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Research article

Enhanced Nonlinear Solvers for Shear-Dependent Viscosity Models in Fluid Dynamics

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Abstract

Nonlinear constitutive relations arise frequently in fluid mechanics, especially in flows of complex or non-Newtonian fluids where viscosity depends on deformation rates. This paper develops and analyzes a robust solution strategy for a representative nonlinear equation obtained from the steady, fully-developed flow of a shear-thinning fluid in a channel. The governing equation reduces to a nonlinear ordinary differential equation whose nonlinearity couples momentum transport with a rate-dependent effective viscosity. We introduce a hybrid fixed-point and Newton correction scheme, prove its convergence properties under physically realistic conditions, and evaluate its performance against standard iterative methods. The proposed approach shows significant improvements in convergence speed and stability, particularly in regimes where classical Newton iteration fails due to strong degeneracy in the viscosity law.

Keywords: Fluid dynamics, Nonlinear equation, Differential equation

Mathematics Subject Classification (2020): 76A05, 65H10, 76M12, 35Q35

1 Introduction

Nonlinear equations arise naturally and unavoidably in many branches of fluid mechanics, particularly whenever constitutive relations deviate from the classical Newtonian assumption. While Newtonian fluids obey a linear relationship between shear stress and strain rate, most real-world fluids—including polymer melts, blood, drilling muds, suspensions, and various industrial slurries—display rheological behaviors that are inherently nonlinear in nature. These behaviors include shear-thinning, shear-thickening, viscoplasticity, thixotropy, or viscoelastic response, all of which introduce strong nonlinear couplings into the governing momentum and energy equations. As a result, even simple geometrical configurations can lead to nonlinear algebraic or differential equations whose analytical solutions are inaccessible and whose numerical solution demands specially crafted iterative strategies.

The study of such nonlinear rheological systems has a long history, particularly following the formulation of generalized Newtonian and viscoelastic models during the twentieth century. The power-law model, introduced as an empirical description of shear-dependent viscosity, remains among the most widely used constitutive relations for representing polymeric and biological flows [1, 2]. Other important models include the CarreauYasuda and Cross models [3,4], which capture transitions between Newtonian plateaus and power-law regimes, as well as viscoplastic models such as the Bingham, Casson, and HerschelBulkley relations [5], which introduce yield stresses and subsequently complicate numerical algorithms due to non-differentiability. Each of these models leads to governing equations in which the viscosity or stress depends on the unknown kinematic variables, thereby introducing nonlinearity directly into the momentum balance.

The mathematical consequences of these nonlinearities have driven decades of research in numerical methods for fluid mechanics. Classical approaches include Newton–Raphson methods, fixed-point iteration, Picard linearization, and various quasi-Newton techniques [6–8]. Although Newton iteration remains the method of choice due to its quadratic convergence near the solution, its performance deteriorates severely when the Jacobian becomes nearly singular or when the initial guess lies outside the basin of attraction. Such behavior is particularly pronounced for strongly shear-thinning fluids, in which the effective viscosity may decrease rapidly toward zero, amplifying numerical stiffness and limiting the efficiency of direct application of Newtons method. On the other hand, pure fixed-point iteration offers better global robustness but often converges slowly, especially when the underlying operator exhibits weak contractivity. Thus, the search for iterative techniques that preserve the global stability of fixed-point iteration while retaining the rapid local convergence of Newton corrections remains an active and important topic.

A large body of literature has focused on improving nonlinear solvers for viscoplastic and generalized Newtonian flows. Studies have proposed regularization techniques, continuation frameworks, damped Newton variants, and hybrid iterative schemes to cope with degeneracy and nonsmoothness [9, 10]. Nevertheless, many of these methods face limitations when dealing with severely shear-dependent viscosity laws or when the governing nonlinear algebraic equations arise repeatedly in discretized momentum equations. Even in one-dimensional, steady, fully developed flows, the resulting algebraic equations can be highly nonlinear and must be solved with great efficiency and reliability to ensure the success of larger fluid simulations.

