Analytical and Numerical Solutions for Nonlinear Equations
©Available online at https://ansne.du.ac.ir/
Online ISSN: 3060–785X
2025, Volume 10, Issue 1, pp. 134–152
Research article

Nonlinear Stationary States of a Wheeler--DeWitt-Type Equation: a Minisuperspace Study with a Self-Interacting Scalar Clock

İzzet Sakallı*

Physics Department, Eastern Mediterranean University, Famagusta 99628, North Cyprus via Mersin 10, Turkey

*** Corresponding author(s):** izzet.sakalli@emu.edu.tr

Received: 12/10/2025 **Revised:** 30/11/2025 **Accepted:** 15/12/2025 **Published:** 17/12/2025 doi 10.22128/ansne.2025.3096.1165

Abstract

Quantum gravity phenomenology in black hole physics and cosmology demands frameworks that can bridge quantum effects with classical singularities while remaining amenable to observational constraints. Motivated by effective quantum corrections in black hole thermodynamics and the quest for singularity resolution mechanisms analogous to those explored in loop quantum cosmology, we investigate a minimal nonlinear variant of the Wheeler–DeWitt framework in a flat FRW minisuperspace where a homogeneous scalar field serves as an internal clock. After deparametrization, the physical wavefunction for the scale factor obeys a stationary nonlinear eigenvalue problem on the half-line, with a confining effective potential and a local cubic self-interaction that models quantum-gravitational backreaction or mean-field matter effects. We prove existence of a nodeless ground state under mild assumptions on the potential, develop a controlled perturbation theory at weak coupling, and compute solution families using high-accuracy Chebyshev pseudo-spectral discretizations and continuation methods. Across representative potentials we find a robust, sharply peaked probability density at a nonzero scale factor—a nonsingular quantum bounce—whose location shifts monotonically with the sign and strength of the nonlinearity. This behavior parallels bounce mechanisms in loop-inspired dynamics but emerges from a variational, local nonlinearity in the deparametrized equation. We discuss the interpretation within relational time, connections to effective quantum corrections in gravitational systems, and implications for excited states and time-dependent evolution, providing a transparent analytical and numerical baseline for exploring singularity avoidance in quantum cosmology.

Keywords: Quantum cosmology, Wheeler-DeWitt equation, Quantum bounce, Nonlinear Schrödinger equation, Singularity resolution.

Mathematics Subject Classification (2020): 83C45, 83F05, 35Q40, 35P30

1 Introduction

Quantum gravity confronts us with two fundamental and intertwined challenges: the absence of an external time parameter and the necessity to reconcile quantum dynamics with dynamical spacetime geometry [1–6]. Canonical approaches to quantum gravity address these challenges by casting the dynamics into constraints, most prominently embodied in the Wheeler–DeWitt (WDW) equation, which in minisuperspace models reduces to a differential equation for the wavefunction of the universe [7–11]. Within this constrained framework, the very notion of time must be reconstructed relationally by selecting a physical degree of freedom to serve as an internal clock and describing the evolution of the remaining variables with respect to it [12–15]. This deparametrization procedure transforms the timeless WDW constraint into a Schrödinger-type equation in relational time, providing a pathway to extract physical predictions from the quantum-gravitational formalism.

A particularly tractable and widely studied arena for implementing relational time is the spatially flat Friedmann-Robertson-Walker (FRW) minisuperspace, where a homogeneous scalar field serves as the internal clock and the scale factor represents the dynamical geometric degree of freedom [10, 16]. Boundary conditions at vanishing spatial volume play a crucial role in determining the physical content of the theory. The DeWitt boundary condition, which suppresses probability support at $a = 0$, represents a canonical choice motivated by singularity avoidance [7]. Alternative prescriptions such as the no-boundary proposal [17] and tunneling scenarios [18] highlight the sensitivity of physics near the would-be classical singularity to both quantization choices and boundary data, underscoring the importance of understanding quantum dynamics in this regime.

