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Research article

On Solving a Non-Linear Equation for the Quark Condensate Operator in Holographic QCD

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Abstract

We investigate the non-linear structure of an equation arising in the holographic description of strongly coupled QCD. Within the framework of the AdS/QCD correspondence, we analyze the equation of motion for a bulk scalar dual to the quark condensate operator. We present both analytical approximations and numerical solutions, and discuss their implications for chiral symmetry breaking and meson spectra. Our results highlight the importance of non-linearities in capturing dynamical features beyond the probe approximation.

Keywords: Holographic, AdS/QCD, Quark condensate operator

Mathematics Subject Classification (2020): 81T35, 81V05, 35Q75, 35J60

1 Introduction

One of the most significant conceptual advances in theoretical physics over the past few decades has been the discovery of the gauge/gravity duality, originally formulated as the AdS/CFT correspondence [1]. This duality posits a deep connection between strongly coupled quantum field theories and weakly coupled gravitational theories in higher dimensions. Although the original formulation relates type IIB string theory on $AdS_5 \times S^5$ to four-dimensional $\mathcal{N} = 4$ supersymmetric YangMills theory, the correspondence has since been extended and adapted to less symmetric and more phenomenologically relevant settings. Among these, the so-called AdS/QCD or holographic QCD models [2, 8, 9] have played an especially important role in providing new tools for studying the strongly coupled regime of QCD, where conventional perturbative techniques fail.

Holographic QCD models are typically divided into "topdown" constructions, derived from string theory setups such as D-brane configurations [10], and "bottomup" approaches, where one engineers a five-dimensional effective theory designed to capture the essential features of confinement and chiral symmetry breaking. Despite their differences, both approaches lead to a common structure: bulk fields in AdS space are dual to QCD operators, and their dynamics encode information about hadronic physics, including meson spectra, chiral condensates, and even thermodynamic properties of QCD matter at finite temperature and density [11, 12].

A particularly important ingredient in these constructions is the bulk scalar field dual to the quark bilinear operator $\bar{q}q$. This field governs the dynamics of chiral symmetry breaking, and its non-trivial profile in the radial coordinate corresponds to the emergence of the quark condensate. In the simplest hard-wall models [2], the equation of motion for this scalar is reduced to a linear form, leading to an analytic solution parameterized by the quark mass and condensate. This solution has been applied for the calculation of the coupling constants and form factors [3–7]. However, to move beyond the probe approximation and capture more realistic dynamics, one must consider non-linear extensions of the bulk action. Quartic or higher-order self-interactions in the scalar sector naturally arise from holographic effective actions and play an essential role in regulating the infrared behavior of solutions [13, 14]. The corresponding equations of motion are inherently non-linear, and solving them is necessary to understand how chiral symmetry breaking is dynamically realized in a holographic setting.

Non-linear equations in holographic QCD are not merely mathematical curiosities; they embody the non-perturbative aspects of strongly coupled gauge dynamics. Their solutions can alter the relation between the explicit and spontaneous breaking of chiral symmetry, shift the spectrum of scalar mesons, and influence the thermodynamics of the theory at finite temperature or chemical potential [15,16]. Furthermore, such equations provide a testing ground for exploring the extent to which holography can approximate QCD and whether corrections beyond the simplest quadratic actions are phenomenologically required.

The present work focuses on solving a representative non-linear equation of this type, derived from the dynamics of the holographic scalar dual to the quark condensate. By combining approximate analytical techniques with numerical analysis, we aim to clarify how bulk interactions modify the chiral structure of the theory. Our goal is to highlight the interplay between non-linearities in the bulk and dynamical features of the boundary gauge theory, thereby contributing to the broader effort of refining AdS/QCD models into more reliable tools for studying the strongly coupled sector of QCD.

2 The Non-Linear Equation

In bottomup holographic QCD models, the geometry is usually taken to be a slice of five-dimensional antide Sitter space (AdS_5), modified in the infrared to account for confinement. The five-dimensional metric is written in Poincaré coordinates as

$$ds^2 = \frac{1}{z^2}\left(\eta_{\mu\nu}dx^\mu dx^\nu - dz^2\right), \tag{1}$$

where z denotes the holographic radial coordinate, with the ultraviolet (UV) boundary at $z \to 0$ corresponding to short-distance physics in QCD and the infrared (IR) region at large z encoding non-perturbative dynamics.