The aim of this work is to revisit a prototypical nonlinear equation arising in planar Poiseuille flow of a generalized Newtonian (specifically, shear-thinning power-law) fluid. This scalar nonlinear equation, which relates shear rate to an imposed stress distribution, is a fundamental component of numerical solvers for these systems and provides a clear, interpretable setting to explore nonlinear solution strategies. We propose a hybrid iterative method that blends Newton iteration with a fixed-point mapping derived from the constitutive law itself. This approach retains the global robustness of the fixed-point formulation while recovering the accelerated convergence of Newtons method near the solution. We analyze its stability properties, demonstrate its superiority to classical iterative techniques for strongly non-Newtonian regimes, and argue that the methodological framework generalizes naturally to more complex constitutive equations and multidimensional discretizations.

Beyond its practical computational benefits, the hybrid strategy contributes to the broader understanding of how physically motivated reformulation of nonlinear constitutive laws can lead to more efficient numerical solvers. Such insights are increasingly relevant as modern fluid simulations demand higher accuracy, stronger robustness, and the ability to handle complex rheological behaviors under challenging flow conditions. The results presented in this paper therefore form a basis for future extensions to time-dependent flows, viscoelastic constitutive models, and large-scale computational fluid dynamics (CFD) frameworks.

2 Governing Equation

The nonlinear equation at the center of this study arises from the classical configuration of steady, incompressible, fully developed flow between two infinite parallel plates. Although the geometry is simple, the presence of a generalized Newtonian constitutive law introduces a nonlinearity that fundamentally alters the character of the governing momentum equation. Let the plates be located at $y = -H$ and $y = H$, and assume that the flow is driven by a constant imposed pressure gradient in the streamwise x-direction. Under the assumption of full development, the velocity field reduces to the unidirectional form $u = u(y)$, while all derivatives with respect to x vanish. The incompressibility condition $\nabla \cdot \boldsymbol{u} = 0$ is identically satisfied, and the only nontrivial component of the Navier–Stokes equations is the balance of streamwise momentum.

For a generalized Newtonian fluid, the deviatoric stress tensor is expressed as

$$\boldsymbol{\tau} = 2\mu(\dot{\gamma})\boldsymbol{D}, \tag{1}$$

where $\boldsymbol{D} = \frac{1}{2}(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T)$ is the rate-of-strain tensor and $\dot{\gamma} = \sqrt{2\boldsymbol{D} : \boldsymbol{D}}$ is the associated shear rate invariant. In the present geometry, the only non-zero component of $\boldsymbol{D}$ is $D_{xy} = \frac{1}{2}\frac{du}{dy}$, and therefore the shear rate reduces to

$$\dot{\gamma}(y) = \left|\frac{du}{dy}\right|. \tag{2}$$

The sign of du/dy determines the direction of shear but does not affect the magnitude of the viscosity, which depends solely on the absolute shear rate.

We adopt the classical power-law viscosity model,

$$\mu(\dot{\gamma}) = K\dot{\gamma}^{n-1}, \tag{3}$$

where $K > 0$ is the consistency index and the flow behavior index n characterizes the rheology: $n < 1$ corresponds to shear-thinning, $n = 1$ recovers Newtonian behavior, and $n > 1$ describes shear-thickening fluids. Substituting the velocity field into the constitutive equation yields the shear stress

$$\tau_{xy}(y) = K\left|\frac{du}{dy}\right|^{n-1}\frac{du}{dy}. \tag{4}$$

This nonlinearity is central to the analysis that follows.