Beyond the linear WDW equation, several physically motivated mechanisms suggest that effective nonlinearities may arise in the deparametrized dynamics while preserving the fundamental linearity of the underlying theory. First, BornOppenheimer and semiclassical treatments of gravity coupled to quantum matter generate self-interaction terms for the slow geometric sector through backreaction of fast matter or perturbative modes [10, 19]. To leading order, these backreaction terms are local and cubic in the wave amplitude. Second, mean-field closures, which are standard procedures in many-body quantum physics, replace bilinear couplings by local or nonlocal functionals of the wavefunction density, yielding nonlinear Schrödinger (NLS) or GrossPitaevskii (GP) type equations as accurate effective descriptions in appropriate regimes [20–22]. We emphasize that such nonlinearities are effective rather than fundamental: unlike the Weinberg class of nonlinear quantum theories [23], which face severe phenomenological and causality constraints [24, 25], the nonlinear terms we consider emerge from coarse-graining or conditionalization on an internal clock within a fully linear underlying quantum theory.

Nonperturbative quantum-gravity programs also suggest departures from the linear WDW picture in simplified models. Loop Quantum Cosmology (LQC) [26], for instance, replaces the WDW differential operator by a difference operator through holonomy corrections and predicts a quantum bounce that resolves the classical big-bang singularity [27–29]. While our analysis does not assume LQC dynamics or polymer quantization, it addresses a complementary question: to what extent can a nonlinear yet local and variational effective equation on the half-line reproduce the qualitative features of a quantum bounce, and how do those features depend on the sign and strength of the nonlinearity? Understanding this question is important for several reasons. It clarifies which aspects of bounce phenomenology are robust across different quantization schemes, it provides a transparent mathematical framework that separates the roles of kinetic dispersion and nonlinear density-dependent effects, and it establishes a baseline against which more elaborate constructions incorporating anisotropies, inhomogeneities, or microscopic derivations of the nonlinearity can be compared.

Motivated by recent developments in black hole thermodynamics with quantum corrections, gravitational lensing in modified gravity theories, and the search for observable signatures of quantum-gravitational effects [1, 4–6], we investigate a minimal nonlinear variant of the WDW framework that captures essential features of singularity resolution while remaining analytically tractable and numerically precise. Mathematically, the deparametrized stationary problem reduces to finding normalized solutions of a one-dimensional nonlinear Schrödinger or GrossPitaevskii equation in a confining potential, posed on the half-line $a \in [0, \infty)$ with the DeWitt boundary condition $\psi(0) = 0$. Compared to the textbook whole-line case, the half-line geometry and boundary condition significantly impact both existence theorems and numerical treatment. By exploiting the variational structure of the energy functional, GagliardoNirenberg (GN) inequalities in one spatial dimension, and compactness arguments arising from confinement, we establish rigorous existence of a normalized, nodeless ground state for repulsive interactions $g > 0$ and for attractive interactions $g < 0$ down to a finite negative threshold $-g_{\text{crit}}$. We connect these analytical results to high-precision numerics using Chebyshev spectral discretizations on adaptively truncated domains, constrained energy minimization through imaginary-time evolution, and NewtonKrylov continuation with pseudo-arclength parameterization to track solution branches and traverse saddle-node folds [31–33].

Our numerical analysis, performed entirely using Maple 2024 [34], reveals that for representative confining potentials the ground-state probability density $|\psi(a)|^2$ exhibits a sharp peak at a nonzero scale factor $a_\star > 0$, realizing a nonsingular quantum bounce in the DeWitt framework. Repulsive nonlinearity $g > 0$ shifts the most-probable scale factor outward and broadens the probability distribution, while attractive nonlinearity $g < 0$ contracts the distribution until a saddle-node termination occurs at $g_{\text{fold}} \approx -0.62$ for our baseline quartic potential. The small-coupling behavior agrees with first-order perturbation theory to within 3×10^{-4} relative error, and all qualitative trends persist across variations in potential shape and boundary conditions, lending confidence to both the analytical framework and numerical implementation.

The present work is deliberately minimal in scope. We focus exclusively on stationary states in a minisuperspace model with a single geometric degree of freedom, a homogeneous scalar clock, a local cubic nonlinearity, and generic confining potentials. This restriction already captures substantial physics and mathematics while maintaining analytical tractability and numerical precision with spectral accuracy. Natural extensions that build upon this foundation include the study of excited nodal states and their stability properties, time-dependent evolution in relational time governed by $i\partial_\phi \psi = (-\partial_a^2 + U + g|\psi|^2)\psi$, incorporation of anisotropies and additional matter sectors, coupling to inhomogeneous perturbations to predict primordial power spectra, and embedding the effective nonlinearity in concrete quantization schemes such as polymerization in LQC to relate its strength g to microscopic parameters and potentially observable cosmological consequences [10, 15, 29]. Throughout this investigation, we maintain focus on the conceptual lessons for the problem of time in quantum gravity and the status of quantum bounces beyond specific regularization schemes.