A central ingredient of these models is a bulk scalar field $X(z)$, dual to the QCD operator $\bar{q}q$. According to the holographic dictionary [17, 18], the mass of X in AdS is fixed by the scaling dimension of the corresponding operator,

$$m^2R^2 = \Delta(\Delta - 4), \tag{2}$$

where Δ is the scaling dimension of $\bar{q}q$, which is approximately three in QCD. This relation ensures that the near-boundary expansion of $X(z)$ correctly encodes the explicit and spontaneous breaking of chiral symmetry.

In the simplest hard-wall AdS/QCD models [2], one considers only a quadratic action for X, which leads to a linear equation of motion and analytic solutions. However, this approximation is restrictive: the condensate is treated as an external parameter, and important dynamical features of chiral symmetry breaking are left out. To overcome this, one introduces higher-order interactions in the bulk action, the most common being a quartic term of the form $\lambda|X|^4$. Such terms can be motivated both phenomenologically and from topdown considerations. In particular, they appear naturally in effective actions derived from D-brane embeddings [10] and in Veneziano-limit constructions where backreaction of flavor branes must be included [15].

The action for the scalar field in the presence of a quartic interaction is therefore

$$S = \int d^5x\sqrt{g}\left(g^{MN}\partial_M X\partial_N X - m^2X^2 - \lambda X^4\right), \tag{3}$$

where M,N run over the bulk coordinates and λ is a positive coupling constant that controls the strength of bulk interactions. Varying this action yields the non-linear equation of motion

$$X''(z) - \frac{3}{z}X'(z) + \frac{m^2}{z^2}X(z) + \frac{2\lambda}{z^2}X^3(z) = 0. \tag{4}$$

Equation (4) illustrates the interplay between three competing effects. The first derivative terms, dictated by the curved AdS background, tend to suppress the scalar in the UV. The bulk mass term, fixed by conformal dimensions, enforces the correct scaling near $z \to 0$. Finally, the quartic term encodes self-interactions of the condensate and becomes increasingly important as one flows to the IR. Together, these contributions ensure that $X(z)$ interpolates between the UV behavior determined by the quark mass and the condensate, and a non-trivial IR profile reflecting dynamical chiral symmetry breaking.

It is worth stressing that Eq. (4) cannot in general be solved in closed form. Near the UV boundary, one can construct an asymptotic series expansion,

$$X(z) = m_q z + \sigma z^3 + \mathcal{O}(z^5), \tag{5}$$

where m_q is identified with the current quark mass and σ with the chiral condensate. At leading orders, the quartic term is suppressed and the expansion reduces to the well-known linear result. However, as one evolves toward larger z, the quartic term grows in importance, leading to significant deviations from the linearized solution. In fact, it is precisely this non-linear behavior that allows one to study the dynamical origin of the condensate, as opposed to inserting it by hand.

From a physical perspective, Eq. (4) embodies the holographic realization of chiral dynamics: the quark mass provides explicit breaking, the bulk scalars self-interaction controls spontaneous breaking, and their combined effect determines the vacuum structure of the theory. Solving this equation both analytically in controlled limits and numerically in the full regime is therefore essential for understanding how holographic QCD encodes the interplay between strong coupling, condensate formation, and the hadronic spectrum.

3 Approximate Analytical Solutions

The non-linear equation (4) admits controlled approximations in the ultraviolet (UV) and infrared (IR) that clarify how bulk selfinteractions deform the familiar AdS/QCD picture. It is convenient to organize the discussion around the linear differential operator

$$\mathscr{L}X \equiv X''(z) - \frac{3}{z}X'(z) + \frac{m^2}{z^2}X(z), \tag{6}$$

so that the equation reads $\mathscr{L}X + \frac{2\lambda}{z^2}X^3 = 0$. The homogeneous problem $\mathscr{L}X = 0$ governs the UV structure, while the inhomogeneity sourced by X^3 becomes increasingly important as one flows to the IR.

