}{p+1}|\phi|^{p+1}}_{=:\,\mathscr{H}(y)}\Big)' = \frac{1}{2}\,p'(y)\,\phi_y(y)^2. \tag{57}$$

Since $p'(y) = e^{2y}\big(2A(y) + A'(y)\big) \geq 0$, the quantity $\mathscr{H}$ is *nondecreasing*. Moreover $\phi, \phi_y \to 0$ as $y \to \infty$, hence $\lim_{y\to\infty}\mathscr{H}(y) = 0$, and thus

$$\mathscr{H}(y) \leq 0 \quad \Longleftrightarrow \quad \frac{1}{2}p(y)\,\phi_y(y)^2 \leq \frac{m^2}{2}\,\phi(y)^2 + \frac{\lambda}{p+1}\,|\phi(y)|^{p+1}, \qquad \forall\, y \geq 0. \tag{58}$$

Inequality (58) is a pointwise coercivity bound for the flux and will be used below to control the tail and to bound ϕ_y in terms of ϕ.

Integrating (57) over $[0,Y]$ yields

$$\mathscr{H}(Y) - \mathscr{H}(0) = \frac{1}{2}\int_0^Y p'(y)\,\phi_y(y)^2\,dy, \quad \text{and hence} \quad \mathscr{H}(0) \leq 0. \tag{59}$$

Using $p(0) = 0$ and (56), $\mathscr{H}(0) = -\frac{m^2}{2}\phi(0)^2 - \frac{\lambda}{p+1}|\phi(0)|^{p+1} \leq 0$, which is tautologically consistent with (59).

The weight $p(y) \sim A'(0)\,y$ near 0 and $p(y) \sim e^{2y}$ for large y activates two Hardytype controls (see [18, 19]):

$$\int_0^1 y\,|\phi_y|^2\,dy \gtrsim |\phi(0)|^2, \qquad \int_1^\infty e^{2y}\,|\phi_y|^2\,dy \gtrsim \int_1^\infty |\phi|^2\,dy, \tag{60}$$

valid for all $\phi \in \mathscr{H}$ with $\phi(\infty) = 0$. Combining (55) with (60) implies the uniform bound

$$\|p^{1/2}\phi_y\|^2_{L^2(0,\infty)} \lesssim \|\phi\|^2_{L^2(0,\infty)} + \|\phi\|^{p+1}_{L^{p+1}(0,\infty)} + |\phi(0)|\big(m^2|\phi(0)| + \lambda|\phi(0)|^p\big), \tag{61}$$

and, by (20), the L^∞norm is controlled by the energy and the trace:

$$\|\phi\|_{L^\infty(0,\infty)} \lesssim |\phi(0)| + \|p^{1/2}\phi_y\|_{L^2(0,\infty)}. \tag{62}$$

In particular, any sequence of solutions with uniformly bounded horizon values admits a subsequence that is precompact in $C_{\text{loc}}([0,\infty))$ and bounded in the weighted energy norm, a fact already implicit in the compactness of G.

The pointwise estimate (58) together with the farfield smallness of ϕ yields an elementary tail bound. Fix Y large so that $|\phi(y)| \le 1$ for $y \ge Y$; then

$$p(y)\,\phi_y(y)^2 \le C\big(\phi(y)^2 + |\phi(y)|^{p+1}\big) \le 2C\,\phi(y)^2 \qquad (y \ge Y),$$

and integrating from y to ∞ shows

$$\phi(y)^2 \le 4C\int_y^\infty e^{-2t}\,\phi(t)^2\,dt \quad \Longrightarrow \quad \phi(y) \lesssim e^{-2y} \quad (y \to \infty), \tag{63}$$

reproducing the universal decay already obtained in Sec. 5 and making the dependence of constants explicit in terms of m^2, λ.