The paper is organized as follows. Section 2 presents the classical minisuperspace model, canonical quantization leading to the WDW equation, deparametrization with respect to the scalar clock, and the emergence of effective nonlinearities from physically motivated mechanisms. Section 3 develops the functional framework, proves existence of nodeless ground states under mild confinement assumptions, and describes local bifurcation from the linear spectrum together with qualitative properties of solutions. Section 4 details the numerical methods including Chebyshev spectral collocation, imaginary-time energy minimization, and NewtonKrylov continuation with adaptive resolution control, emphasizing the use of Maple 2024 as the computational platform. Section 5 presents quantitative results for representative confining potentials, validates perturbation theory at weak coupling, and documents the phenomenology of repulsive and attractive nonlinearities including the quantum bounce and fold termination. Section 6 discusses physical interpretation within the relational-time framework, connections to LQC dynamics, hydrodynamic semiclassical pictures, and comprehensive verification and validation procedures ensuring numerical reliability. Section 7 concludes with a summary of findings and perspectives on future extensions.

2 Minisuperspace Model and Deparametrization

In this section we establish the classical minisuperspace model, outline the canonical quantization choices that lead to a WDW operator on the half-line, and describe the deparametrization procedure with respect to a homogeneous scalar field serving as an internal clock. We motivate the emergence of effective nonlinearities from quantum-gravitational backreaction and mean-field effects, discuss boundary conditions and self-adjointness, and introduce a convenient nondimensionalization that casts the problem into the canonical form analyzed throughout this work. Our presentation builds upon standard treatments of canonical quantum cosmology [7–11, 16] while incorporating insights from loop-inspired effective dynamics [27–29].

We consider a spatially flat Friedmann-Robertson-Walker (FRW) metric ansatz

$$ds^2 = -N^2(t)\,dt^2 + a^2(t)\,d\vec{x}^2, \qquad a(t) > 0, \tag{1}$$

minimally coupled to a homogeneous scalar field $\phi(t)$ with potential $V(\phi)$. Working in Planck units ($\hbar = c = 8\pi G = 1$), the reduced Lagrangian reads

$$L = \frac{1}{2N}\left(-a\dot{a}^2 + a^3\dot{\phi}^2\right) - N a^3 V(\phi) - N\Lambda a^3, \tag{2}$$

where we include a cosmological constant Λ for generality.[1] The canonical momenta are $p_a = -a\dot{a}/N$ and $p_\phi = a^3\dot{\phi}/N$, yielding the Hamiltonian as a product of the lapse function times the Hamiltonian constraint,

$$H = N\mathscr{C}, \qquad \mathscr{C} = -\frac{p_a^2}{2a} + \frac{p_\phi^2}{2a^3} + a^3 V(\phi) + \Lambda a^3 \approx 0. \tag{3}$$

The constraint $\mathscr{C} = 0$ encodes the entire dynamical content of the theory; physical histories must satisfy this condition for all times t. This constraint structure is characteristic of diffeomorphism-invariant theories and constitutes the starting point for canonical quantization in quantum cosmology.

Canonical quantization promotes the phase-space variables $(a, p_a; \phi, p_\phi)$ to operators acting on wavefunctions $\Psi(a,\phi)$ endowed with an appropriate inner product. A natural factor ordering is the Laplace–Beltrami choice associated with the minisuperspace metric on the

[1] A spatial curvature term can be reintroduced straightforwardly; our numerical examples focus on the flat case $k = 0$.

configuration space (a,ϕ) [10, 11]. For our purposes, we adopt an ordering that renders the WDW operator symmetric on $L^2([0,\infty)\times\mathbb{R}, da\,d\phi)$ and isolates a one-dimensional Schrödinger operator in the scale factor a:

$$\widehat{\mathscr{C}}\Psi(a,\phi) = \left[-\partial_a^2 + U(a,\phi) - \partial_\phi^2\right]\Psi(a,\phi) = 0, \tag{4}$$

where $U(a,\phi)$ collects the effects of spatial curvature, the cosmological constant Λ, the scalar potential $V(\phi)$, and factor-ordering contributions.[2] Throughout this work we focus on regimes where U can be treated as an effective confining potential in a for fixed ϕ or, after deparametrization, as a fixed function $U(a)$ encoding the geometry of the quantum state space.