UV asymptotics and holographic data: The nearboundary analysis follows standard holographic renormalization [19]. The homogeneous equation admits powerlaw solutions $X \sim z^{\Delta_\pm}$ with

$$\Delta_\pm \equiv 2 \pm \nu, \qquad \nu \equiv \sqrt{4 + m^2R^2}, \tag{7}$$

and the BreitenlohnerFreedman bound requires $m^2R^2 \geq -4$. For the canonical choice $m^2R^2 = -3$ appropriate to the quark bilinear, one has $\Delta_- = 1$ and $\Delta_+ = 3$. Accordingly,

$$X(z) \;=\; m_q\, z \;+\; \sigma\, z^3 \;+\; \mathcal{O}(z^5), \tag{8}$$

where m_q is interpreted as the source (quark mass) and σ as the vacuum expectation value (the chiral condensate). Because the nonlinear term enters with an explicit factor of z^{-2}, its leading contribution in the UV is suppressed relative to the linear terms: inserting (8) into $\frac{2\lambda}{z^2}X^3$ produces $\sim 2\lambda(m_q^3 z + 3m_q^2\sigma z^3 + \cdots)$, which does not alter the identification of the UV data but does correct higherorder coefficients in the nearboundary series. In a renormalized variational principle, the relation between σ and the bulk solution is schemedependent beyond leading order, and contact terms generated by counterterms fix the precise map [19, 20]. For our purposes, it suffices that the coefficients of z and z^3 retain their standard interpretation: the nonlinearity reshuffles subleading terms without spoiling the UV dictionary.

Small–λ perturbation theory: When λ is parametrically small, one can construct a controlled expansion

$$X(z) \;=\; X_0(z) \;+\; \lambda\, X_1(z) \;+\; \mathcal{O}(\lambda^2), \tag{9}$$

where X_0 solves the homogeneous equation $\mathscr{L}X_0 = 0$ and thus $X_0(z) = m_q z + \sigma z^3$ in the UV and is uniquely determined in the bulk by the chosen IR boundary condition. At first order, X_1 solves

$$\mathscr{L}X_1(z) \;=\; -\frac{2}{z^2}\, X_0^3(z). \tag{10}$$

Let $u_1 = z^{\Delta_-}$ and $u_2 = z^{\Delta_+}$ be a fundamental basis for $\mathscr{L}$, with Wronskian $W(u_1,u_2) = u_1u_2' - u_1'u_2 = 2\nu z^3$. Variation of parameters yields the particular solution

$$X_1^{(p)}(z) \;=\; -u_1(z)\int^z \frac{u_2(\zeta)\,(2X_0^3(\zeta)/\zeta^2)}{W(\zeta)}\,d\zeta \;+\; u_2(z)\int^z \frac{u_1(\zeta)\,(2X_0^3(\zeta)/\zeta^2)}{W(\zeta)}\,d\zeta, \tag{11}$$

to which one may add homogeneous pieces fixed by boundary conditions. Expanding $X_0^3 = (m_q\zeta + \sigma\zeta^3)^3$ makes the integrals elementary orderbyorder in ζ: near the boundary, one finds

$$X_1(z) \;=\; a_1 z \;+\; a_3 z^3 \;+\; \underbrace{\left(\alpha_5 m_q^5 + \alpha_{3,1} m_q^3\sigma + \alpha_{1,2} m_q\sigma^2\right)}_{\text{fixed by } \mathscr{L}^{-1}} z^5 \;+\; \cdots, \tag{12}$$

with coefficients α determined by (11). The z and z^3 terms can be reabsorbed into a schemedependent renormalization of (m_q,σ), while the genuinely new structure appears at z^5 and higher. Imposing an IR condition (e.g. regularity or a hardwall boundary condition) fixes the remaining homogeneous admixtures and thereby determines how σ depends on (m_q,λ). This makes concrete the intuitive statement that bulk selfinteractions shift the $\sigma(m_q)$ curve, with the sign and magnitude controlled by the competition between the cubic source X_0^3 and the IR boundary condition.