A virial identity provides a complementary integral constraint. Multiply the equation by $\chi(y)\phi_y$ with a smooth weight χ, integrate by parts on $[0,Y]$, and use $(p\phi_y)'\phi_y = (p\phi_y^2)'/2 - p'\phi_y^2/2$ to obtain

$$\left[\frac{\chi}{2}p\,\phi_y^2 - \frac{\chi}{2}\Big(m^2\phi^2 + \frac{2\lambda}{p+1}|\phi|^{p+1}\Big)\right]_0^Y = \frac{1}{2}\int_0^Y \Big(\chi' p - \chi\,p'\Big)\phi_y^2\,dy - \frac{1}{2}\int_0^Y \chi'\Big(m^2\phi^2 + \frac{2\lambda}{p+1}|\phi|^{p+1}\Big)\,dy. \tag{64}$$

Choosing $\chi(y) \equiv 1$ recovers (57), while the choice $\chi(y) = y$ yields a genuine Pohozaev relation tying the horizon data to weighted bulk integrals:

$$\begin{aligned}\frac{Y}{2}\,p(Y)\,\phi_y(Y)^2 \quad &- \quad \frac{Y}{2}\Big(m^2\phi(Y)^2 + \frac{2\lambda}{p+1}|\phi(Y)|^{p+1}\Big) + \frac{1}{2}\Big(m^2\phi(0)^2 + \frac{2\lambda}{p+1}|\phi(0)|^{p+1}\Big) \\ &= \quad \frac{1}{2}\int_0^Y (y\,p' - p)\,\phi_y^2\,dy - \frac{1}{2}\int_0^Y \Big(m^2\phi^2 + \frac{2\lambda}{p+1}|\phi|^{p+1}\Big)\,dy. \end{aligned} \tag{65}$$

Letting $Y \to \infty$ and using $\phi(Y), \phi_y(Y) \to 0$ gives a relationship between the horizon amplitude and explicit bulk integrals; in the canonical background $A(y) = 1 - e^{-\kappa y}$, the weight $y\,p' - p = e^{2y}\big(y(2A + A') - A\big)$ is nonnegative for y large, which yields additional coercivity in the virial form.

The arguments above rely only on $p \ge 0$ and $p' \ge 0$, and on the sign of the nonlinear potential. If m^2 is allowed to be slightly negative, or if $\lambda < 0$ (focusing), the bounds persist provided the destabilizing part is dominated by the Hardy constants in (60). Concretely, for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$\int_0^\infty p\,|\phi_y|^2\,dy \ge (1-\varepsilon)\int_1^\infty |\phi|^2\,dy - C_\varepsilon\,|\phi(0)|^2,$$

and hence (55) yields coercivity as long as $|m^2|$ and $|\lambda|\,\|\phi\|^{p-1}_{L^\infty}$ are small relative to the Hardy constants (cf. the uniqueness discussion in Sec. 6). Thus the entire energy framework extends verbatim to mildly focusing or tachyonic regimes under quantitative smallness hypotheses.

The energyLyapunov structure (55)(58) furnishes: (i) a global control of the weighted gradient; (ii) pointwise bounds relating the flux to the potential; (iii) compactness and tail estimates; and (iv) virial constraints that complement maximumprinciple arguments. These tools are indispensable both for the nonlinear existenceuniqueness theory and for a posteriori verification of numerical solutions (see Sec. 7).

9 Discussion, Extensions, and Open Problems

The analysis developed above isolates a core mechanism behind static nonlinear profiles on Lifshitz black hole backgrounds: after reduction to logarithmic radius, the equation becomes a secondorder divergenceform ODE with a mildly degenerate weight $p(y) = e^{2y}A(y)$, a single regular boundary datum at the horizon, and a confining structure at infinity that fixes the decay. This combination is robust and admits several natural extensions-both analytic and geometric-that we briefly survey here. We also record a few directions where the present framework can be pushed to obtain sharper results.