To extract physical dynamics from the timeless WDW constraint, we employ the homogeneous scalar field ϕ as an internal time variable, that is, a relational clock with respect to which the evolution of the scale factor a is measured. This deparametrization procedure is justified when ϕ varies monotonically along classical solutions in the regime of interest, a condition typically satisfied for free or sufficiently shallow potentials [8, 14, 29]. Seeking states of the form

$$\Psi(a,\phi) = \psi(a;\phi), \tag{5}$$

and solving the constraint (4) for the relational time derivative $-i\partial_\phi$ yields a Schrödinger-type equation in relational time,

$$i\,\partial_\phi\,\psi(a;\phi) = \widehat{H}_a\,\psi(a;\phi), \qquad \widehat{H}_a := \sqrt{-\partial_a^2 + U(a,\phi)}. \tag{6}$$

In the minisuperspace and ordering adopted here, and in regimes where U is effectively ϕ-independent, it is common and sufficient for the present study to replace the square-root Hamiltonian by an equivalent second-order equation and to seek stationary solutions in ϕ of the form

$$\Psi(a,\phi) = \psi(a)\,e^{i\omega\phi}, \qquad \omega\in\mathbb{R}. \tag{7}$$

For the linear theory, this ansatz reduces the WDW equation to a one-dimensional eigenvalue problem [30] for $\psi(a)$ with eigenvalue ω^2,

$$\left[-\frac{d^2}{da^2} + U(a)\right]\psi(a) = \omega^2\,\psi(a), \tag{8}$$

posed on the half-line $a\in[0,\infty)$. This eigenvalue problem forms the baseline against which we measure the effects of effective nonlinearities introduced below.

While the full quantum theory described by the WDW equation is fundamentally linear, the deparametrized dynamics can acquire effective nonlinearities through several physically motivated mechanisms that preserve the underlying linearity at a more fundamental level. A first mechanism is semiclassical backreaction, where a Born–Oppenheimer separation between slow geometric modes, represented by the scale factor, and fast matter or perturbative modes induces self-interaction terms for the slow sector after integrating out the fast degrees of freedom [10,19]. To leading order, these backreaction terms are local and cubic in the wave amplitude, precisely matching the form we adopt in our analysis. A second mechanism arises from mean-field closures familiar from many-body quantum systems, where bilinear couplings between constituents are replaced by functionals of the local density, leading to nonlinear Schrödinger (NLS) or GP equations that provide accurate effective descriptions in appropriate regimes [20–22]. Our use of a local cubic term $g|\psi|^2\psi$ should be interpreted in this spirit as an effective mean-field description capturing collective quantum-gravitational effects. A third mechanism is effective polymerization from LQC quantizations that modify the gravitational kinetic term through holonomy corrections. In minisuperspace reductions, these modifications can be captured by nonstandard dispersion relations or, after suitable approximations, by density-dependent potentials, again yielding an effective nonlinearity in the scale-factor sector [27,29]. While our framework does not assume LQC dynamics, it provides a complementary perspective on how local nonlinearities can generate bounce phenomenology similar to that found in loop-inspired approaches.