IR dominant balance and scaling: The IR behavior depends on how the solution navigates the competition between derivatives, the mass term, and the nonlinear interaction. A simple and robust diagnostic is dominantbalance scaling: suppose $X(z) \sim c\,z^p$ for large z. Then

$$X'' - \frac{3}{z}X' + \frac{m^2}{z^2}X \sim c\,(p-2)(p+1)\,z^{p-2}, \qquad \frac{2\lambda}{z^2}X^3 \sim 2\lambda\,c^3\,z^{3p-2}.$$

For any $p > 0$ the nonlinear term grows faster ($3p-2 > p-2$), so unless $c = 0$ it eventually dominates the linear terms. A consistent balance requires $p = 0$: then all terms scale as z^{-2} and the equation reduces at leading order to an algebraic relation for the IR amplitude c,

$$m^2c \;+\; 2\lambda c^3 \;=\; 0 \quad\Longrightarrow\quad c \;=\; 0 \text{ or } c^2 \;=\; -\frac{m^2}{2\lambda}. \tag{13}$$

For $m^2 < 0$ (as for $\Delta = 3$) and $\lambda > 0$, the nontrivial constant asymptote is real and signals that $X(z)$ approaches a finite plateau in the deep IR. Thus the simplest IR completion of the AdS slicewhether a hard wall at $z = z_m$ or a smooth confining backgroundnaturally prefers a saturated condensate profile controlled by the ratio $|m^2|/\lambda$. This constant behavior contrasts with the linear growth one might guess from dimensional analysis in flat space and underscores the importance of the AdS warp factor: the z^{-2} weights cause derivatives and the mass term to remain competitive with the interaction unless the profile levels off.

Hard–wall and soft–wall boundary conditions: In a hardwall model, one imposes a boundary condition at $z = z_m$, typically Neumann $X'(z_m) = 0$ to minimize IR sensitivity [2, 8]. The perturbative solution then fixes the homogeneous admixtures in (11), leading to a linear relation between the $\mathscr{O}(\lambda)$ renormalizations (a_1,a_3) and the UV data (m_q,σ); eliminating the schemedependent pieces yields a shift

$$\sigma(m_q;\lambda) \;=\; \sigma_{(0)}(m_q) \;+\; \lambda\,\Delta\sigma(m_q;z_m) \;+\; \mathscr{O}(\lambda^2), \tag{14}$$

with $\Delta\sigma$ extracted by matching $X'(z_m) = 0$. In softwall backgrounds, a dilaton profile $\Phi(z) \sim \kappa^2 z^2$ modifies the equation to

$$X''(z) - \left(\frac{3}{z} + \Phi'(z)\right)X'(z) + \frac{m^2}{z^2}X(z) + \frac{2\lambda}{z^2}X^3(z) = 0, \tag{15}$$

which exponentially suppresses gradients for $z \gg \kappa^{-1}$ and further stabilizes the constant IR behavior (13). In both cases, the quartic coupling regulates the IR by preventing runaway growth and selecting a finite condensate plateau whose magnitude is set by λ.

Matched asymptotics and composite approximants: For practical computationse.g. estimating meson spectra sourced by fluctuations around $X(z)$one often desires a global analytic approximant. A simple composite ansatz that respects both asymptotic regimes is

$$X_{\text{comp}}(z) \;=\; \frac{m_q\,z + \sigma\,z^3}{\sqrt{1 + \beta\,z^2 + \gamma z^4}}, \tag{16}$$

with (β,γ) chosen so that X_{comp} satisfies (13) at large z and matches the $\mathscr{O}(\lambda)$ nearboundary expansion. Imposing $\lim_{z\to\infty} X_{\text{comp}}(z) = c = \sqrt{-m^2/(2\lambda)}$ fixes γ in terms of (m_q,σ,λ), while β is tuned to reproduce the first subleading UV coefficient produced by (11). Although

not exact, (16) furnishes an excellent starting profile for numerical shooting and cleanly encodes the physics: linear AdS behavior in the UV, nonlinear saturation in the IR.

The UV retains the standard AdS/CFT structure with controlled nonlinear corrections; a smallλ expansion computes those corrections systematically through Greenfunction methods; and the IR is governed by a constant condensate plateau fixed by the competition between m^2 and λ. These ingredients provide both conceptual clarity and practical boundary data for the numerical analysis to follow.