The streamwise component of the steady Navier–Stokes equation reduces to

$$\frac{d\tau_{xy}}{dy} = \frac{dp}{dx}, \tag{5}$$

where $\frac{dp}{dx}$ is constant due to the fully developed assumption. For convenience, we define

$$G = \frac{dp}{dx}, \tag{6}$$

which is typically negative for flow in the positive x direction, though its sign is irrelevant for the structure of the following nonlinear equation. Integrating (5) once with respect to y yields

$$\tau_{xy}(y) = Gy + C, \tag{7}$$

where C is an integration constant determined by symmetry. Because the geometry is symmetric with respect to the channel centerline $y = 0$, the shear stress must vanish there, leading to $C = 0$ and hence

$$\tau_{xy}(y) = Gy. \tag{8}$$

Thus, the shear stress varies linearly across the channel, a classical result obtained independently of the material rheology. The nonlinear behavior enters only when relating the shear stress to the velocity gradient.

Combining (4) and (8) produces a nonlinear algebraic equation for the shear rate $s(y) = du/dy$:

$$K|s(y)|^{n-1}s(y) = Gy. \tag{9}$$

Because the flow is antisymmetric about $y = 0$, we may without loss of generality restrict attention to the half-channel $y > 0$, for which $s(y) > 0$. In this region, the absolute value may be removed, and (9) becomes

$$Ks(y)^n = Gy. \tag{10}$$

This equation encapsulates the essential mathematical challenge: for each spatial location, the shear rate must satisfy a nonlinear power-law relation driven by the local shear stress Gy. Solving (10) repeatedly across the channel is a key subroutine in numerical algorithms for generalized Newtonian flows. Although the equation admits the closed-form analytic solution

$$s(y) = \left(\frac{Gy}{K}\right)^{1/n}, \tag{11}$$

its structure embodies the type of nonlinear algebraic equation that arises far more generally in complex rheological models—in particular, where constitutive relations lack such simple inverses or involve additional coupling terms. Consequently, (9) serves as a representative model problem for analyzing iterative techniques applicable to a wider class of nonlinear equations encountered in computational rheology.

To connect the shear rate back to the velocity field, one integrates the relation $du/dy = s(y)$, imposing the no-slip boundary condition $u(H) = 0$ at the wall. However, the focus of the present work is not the velocity solution itself but rather the efficient and robust solution of the nonlinear algebraic equation for $s(y)$, which is the computationally dominant operation when the constitutive relation is significantly more complex than the power-law form. The derivation above therefore establishes the mathematical foundation from which modern nonlinear solvers must proceed.

3 Numerical Strategy

The nonlinear algebraic equation derived in the previous section,

$$Ks^n = H, \qquad H = Gy, \tag{12}$$

must be solved for the shear rate s at each spatial location. Although for the power-law model this relation can be inverted analytically, the nonlinear structure of (12) closely resembles those encountered in more elaborate generalized Newtonian or viscoplastic models, where direct analytic inversion is no longer feasible. Therefore, rather than exploiting the explicit form $s = (H/K)^{1/n}$, we treat (12) as a representative nonlinear equation requiring iterative solution. This approach allows us to investigate solver performance in a controlled environment while developing techniques applicable to constitutive relations that are significantly more complicated.

To cast (12) into a root-finding form, we define the function

$$f(s) = Ks^n - H. \tag{13}$$

The goal is to determine $s > 0$ such that $f(s) = 0$. For $n < 1$, corresponding to shear-thinning behavior, the function f is highly nonlinear: its slope $f'(s) = Kns^{n-1}$ becomes unbounded near $s = 0$ when $n-1 < 0$, and the curvature becomes steep as s increases. These features pose substantial difficulties for iterative solvers, especially in the context of large-scale simulations where the nonlinear equation must be solved repeatedly as the stress field updates.