Guided by these considerations, we augment the linear eigenvalue problem (8) with a local cubic self-interaction of strength $g\in\mathbb{R}$ and study the stationary nonlinear eigenvalue problem

$$\left[-\frac{d^2}{da^2} + U(a)\right]\psi(a) + g\,|\psi(a)|^2\psi(a) = \omega^2\,\psi(a), \tag{9}$$

[2] One may absorb measure factors into U so that the inner product is flat in a; this choice is convenient for the variational analysis below. Other orderings such as Laplace–Beltrami with a-dependent measure lead to equivalent half-line problems after a similarity transform [10].

which serves as the working equation throughout this paper. The sign of g distinguishes repulsive interactions with $g > 0$ from attractive interactions with $g < 0$, with significant consequences for the existence and properties of stationary states as we demonstrate in later sections. The repulsive case corresponds physically to quantum-gravitational effects that stiffen the effective potential near small scale factors, pushing the wavefunction away from the classical singularity, while the attractive case represents mechanisms that enhance localization until a critical threshold is reached. The choice of a local cubic self-interaction $g|\psi|^2\psi$ merits further justification. This form is not unique but represents the leading-order term in a general class of effective density-dependent corrections consistent with the symmetries of the problem. Any local nonlinearity respecting the $U(1)$ phase invariance $\psi \mapsto e^{i\theta}\psi$ takes the form $f(|\psi|^2)\psi$ for some real function f. Expanding $f(|\psi|^2) = f_0 + f_1|\psi|^2 + f_2|\psi|^4 + \cdots$ and absorbing the constant f_0 into the effective potential $U(a)$, the cubic term $f_1|\psi|^2\psi$ emerges as the leading nonlinear contribution. Higher-order terms such as the quintic $|\psi|^4\psi$ are permitted by symmetry and may arise at subleading orders in perturbative expansions of backreaction effects [10, 19], but they remain subdominant for weak to moderate coupling strengths. The cubic form also arises naturally in mean-field approximations to many-body quantum systems where pairwise interactions are replaced by an effective density-dependent potential [20, 21]. From a mathematical standpoint, the GP equation with cubic nonlinearity possesses a well-developed variational theory with established existence, uniqueness, and stability results in one spatial dimension [22, 36]. Our analytical framework extends to power nonlinearities $|\psi|^{2\sigma}\psi$ with $\sigma \in (0,2)$, as noted in Section 3, with the critical threshold g_{crit} depending on the exponent σ through modified constants in the GN inequality. The cubic case $\sigma = 1$ thus serves as a representative and analytically tractable member of this broader class, capturing the essential physics of density-dependent quantum-gravitational corrections while maintaining mathematical control.

Because a represents a scale factor, the physical domain is restricted to the half-line $a \in [0,\infty)$. Self-adjoint realizations of the Schrödinger operator $-\partial_a^2 + U(a)$ on this domain require specification of boundary conditions at $a = 0$ [10, 11]. The three most commonly discussed choices in quantum cosmology are the DeWitt boundary condition $\psi(0) = 0$, which suppresses probability support at vanishing spatial volume and is motivated by the desire to avoid the classical singularity [7], the Robin family of conditions $\psi'(0) = \kappa\,\psi(0)$ with $\kappa \in \mathbb{R}$, encoding alternative factor orderings or different physical prescriptions near the origin $a = 0$, and the Neumann condition $\psi'(0) = 0$, which is less common for the scale factor but sometimes employed as a limiting case of the Robin family. In all cases, normalizability in $L^2([0,\infty))$ requires $\psi(a) \to 0$ as $a \to \infty$ for confining potentials U. Throughout this work we adopt the DeWitt condition $\psi(0) = 0$ as our default choice, both for its conceptual clarity in implementing singularity avoidance and because it simplifies the variational analysis presented in Section 3.

The effective potential $U(a)$ compiles contributions from spatial curvature, the cosmological constant Λ, factor-ordering terms, and after deparametrization from the scalar field self-interaction $V(\phi)$. Our analysis requires only that U be sufficiently confining to yield a discrete spectrum in the linear problem and to ensure compactness in the nonlinear variational formulation. For concreteness and to facilitate reproducibility of our numerical results, we employ a quartic confining form,

$$U(a) = \alpha\, a^2 + \beta\, a^4, \qquad \alpha > 0,\ \beta > 0, \tag{10}$$

which captures the essential features of a harmonic core stiffened by anharmonic corrections at large scale factors. This potential provides a realistic effective description of the combined effects of the cosmological constant, scalar field dynamics, and quantum corrections encoded in the factor ordering.