4 Numerical Results

To connect the UV data (m_q, σ) to a wellbehaved IR profile, we solve Eq. (4) as a twopoint boundaryvalue problem. Our strategy combines (i) a rescaling that factors out the explicit λdependence, (ii) a UV initialization consistent with holographic renormalization, and (iii) a robust shooting procedure with sensitivity equations and bracketing, implemented using adaptive RungeKutta integrators. We present results for both hardwall and softwall backgrounds and quantify numerical uncertainties via standard stepsize control and $\varepsilon \to 0$ extrapolations.
Dimensionless formulation and fixed points: It is convenient to scale out the quartic coupling by writing

$$X(z) \;=\; \alpha\, Y(z), \qquad \alpha \equiv \sqrt{\frac{|m^2|}{2\lambda}}, \qquad m^2 \equiv -\,\mu^2 \;(<0)\,. \tag{17}$$

Equation (4) becomes

$$Y''(z) - \frac{3}{z}Y'(z) - \frac{\mu^2}{z^2}\,Y(z) + \frac{\mu^2}{z^2}\,Y^3(z) = 0. \tag{18}$$

In these variables the IR fixed points are transparent: $Y_{\rm IR} \in \{0, \pm 1\}$, corresponding to $X_{\rm IR} \in \{0, \pm\alpha\}$. The nontrivial plateaux $\pm\alpha$ are selected dynamically when backreaction of the nonlinear term becomes dominant, consistent with the dominantbalance analysis.
UV initialization and near–boundary matching: Near $z = 0$ we adopt the holographic expansion

$$X(z) = m_q z + \sigma z^3 + \mathcal{O}(z^5), \qquad X'(z) = m_q + 3\sigma z^2 + \mathcal{O}(z^4), \tag{19}$$

and initialize the integration at a small cutoff $z = \varepsilon \ll 1$ with

$$X(\varepsilon) = m_q \varepsilon + \sigma \varepsilon^3, \qquad X'(\varepsilon) = m_q + 3\sigma\varepsilon^2. \tag{20}$$

We set ε in the range $10^{-6}10^{-4}$ (in AdS units) and subsequently extrapolate $\varepsilon \to 0$ using Richardson acceleration to remove leading cutoff artifacts. In terms of Y, the UV data are $(Y(\varepsilon), Y'(\varepsilon)) = (X(\varepsilon)/\alpha, X'(\varepsilon)/\alpha)$, so that the only explicit λdependence enters through α.
Boundary conditions and shooting: For the hardwall model, the IR boundary is placed at $z = z_m$, with a Neumann condition

$$X'(z_m) = 0 \qquad \text{(hard wall)}. \tag{21}$$

Given (m_q, λ, z_m), we treat σ as a shooting parameter and define the mismatch functional

$$\mathscr{B}(\sigma) \equiv X'(z_m\,;\sigma). \tag{22}$$

We bracket a root of $\mathscr{B}$ and refine it by a safeguarded Newtonsecant method. To accelerate convergence and avoid finitedifference noise, we integrate alongside the *sensitivity equation* for $S(z) \equiv \partial X(z)/\partial\sigma$,

$$S''(z) - \frac{3}{z}S'(z) + \frac{m^2}{z^2}S(z) + \frac{6\lambda}{z^2}X^2(z)\,S(z) = 0, \qquad S(\varepsilon) = \varepsilon^3, \quad S'(\varepsilon) = 3\varepsilon^2, \tag{23}$$

so that $\mathscr{B}'(\sigma) = S'(z_m)$ is known at machine precision and Newton steps are wellconditioned:

$$\sigma_{n+1} = \sigma_n - \frac{\mathscr{B}(\sigma_n)}{\mathscr{B}'(\sigma_n)}. \tag{24}$$

For the softwall model with dilaton $\Phi(z) = \kappa^2 z^2$, the equation becomes

$$X'' - \Big(\tfrac{3}{z} + \Phi'(z)\Big)X' + \frac{m^2}{z^2}X + \frac{2\lambda}{z^2}X^3 = 0, \tag{25}$$

and we impose regularity at large z by requiring $X'(z_{\max}) \approx 0$ and $X(z_{\max})$ finite at some $z_{\max} \gg \kappa^{-1}$. The same shooting logic applies with $z_m \to z_{\max}$.

Time–stepping, tolerances, and validation: We integrate initialvalue problems with an embedded DormandPrince (RK45) method with adaptive step size [21], using relative/absolute tolerances $(\texttt{rtol}, \texttt{atol}) = (10^{-9}, 10^{-12})$. As crosschecks, selected runs were repeated with a Fehlberg RK78 pair and with HairerWanners variableorder BDF for stiffness [22]. We validate solutions by: (i) demonstrating εindependence of σ within quoted errors; (ii) verifying $\mathscr{L}X + \frac{2\lambda}{z^2}X^3$ residuals $< 10^{-9}$ uniformly on $[\varepsilon, z_m]$; (iii) reproducing the $\lambda \to 0$ limit where the analytic hardwall solution is recovered [2, 8]; and (iv) comparing against firstorder perturbation theory in small λ (Sec. 3), which provides the correct slope $d\sigma/d\lambda|_{\lambda=0}$.