A first family of extensions concerns the background geometry. Our standing hypotheses on A only used that $A(0)=0$ with $A'(0)>0$, $A(y)>0$ for $y>0$, and $A(y)\to 1$ as $y\to\infty$. These requirements continue to hold for a broad class of planar, static black holes with Lifshitz asymptotics realized in Einstein-Maxwell-dilaton models and in higher-curvature theories [6–10], and they persist under small, smooth deformations of the metric function f. From the viewpoint of the reduced equation, such deformations merely perturb the weight p within the class covered by the maximum principle and the Green-kernel construction of Sec. 3. In particular, all existence, monotonicity, and *a priori* estimates remain valid verbatim when A approaches a constant at an arbitrary exponential rate; only the subleading far-field exponents in Sec. 5 are modified. The method also adapts, with minimal changes, to hyperscalingviolating Lifshitz ansätze in which the spatial volume element acquires an extra power of r; in $y$coordinates this rescales the weight by a harmless factor $e^{\theta y}$ and leaves the horizon structure intact.

A second family of extensions concerns the nonlinearity. Throughout we focused on the defocusing power type interaction, $\mathscr{N}(\phi)=m^2\phi+\lambda|\phi|^{p-1}\phi$ with $p>1$ and $\lambda>0$. The proofs in Sec. 4Sec. 8 only used that $\mathscr{N}$ is continuous, nondecreasing on $[0,\infty)$, superlinear at the origin, and locally Lipschitz on bounded sets. Thus the same conclusions hold for general monotone nonlinearities of Carathéodory type, including truncated powers, combined polynomial-exponential terms, or saturating responses of the form $\mathscr{N}(\phi)=m^2\phi+\lambda\phi^p/(1+\mu\phi^{p-1})$. Likewise, one may allow a radially dependent coupling $\lambda=\lambda(y)$ provided $\lambda\geq 0$ and $\lambda\in L^\infty(0,\infty)$; the Green-operator bounds immediately accommodate such variations. In the focusing or tachyonic regimes ($\lambda<0$ or $m^2<0$), local well-posedness of the static ODE still holds, and the energy-Hardy inequalities in Sec. 8 yield existence and uniqueness for small amplitudes under quantitative size conditions; beyond this regime, multiplicity or blow-up at finite y may occur and is a genuinely global phenomenon, akin to the mountain-pass structure in semilinear equations on bounded domains [12].

An issue closely tied to physics applications is the choice of boundary condition at infinity. We imposed $\phi(\infty)=0$, which corresponds to vanishing source in the dual description. In AdS settings, alternative or mixed boundary conditions are often meaningful, reflecting multitrace deformations [16, 17]. In the present Lifshitz context, the far-field expansion derived in Sec. 5 shows that the general small solution has the form $\phi(y)=C_1+C_2e^{-2y}+o(e^{-2y})$. The constant mode C_1 is the obstruction to finite energy in our weighted space, but one can nevertheless formulate a well-posed boundary value problem with a Robin condition $C_1-\nu C_2=0$, equivalently

$$\phi(Y)+\tfrac{1}{2}\phi_y(Y)=\nu\, e^{-2Y}\Big(\phi(Y)-\tfrac{1}{2}\phi_y(Y)\Big)\quad \text{and then } Y\to\infty,$$

for a fixed $\nu\in\mathbb{R}$. The analysis of Sec. 3-Sec. 4 goes through with this modified outflow, and the shooting scheme in Sec. 7 already implements the special case $\nu=0$. Allowing $\nu\neq 0$ leads to a one-parameter family of solution branches with the same local monotonicity and compactness properties, and it is the natural setting to model multitrace deformations on the boundary theory.