The classical Newton–Raphson method is the traditional choice for nonlinear problems of this type, offering quadratic convergence when the iterate lies sufficiently close to the true solution. Applied to (13), Newtons method generates the sequence

$$s_{k+1} = s_k - \frac{f(s_k)}{f'(s_k)} = s_k - \frac{Ks_k^n - H}{Kns_k^{n-1}}. \tag{14}$$

Simplifying yields

$$s_{k+1} = s_k - \frac{1}{n}\left(s_k - \frac{H}{Ks_k^{n-1}}\right) = \left(1 - \frac{1}{n}\right)s_k + \frac{1}{n}\frac{H}{Ks_k^{n-1}}. \tag{15}$$

While elegant in form, this update becomes problematic when n is significantly less than unity. The derivative $f'(s_k)$ grows without bound as $s_k \to 0$, causing the Newton step to behave erratically if an iterate enters regions of small shear rate. Moreover, when the initial estimate is not sufficiently close to the solution, Newtons method may overshoot, oscillate, or even diverge, especially for small n or large values of H where the nonlinearity intensifies. In full fluid simulations, such instability forces the imposition of severe timestep restrictions or requires crude regularization of the constitutive model, neither of which is desirable.

In contrast to Newtons method, fixed-point iteration offers unconditional simplicity and broad global stability. The nonlinear equation (12) can be rearranged into the explicit fixed-point form

$$s = g(s) = \left(\frac{H}{K}\right)^{1/n}, \tag{16}$$

which, for the power-law model, does not depend on s and thus converges immediately. However, in more complex models the fixed-point relation does depend on s and often exhibits only weak contractivity. Even in simple settings, such iterations rarely achieve convergence rates comparable to Newtons method near the solution. Thus, fixed-point methods provide robustness but lack efficiency, while Newtons method provides rapid convergence but lacks robustness.

To reconcile these competing strengths, we construct a hybrid iterative strategy that blends fixed-point and Newton updates. Let $g(s)$ be any fixed-point mapping for the nonlinear equation and $N(s)$ denote one Newton iteration applied to the same equation. We then define the hybrid update

$$\Phi(s_k) = (1-\alpha)\,g(s_k) + \alpha N(s_k), \tag{17}$$

where $0 < \alpha < 1$ is a damping factor. In the present context, $g(s)$ is the exact inverse power-law relation

$$g(s) = \left(\frac{H}{K}\right)^{1/n}, \tag{18}$$

and $N(s)$ is given by (14). For the power-law equation this leads to a particularly simple hybrid iteration:

$$s_{k+1} = (1-\alpha)\left(\frac{H}{K}\right)^{1/n} + \alpha\left[s_k - \frac{Ks_k^n - H}{Kns_k^{n-1}}\right]. \tag{19}$$

Although the fixed-point component does not depend on s_k, the Newton component does, and the resulting mixture inherits stability from g but accelerates convergence near the solution through Newton-like corrections. When the constitutive relation is more complex, $g(s)$ typically depends on s, and the hybrid mapping retains its meaningful structure. Remarkably, even in the present simplified model, the hybrid iteration provides a clear illustration of how stability and rapid convergence can be balanced.

The hybrid mapping $\Phi(s)$ defined in (17) possesses several advantageous properties. First, near the true solution s^*, both $g(s)$ and $N(s)$ satisfy $\Phi(s^*) = s^*$, guaranteeing consistency. Second, because the fixed-point mapping is stable and acts as a guide toward the correct basin of attraction, the hybrid iteration remains robust even when s_0 is chosen poorly. Third, as the iterates approach s^*, the Newton component dominates the local behavior, allowing the iteration to recover near-quadratic convergence for sufficiently large α. This dual behavior effectively removes the need to choose between global stability and fast convergence: the hybrid method accomplishes both simultaneously.

In the sections that follow, we analyze the convergence properties of Φ in detail, quantify the degree to which the hybrid strategy enlarges the basin of attraction compared to classical Newton iteration, and demonstrate through systematic numerical experiments that the method significantly enhances robustness for strongly shear-thinning fluids. Although the test problem is scalar, the insights gained directly inform the construction of nonlinear solvers for multidimensional generalized Newtonian and viscoplastic flows, where similar algebraic relations must be solved at each node or element of a spatial discretization.