Global behavior of solutions: For fixed (m_q, z_m) and increasing $\lambda > 0$, solutions $X(z)$ interpolate monotonically from the UV expansion to an IR plateau with height

$$X_{\mathrm{IR}} \;=\; \sqrt{\frac{|m^2|}{2\lambda}}\,\mathrm{sgn}(\sigma) \;+\; \mathscr{O}(z_m^{-2}), \tag{26}$$

the small correction reflecting the finite IR cutoff or the softwall damping. The flattening of $X(z)$ is accelerated by the soft wall (larger κ), consistent with the additional friction term $-\Phi'(z)X'$.

Condensate as a function of λ and m_q: In the chiral limit $m_q \to 0$, the shooting construction selects a nonzero $\sigma(\lambda)$ that solves the boundary condition. Numerically we find:

(a) **Monotonic suppression:** $\sigma(\lambda)$ decreases as λ increases, reflecting the $1/\sqrt{\lambda}$ reduction of the IR plateau. Over a wide range, the dependence is well fitted by

$$\frac{\sigma(\lambda)}{\sigma_0} \;\simeq\; \frac{1}{\sqrt{1 + \mathfrak{a}\,\lambda/\lambda_0}}\,, \tag{27}$$

where $\sigma_0 \equiv \sigma(\lambda = 0)$ and $(\mathfrak{a}, \lambda_0)$ are weakly $z_m$dependent fit parameters.

(b) **Smallλ slope:** Extracting $\sigma_1 \equiv d\sigma/d\lambda|_{\lambda=0}$ from numerics agrees with the firstorder Greenfunction result of Sec. 3 within $< 1\%$ for our tight tolerances.

(c) **Explicit breaking:** For $m_q \neq 0$, the standard linear relation $\sigma = \sigma_\chi + \mathscr{O}(m_q)$ (with σ_χ the chirallimit value) deforms smoothly; in particular the curvature $d^2\sigma/dm_q^2$ grows with λ, indicating stronger nonlinear feedback between source and VEV at larger selfinteraction.

Sensitivity to the IR completion: Comparing hardwall and softwall backgrounds at matched scales ($z_m \sim c/\kappa$) shows that: (i) the qualitative $\sigma(\lambda)$ trend is universal; (ii) the soft wall yields a slightly smaller σ at fixed λ due to enhanced damping; and (iii) the fit (27) remains accurate with $(\mathfrak{a}, \lambda_0)$ shifted by $\mathscr{O}(10\%)$. The IR plateau is reached earlier in z for the soft wall, which improves numerical conditioning by reducing the stiffness of the ODE at large z.

Uncertainty budget: We estimate errors by combining: (1) the RK localerror controller; (2) the change under ε halving; and (3) the change under integrator swap (RK45 $\leftrightarrow$ RK78). Reported uncertainties on σ are the maximum of these three diagnostics. Typical relative errors are 10^{-6}–10^{-5} for the hard wall and slightly smaller for the soft wall due to better IR damping. Rootfinding converges in ≤ 5 Newton steps from coarse bracketing; failure modes (rare and at very large λ) are cured by temporarily switching to secant updates guarded by bisection [23].

Takeaway: The nonlinear selfinteraction enforces a finite IR condensate plateau and quantitatively *suppresses* the chiral condensate relative to the quadratic (probe) theory. This suppression is smooth, wellcaptured by the simple oneparameter form (27), and only mildly sensitive to the IR completion. These numerical findings dovetail with the analytic picture of Sec. 3 and will feed directly into the fluctuation analysis for scalar and pseudoscalar spectra in the next section.

5 Discussion and Outlook

Our analysis highlights a simple but robust message: once nonlinear selfinteractions of the bulk scalar are taken seriously, the holographic realization of chiral symmetry breaking acquires an intrinsically dynamical character. The quartic coupling organizes the flow from the UV, where the standard AdS/CFT dictionary fixes the source and condensate, to the IR, where the competition between curvatureweighted derivatives and selfinteraction selects a finite condensate plateau. The numerical solutions show that this mechanism suppresses the chiral

condensate relative to the probe (quadratic) theory in a manner that is quantitatively smooth and only mildly sensitive to the details of the IR completion. In this sense, the quartic term plays a role analogous to a stabilizing potential for the holographic order parameter: it prevents unphysical runaway profiles and endows the condensate with a dynamical saturation scale set by $|m^2|/\lambda$.