To render equation (9) dimensionless and reduce parameter redundancy, we introduce a characteristic length scale ℓ via the transformation $a = \ell x$ and rescale the wavefunction by $\psi(a) = \ell^{-1/2}\,\varphi(x)$, preserving the L^2 norm such that $\|\varphi\|_{L^2(0,\infty)} = \|\psi\|_{L^2(0,\infty)}$. Choosing ℓ to normalize the quadratic part of U and denoting the rescaled potential by the same symbol U, the stationary problem becomes

$$\left[-\frac{d^2}{dx^2} + U(x)\right]\varphi(x) + g\,|\varphi(x)|^2\varphi(x) = \Omega\,\varphi(x), \qquad x \in [0,\infty), \tag{11}$$

with $\Omega = \omega^2$ in our normalization. For the quartic potential (10), one convenient choice is $\ell = \alpha^{-1/2}$, yielding $U(x) = x^2 + \tilde{\beta}\,x^4$ with $\tilde{\beta} = \beta/\alpha^2$. This rescaling simultaneously absorbs the natural dimensional scaling of the nonlinearity coefficient in one spatial dimension, rendering g a dimensionless coupling parameter throughout our analysis. We emphasize that dimensional consistency is maintained throughout our formulations. Working in Planck units ($\hbar = c = 8\pi G = 1$), all fundamental constants are set to unity, rendering gravitational and quantum-mechanical quantities dimensionless in natural units. The rescaling $a = \ell x$ with $\ell = \alpha^{-1/2}$ converts the scale factor to a dimensionless coordinate x, while the wavefunction transformation $\psi(a) = \ell^{-1/2}\,\varphi(x)$ ensures that φ satisfies $\int_0^\infty |\varphi(x)|^2\,dx = 1$ with dimensionless integrand. In the stationary equation (9), each term—the kinetic contribution $-d^2\varphi/dx^2$, the potential term $U(x)\varphi$, the nonlinear interaction $g|\varphi|^2\varphi$, and the eigenvalue term $\Omega\varphi$—carries identical dimensionality, confirming internal consistency. The coefficient g emerges as a pure number measuring the strength of the effective self-interaction relative to the confining potential, while Ω represents the squared frequency in units set by the potential curvature α. This nondimensionalization permits direct comparison of results across different physical realizations of the effective potential without ambiguity from unit choices. The nondimensionalization clarifies the universal scaling properties of the solution and facilitates comparison across different physical realizations of the effective potential.

Equation (9) arises as the Euler–Lagrange equation of the constrained energy functional

$$\mathscr{E}[\psi] = \int_0^\infty \Big(|\psi'(a)|^2 + U(a)\,|\psi(a)|^2 + \frac{g}{2}\,|\psi(a)|^4\Big)\,da, \qquad \|\psi\|_2^2 = 1, \;\; \psi(0) = 0, \tag{12}$$

where the normalization constraint is enforced via a Lagrange multiplier identified with the eigenvalue ω^2. The functional is invariant under global $U(1)$ phase transformations $\psi \mapsto e^{i\theta}\psi$, which in the time-dependent relational evolution $i\partial_\phi\psi = (-\partial_a^2 + U + g|\psi|^2)\psi$ guarantees conservation of the L^2 norm in relational time ϕ. For power-law potentials such as (10), one can derive virial-type identities relating kinetic, potential, and nonlinear energy contributions by exploiting scaling symmetries of the functional. We employ these identities as sensitive diagnostics of numerical accuracy in Section 4, as they provide independent consistency checks on the computed solutions beyond simple residual norms.

The modeling choices outlined in this section culminate in the nonlinear stationary eigenvalue problem (9) on the half-line with the DeWitt boundary condition. Under mild confinement assumptions on U, this problem admits a well-defined variational formulation with favorable compactness properties in one spatial dimension, enabling us to prove rigorous existence results in Section 3 and to compute solution branches with spectral accuracy in Sections 4 and 5. The framework provides a transparent baseline for exploring how effective quantum corrections modify the probability distribution of the scale factor near the classical singularity, with direct implications for singularity resolution and quantum bounce scenarios in deparametrized quantum cosmology. The combination of analytical rigor through the variational structure and computational precision through spectral methods allows us to extract reliable physical predictions about the nature of quantum states near the cosmological singularity.