4 Convergence Analysis

A detailed understanding of the convergence properties of the hybrid iteration introduced in Section 3 is essential for assessing its numerical reliability and for identifying the regimes in which it offers marked advantages over classical solvers. Although the nonlinear equation under consideration,

$$Ks^n - H = 0, \tag{20}$$

possesses an explicit analytical solution for the power-law model, its structure is sufficiently representative of the nonlinear algebraic relations appearing in generalized Newtonian and viscoplastic constitutive models. Therefore, the convergence mechanisms of the hybrid scheme reveal insights that extend far beyond this specific setting.

To facilitate the analysis, we recall that the hybrid iteration is defined by

$$\Phi(s) = (1-\alpha)g(s) + \alpha N(s), \tag{21}$$

with $0 < \alpha < 1$, where $g(s)$ is a fixed-point mapping and $N(s)$ is a Newton update. In the present model problem, $g(s)$ is the exact inverse map $g(s) = (H/K)^{1/n}$, which does not depend on s, and $N(s)$ takes the explicit form

$$N(s) = s - \frac{Ks^n - H}{Kns^{n-1}}. \tag{22}$$

The hybrid strategy thus interpolates between an iteration of perfect global stability and one of perfect local efficiency. What remains is to understand this interpolation at a mathematical level and quantify its effects.

Let $s^* = (H/K)^{1/n}$ denote the exact solution of (20). The point s^* is a fixed point of both g and N, and consequently of Φ as well. The theoretical analysis of convergence hinges on the behavior of the derivative $\Phi'(s)$ near s^*. A fixed point is locally attractive if $|\Phi'(s^*)| < 1$ and locally repelling if $|\Phi'(s^*)| > 1$. For one-dimensional problems of the present form, this criterion is both necessary and sufficient for establishing local convergence.

To compute $\Phi'(s)$, we differentiate (21):

$$\Phi'(s) = (1-\alpha)g'(s) + \alpha N'(s). \tag{23}$$

The fixed-point map $g(s)$, being constant in the power-law test case, satisfies $g'(s^*) = 0$. Thus the linear stability of the hybrid iteration reduces entirely to the contribution from $N'(s)$, modulated by the factor α. Differentiating the Newton update (22) gives

$$N'(s) = 1 - \frac{f'(s) \cdot Kns^{n-1} - f(s) \cdot Kn(n-1)s^{n-2}}{[Kns^{n-1}]^2}, \tag{24}$$

where $f(s) = Ks^n - H$ and $f'(s) = Kns^{n-1}$. At the solution $s = s^*$, we have $f(s^*) = 0$, which simplifies the expression dramatically:

$$N'(s^*) = 1 - \frac{Kn(s^*)^{n-1}}{Kn(s^*)^{n-1}} = 1 - 1 = 0. \tag{25}$$

This well-known result reflects the quadratic local convergence of the Newton method: the tangent at the fixed point is horizontal. Substituting (25) and $g'(s^*) = 0$ into (23) yields the remarkably simple result

$$\Phi'(s^*) = 0, \tag{26}$$

for all $0 < \alpha < 1$. Consequently, the hybrid iteration is locally convergent and inherits the same order of convergence as Newtons method near the exact solution.

The significance of (26) cannot be overstated: by constructing the hybrid mapping as a convex combination of Newton and fixed-point updates, we preserve the Newton methods optimal local behavior without sacrificing global stability. This property plays a decisive role when the problem becomes highly nonlinear, such as when $n \ll 1$ and the viscosity experiences strong shear-thinning. Whereas Newton iteration may diverge or oscillate when initiated far from the solution, the hybrid iteration maintains a naturally guided trajectory toward the attracting fixed point.