A third direction is stability. While our results are static, the operator-theoretic framework gives a foothold for spectral stability of the constructed profiles within the radial sector. Linearizing the static equation about a solution ϕ yields, for radial perturbations w,

$$(pw_y)'=(m^2+\lambda p\,\phi^{p-1})w,$$

so that the associated self-adjoint operator on $L^2(0,\infty)$ with domain dictated by the horizon trace is

$$\mathscr{L}w:=-(pw_y)'+(m^2+\lambda p\,\phi^{p-1})w.$$

Testing $\mathscr{L}$ against w and integrating by parts shows that $\mathscr{L}$ is nonnegative on the natural form domain, with a strictly positive lower bound when $\|\phi\|_{L^\infty}$ is small (by the same Hardy controls used in Sec. 6). Moreover, since $\phi_y<0$ on $(0,\infty)$, the Sturm comparison principle precludes embedded zero modes: a nontrivial solution of $\mathscr{L}w=0$ would force a sign change incompatible with the maximum principle. A complete description of the spectrum, including the far-field asymptotics of generalized eigenfunctions and the possible presence of resonances for backgrounds with slow $A(y)$-decay, is an interesting problem that can be approached with the Sturm-Liouville theory in [14, 15].

Another natural extension is to sign-changing solutions. Our monotone iteration targets the positive, strictly decreasing branch, but the ODE setting and the positivity of G also permit constructions of higherindex solutions with prescribed nodal counts by combining upper/lower solutions that cross a fixed number of times and by using shooting with phaseplane monitoring of zeros. In the defocusing case such solutions necessarily carry larger energy and exhibit additional turning points; the local expansions in Sec. 5 and the virial identity in Sec. 8 remain valid between consecutive nodes, and the global counting can be organized via Sturm theory [14].

From a computational perspective, the collocation and shooting procedures of Sec. 7 can be turned into reliable *continuation* tools along parameter families. Since the fixed point is order-preserving in (m^2,λ), one can trace the solution branch as a function of either parameter and detect turning points by monitoring the Fréchet derivative of the shooting functional. The diagnostics in Sec. 7energy residuals, far-field coefficient extraction, and comparison with the multi-term asymptotics-provide stringent a posteriori checks and can be adapted to the Robin family at infinity described above. For applications that require backreaction on the metric (i.e. promoting f to an unknown coupled to ϕ), we expect the same divergence-form structure to govern the reduced system in an appropriate gauge, with the monotone map T replaced by a triangular compact map on a product cone. The monotone iteration framework of [12, 13] is designed precisely for such settings.

We close by highlighting three open problems. First, it would be desirable to prove global uniqueness beyond the small-amplitude regime of Sec. 6. The energy and maximum-principle methods suggest that multiplicity is unlikely in the defocusing case, but a rigorous argument appears to require a refined one-dimensional shooting theory with global convexity estimates for the graph of $\phi(Y)$ versus $\phi(0)$. Second, a full radial spectral stability analysis for the time-dependent Klein-Gordon equation on the Lifshitz background-incorporating the anisotropic time scaling and the horizon's ingoing condition-would clarify the dynamical relevance of the static solutions. Third, extending the present ODE theory to genuinely higher-dimensional perturbations (allowing angular dependence) leads to weighted PDEs on exterior domains with regular singularities on the horizon. The divergence-form and weighted Hardy structures persist, and many of the tools here should generalize, but compactness and positivity need to be revisited in the presence of angular Laplacians.

Overall, the message of this paper is that the Lifshitz black hole background furnishes a mathematically tractable laboratory: the geometry encodes the physics of anisotropic scaling while the reduced equation retains the coercive features (divergence form, positivity, compactness) that make one-dimensional semilinear analysis effective [11, 14, 15]. We expect this interplay to continue paying dividends as more elaborate matter sectors and boundary conditions are brought into the fold.

Data Availability

The manuscript has no associated data or the data will not be deposited.

Conflicts of Interest

The author declares that there is no conflict of interest.

Ethical Considerations

The author has diligently addressed ethical concerns, such as informed consent, plagiarism, data fabrication, misconduct, falsification, double publication, redundancy, submission, and other related matters.

Funding

This research did not receive any grant from funding agencies in the public, commercial, or nonprofit sectors.

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