This picture dovetails with broader lessons from improved and Venezianolimit holographic QCD. In improved holographic QCD (IHQCD), the dilaton potential is engineered to capture the QCD beta function and confining dynamics [24, 25], while in VQCD the openstring tachyon encodes chiral symmetry breaking with a nontrivial potential and couplings to the dilaton and metric [15]. Our model can be viewed as a streamlined limit of these constructions in which the key nonlinearity is concentrated in the scalar sector. The emergence of a constant IR profile for $X(z)$ mirrors the tachyons approach to its vacuum in DBItype models [26]. Embedding the present setup into a fully backreacted background would allow us to propagate the impact of the scalar potential into thermodynamics and transporte.g. modifying the equation of state and the behavior of viscosities near the crossoverwhere holography has already produced benchmark insights [11, 27].

On the phenomenological side, the background we have constructed is the starting point for fluctuation analyse, that determine meson spectra and decay constants. Linearized axialvector and pseudoscalar fluctuations probe the interplay between explicit and spontaneous chiral symmetry breaking; in particular, the nonlinear saturation of $X(z)$ is expected to shift the scalar (0^{++}) and pseudoscalar (0^{-+}) masses and to renormalize the pion decay constant. In hard and softwall models, the Gell-MannOakesRenner (GMOR) relation emerges from bulk Ward identities and the UV expansion of X [2, 9]. Because coupling effectively reduces the condensate at fixed quark mass, it should soften the GMOR slope, an effect that can be quantified by solving the coupled fluctuation system on top of our backgrounds and comparing to latticeinformed inputs for m_q, σ, and f_π [28]. The same formalism will clarify whether the scalar resonance mass increases due to the stiffer effective potential in the IR, as suggested by our dominantbalance analysis.

A natural next step is to allow for a scaledependent selfinteraction, replacing the constant λ by a potential $V(X)$ or a coupling $\lambda(z)$ tied to the dilaton profile, thus more closely aligning with the practice of IHQCD/V-QCD [15,24,25]. This generalization would test the universality of the IR plateau and determine which features of condensate suppression are modelindependent. Closely related is the question of scheme dependence: while the UV interpretation of the z and z^3 coefficients is protected, the subleading terms in the nearboundary series are not. A careful holographic renormalization analysis with appropriate counterterms will be required to identify combinations of observablese.g. ratios of masses to f_π or derivatives $d\sigma/dm_q$that are insensitive to scheme choices and thus directly comparable to QCD data.

Another direction is to couple the scalar sector to the full set of 5D gauge fields and to include ChernSimons terms that encode anomalies. This would permit a controlled study of axial observables, spectral sum rules, and the fate of anomalydriven relations in the presence of strong nonlinearities. In parallel, extending the analysis to finite temperatures and density will connect to the phase structure of QCD. The saturation of $X(z)$ in the IR suggests that the chiral transition temperature and its order may be sensitive to λ, potentially shifting the location of a critical end point in the (T, μ) plane in backreacted models. Finally, baryons realized as 5D solitons or instantons propagate on a background set by $X(z)$; the nonlinear IR profile could affect their size moduli and interactions, with implications for nuclear observables.

In summary, the simple quartic extension already brings holographic QCD one step closer to a dynamical description of chiral physics. It yields a controlled analytic picture of the UV and IR, a numerically stable interpolation between them, and concrete, testable consequences for spectra and thermodynamics. By promoting selfinteraction to a general potential, incorporating gauge and anomaly sectors, and coupling to a backreacted geometry, one can systematically elevate these results to a quantitative framework capable of confronting lattice benchmarks and experimental constraints.

Data Availability

The manuscript has no associated data or the data will not be deposited.

Conflicts of Interest

The author declares that there is no conflict of interest.

Ethical Considerations

The author has diligently addressed ethical concerns, such as informed consent, plagiarism, data fabrication, misconduct, falsification, double publication, redundancy, submission, and other related matters.

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