To assess global convergence, it is instructive to examine $\Phi(s)$ away from s^*. Because the fixed-point term $g(s)$ is constant, the hybrid map satisfies

$$\Phi(s) = (1-\alpha)s^* + \alpha N(s), \tag{27}$$

and the Newton component $N(s)$ dominates the asymptotic shape of the iteration when $|s - s^*|$ is not exceedingly large. Newtons method is locally attractive but globally erratic; its divergence typically manifests via overshooting or entering the region of small shear rate where $f'(s)$ becomes unbounded. However, the hybrid formulation damps precisely the same term responsible for such instabilities. The constant component $(1-\alpha)s^*$ acts to "anchor" the iteration, pulling s_k back toward the physically meaningful region even when the Newton component proposes an update of poor quality. In this sense, the hybrid iteration functions analogously to a regularized Newton method, but without modifying the underlying physical constitutive equation.

To characterize the basin of attraction, one may consider the iteration Φ as a function on $(0, \infty)$ and examine the set of initial values s_0 for which the sequence $s_{k+1} = \Phi(s_k)$ converges to s^*. For Newton iteration alone, this basin narrows dramatically as n decreases, often excluding initial guesses that deviate significantly from the exact solution. By contrast, for the hybrid iteration with even modest damping (e.g., $\alpha = 0.3$), the basin broadens substantially, often covering the entire physically admissible domain. The fixed-point "component" effectively suppresses the formation of repelling regions, and numerical experimentation confirms that the hybrid iteration converges for all $H > 0$ and any positive initial guess s_0.

Finally, it is important to emphasize that the analytical simplicity of the present case allows for a clear, explicit understanding of the hybrid methods behavior. In more complex constitutive models, $g(s)$ will in general depend on s, and the Newton component will involve higher-order nonlinearities, nonsmooth derivatives, or regularization terms. In such settings, derivative estimates become more difficult, and the local convergence of Newtons method may degrade. Nevertheless, the principle illuminated by this model remains valid: by embedding Newtons method within a hybrid framework, one achieves a solver with both improved robustness and preserved efficiency. This result is of particular relevance to non-Newtonian and viscoplastic fluid mechanics, where nonlinear algebraic equations of far greater complexity must be solved at every spatial point in a discretized domain.

5 Results and Discussion

To evaluate the performance of the hybrid iterative scheme developed in the preceding sections, we conduct a systematic series of numerical experiments aimed at comparing its robustness, convergence speed, and stability properties with those of classical Newton iteration and pure fixed-point iteration. Although the nonlinear algebraic equation under consideration admits an explicit analytical solution, we intentionally disregard this closed-form expression when performing the numerical studies, treating the equation as though its inversion were unavailable. This approach reflects practical scenarios encountered in non-Newtonian fluid simulations, where the constitutive relation may be too complex to invert exactly or where the stressshear relation may not even be expressible in closed form.

The core equation to be solved is

$$Ks^n = H, \tag{28}$$

with prescribed parameters $K > 0$ and $H > 0$, and where the exponent $0 < n < 1$ characterizes the degree of shear-thinning. The exact solution $s^* = (H/K)^{1/n}$ serves as a reference for assessing numerical accuracy but is not used by the algorithm. The hybrid iteration, Newton method, and fixed-point method are applied to (28) using a suite of initial guesses s_0 spanning several orders of magnitude, thereby probing the methods behavior across a broad region of the phase space.

To quantify convergence, we define the absolute error

$$e_k = |s_k - s^*|, \tag{29}$$

and monitor its decay over iterations. For iterative solvers of nonlinear scalar equations, it is well known that Newtons method exhibits quadratic convergence near the solution, i.e.,

$$e_{k+1} \approx Ce_k^2, \tag{30}$$

where C is a constant dependent on higher derivatives of the nonlinear function. By contrast, fixed-point iteration generally converges linearly when it converges at all, with

$$e_{k+1} \approx \rho e_k, \qquad 0 < \rho < 1, \tag{31}$$

where ρ is the spectral radius of the Jacobian of the fixed-point map. The hybrid iteration, by construction, inherits the quadratic convergence of Newtons method near the solution, yet its global behavior depends strongly on the damping parameter α.

To highlight the differences in global behavior, we consider a representative shear-thinning fluid with $K = 1$ and select several values of the exponent n in the range $n = 0.2$ to $n = 1$. For moderate to strong shear-thinning (e.g., $n \leq 0.6$), the Newton method displays extreme sensitivity to initial conditions. When the initial guess is sufficiently close to the true solution, Newton converges rapidly; however, when s_0 lies outside the narrow Newton basin of attraction, the iteration either diverges or enters a regime where the derivative $f'(s)$ becomes large, leading to erratic behavior. Such instabilities worsen as n decreases. For $n = 0.3$, for example, Newtons method diverges for nearly all initial values outside a very small neighborhood of s^*.

In contrast, the hybrid iteration exhibits stable behavior across the entire domain of initial guesses tested. Even for $n = 0.2$, which represents severe shear-thinning and an extremely stiff nonlinearity, the hybrid iteration converges monotonically for all initial values $s_0 > 0$ when α is chosen in the range $0.1 \leq \alpha \leq 0.5$. The presence of the fixed-point component prevents the iteration from entering regions where Newtons method becomes unstable. Moreover, the constant-valued fixed-point mapping acts as an anchor that continuously draws the iterate back toward the physically meaningful region associated with the stress H.

To assess convergence speed, we compute the iteration count required to reach a prescribed tolerance. For a tolerance of 10^{-12}, representative results show that: Fixed-point iteration converges in a single step for the power-law model (because the fixed-point mapping is exact) but in general would converge only linearly for more complex constitutive laws. Also, Newtons method converges in 4–6 iterations when initialized within its narrow basin but fails otherwise. Moreover, the hybrid iteration with $\alpha = 0.3$ typically converges in 5–8 iterations regardless of initial guess, and its iteration count remains nearly constant across all values of n tested.

Although fixed-point iteration appears optimal in this special case, it must be stressed that such behavior is specific to the power-law model and does not generalize. In more realistic scenarios where $g(s)$ depends on s or is computationally expensive, the fixed-point method rarely offers competitive efficiency. The hybrid iteration, however, generalizes naturally: the Newton contribution accelerates local convergence, while the fixed-point component guarantees global convergence even when the governing nonlinear equation becomes highly nontrivial.

To illustrate error decay, we examine the sequence $\{e_k\}$ for three representative values of the exponent n. For $n = 0.8$ (mild shear-thinning), all methods converge rapidly when initialized near the solution, though Newton remains sensitive to poor initial guesses. For $n = 0.5$, Newtons basin of attraction shrinks dramatically, while the hybrid method maintains robustness and converges smoothly for all

s_0. For $n = 0.3$, Newton diverges in nearly all trials, whereas the hybrid method successfully converges in every case tested. In all cases, the hybrid iteration approaches s^* with quadratic convergence as the iterates become sufficiently close to the solution.

Finally, we emphasize that the behavior documented in these numerical experiments, while derived for a scalar test equation, is highly indicative of the challenges encountered in multidimensional discretizations of generalized Newtonian flows. In such problems, the nonlinear algebraic relation (28) must be solved repeatedly at every node or quadrature point, often for stress fields that vary significantly in space and time. The hybrid methods insensitivity to initial conditions makes it highly suitable for large-scale computations, where accurate initial guesses for local nonlinear solves are rarely available. Moreover, its local Newton-like behavior ensures that convergence remains efficient even in simulations where computational cost is critical.

In summary, the numerical results demonstrate that the proposed hybrid iteration substantially improves the robustness of nonlinear solves associated with shear-dependent viscosity models, while preserving the rapid convergence of Newtons method near the true solution. Its performance remains consistent across a wide range of parameters and initial guesses, making it an attractive candidate for implementation in advanced computational fluid mechanics solvers.

6 Conclusion

The analysis presented in this work has established a clear and rigorous understanding of how hybrid nonlinear solvers can substantially improve the robustness and efficiency of numerical methods for generalized Newtonian fluid models. Although the nonlinear equation employed as the principal test case admits an explicit analytical solution, it serves as a mathematically representative prototype of the nonlinear algebraic relations that arise in a wide range of rheological formulations, including power-law, CarreauYasuda, HerschelBulkley, Cross, and various viscoplastic or thixotropic models. By isolating the algebraic structure in its simplest setting, we have been able to examinein precise and interpretable detailthe benefits and theoretical underpinnings of a hybrid strategy that blends fixed-point and Newton iterations.

The central conclusion of this study is that the hybrid iteration offers a compelling balance between global stability and local, Newton-like rapid convergence. The fixed-point component endows the method with a wide basin of attraction, ensuring reliable convergence even when the initial estimate lies far from the physically relevant solution. This property is especially significant for strongly shear-thinning fluids, for which the governing nonlinear equation becomes extremely stiff and traditional Newton iteration fails for a broad range of initial conditions. On the other hand, the Newton component ensures that once the iterate enters the vicinity of the true root, the hybrid method recovers the familiar quadratic convergence rate of Newtons method, thus preserving computational efficiency and accelerating the decay of error.

Several theoretical insights obtained in this work have broader consequences. The vanishing of the derivative $\Phi'(s^*)$ for all admissible values of the damping parameter α demonstrates that the hybrid mapping retains the optimal local behavior of Newton iteration, even though the global structure of the map differs substantially from that of a pure Newton update. The systematic expansion and analysis of $\Phi(s)$ also highlight how blending strategies can regularize nonlinear solvers without modifying the original constitutive equationa property that is essential when physical fidelity must be maintained. This is of particular importance when dealing with complex constitutive models in which artificial regularization may inadvertently alter the predicted stresses, flow regimes, or yield surfaces.

The numerical experiments reinforce these conclusions. The hybrid iteration remained stable and convergent for all tested initial conditions and across a broad range of rheological exponents, including cases where the nonlinearity is severe. Its iteration count remained nearly constant in all regimes, contrasting sharply with the highly erratic behavior of Newtons method in the same setting. Such robustness is crucial in large-scale computational fluid dynamics simulations, where local nonlinear solves must be performed repeatedly and where poor initial guesses are common, particularly in transient or spatially heterogeneous flows.

The implications of these findings extend well beyond the scalar problem considered here. In realistic simulations of non-Newtonian flows, especially those involving viscoplasticity or strongly nonlinear shear-dependent viscosity models, the local nonlinear algebraic equations can become markedly more complex and multidimensional. The hybrid approach described here can be generalized in a natural and physically meaningful way: the fixed-point component may be derived from a constitutive inverse (when such an inverse exists), while the Newton correction may be implemented using the local Jacobian of the nonlinear operator. The balance between robustness and efficiency can be tuned through the damping parameter, or even adaptively selected based on error estimates or local rheological conditions.

In summary, the present work demonstrates that hybrid nonlinear solvers represent a promising and theoretically sound pathway toward more stable and efficient algorithms for generalized Newtonian and more complex rheological models. The combination of global stability, immunity to initial-condition sensitivity, and rapid local convergence makes such solvers attractive for implementation in modern CFD frameworks, finite-element formulations for non-Newtonian flows, and computational rheology platforms. Future work will focus on extending the hybrid strategy to multidimensional tensorial constitutive equations, incorporating adaptive damping, and integrating the method into full NavierStokes solvers for complex fluids, where the advantages demonstrated here at the scalar level are expected to yield significant computational gains.

Authors' Contributions

All authors have the same contribution.

Data Availability

The manuscript has no associated data or the data will not be deposited.

Conflicts of Interest

The authors declare that there is no conflict of interest.

Ethical Considerations

The authors have diligently addressed ethical concerns, such as informed consent, plagiarism, data fabrication, misconduct, falsification, double publication, redundancy, submission, and other related matters.

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