3 Variational Structure, Existence, and Bifurcation

We now develop a precise functional framework for the stationary problem, prove ground-state existence under mild confinement assumptions, and describe the local bifurcation from the linear spectrum together with qualitative properties (positivity, regularity, monotonicity) of solutions. We also discuss excited states, thresholds at attractive coupling, and stability heuristics.

Let

$$\mathscr{H} := H_0^1(0,\infty) = \{\psi \in H^1(0,\infty) : \; \psi(0) = 0\},$$

equipped with the norm $\|\psi\|_{\mathscr{H}}^2 = \int_0^\infty \big(|\psi'|^2 + |\psi|^2\big)\,da$. Throughout we impose:

(A) $U \in C^1([0,\infty))$ and there exist $p > 0$, $c_1 > 0$, $c_0 \ge 0$ such that

$$U(a) \ge c_1 a^p - c_0 \qquad \text{for all } a \ge 0.$$

Assumption **(A)** ensures coercivity and compactness: the quadratic form

$$\mathfrak{q}[\psi] := \int_0^\infty \big(|\psi'|^2 + U(a)|\psi|^2\big)\,da$$

controls $\|\psi\|_{\mathscr{H}}^2$ and the embedding $\mathscr{H} \hookrightarrow L^q(0,\infty)$ is compact for any $2 \le q < \infty$ (by confinement at infinity). The stationary problem

$$\Big(-\frac{d^2}{da^2} + U(a)\Big)\psi + g|\psi|^2\psi = \omega^2\psi, \qquad \psi \in \mathscr{H}, \;\; \|\psi\|_2 = 1, \tag{13}$$

is the EulerLagrange equation of the constrained energy

$$\mathscr{E}[\psi] = \int_0^\infty \Big(|\psi'|^2 + U|\psi|^2 + \tfrac{g}{2}|\psi|^4\Big)\,da, \qquad \|\psi\|_2^2 = 1. \tag{14}$$

Theorem 1 (Existence of a nodeless minimizer)**.** *Under* **(A)***, there exists $g_{\rm crit} \in (0,\infty)$ such that for all $g > -g_{\rm crit}$ the constrained minimization of $\mathscr{E}$ over $\mathscr{H}$ with $\|\psi\|_2 = 1$ admits a minimizer $\psi_0 \in \mathscr{H}$. The minimizer can be taken real, strictly positive on $(0,\infty)$, and nodeless. It solves* (13) *with some Lagrange multiplier $\omega_0^2 \in \mathbb{R}$.*

Proof sketch. Boundedness from below: In one dimension the GagliardoNirenberg inequality yields

$$\|\psi\|_4^4 \le C_{\rm GN}\|\psi'\|_2\,\|\psi\|_2^3 = C_{\rm GN}\|\psi'\|_2$$

under the normalization $\|\psi\|_2 = 1$. Hence, for any $\varepsilon > 0$,

$$\frac{g}{2}\|\psi\|_4^4 \;\ge\; -\frac{|g|}{2}C_{\rm GN}\|\psi'\|_2 \;\ge\; -\varepsilon\|\psi'\|_2^2 - \frac{C_{\rm GN}^2 g^2}{16\varepsilon}.$$

Choosing ε small and using **(A)**, we obtain $\mathscr{E}[\psi] \ge c\|\psi\|_{\mathscr{H}}^2 - C$ for g above a negative threshold, which defines $g_{\rm crit}$.

Existence: Take a minimizing sequence $\{\psi_n\} \subset \mathscr{H}$ with $\|\psi_n\|_2 = 1$. Coercivity gives boundedness in $\mathscr{H}$; by compact embedding and the BrezisLieb lemma [35] (plus locality of the quartic term), we extract a strongly convergent subsequence in L^2 and L^4 and a weakly convergent subsequence in $\mathscr{H}$, yielding a minimizer.

Positivity and absence of nodes: The equation is real and $U(1)$-invariant; replacing ψ by $|\psi|$ lowers (or preserves) the energy, so a minimizer can be taken nonnegative. The strong maximum principle and Hopf boundary lemma on the half-line imply $\psi_0 > 0$ on $(0,\infty)$ and $\psi_0'(0) > 0$. Nodelessness follows by the standard Sturm comparison for the associated ODE [22]. □

Remarks. (i) The argument extends verbatim to power nonlinearities $|\psi|^{2\sigma}\psi$ with $\sigma \in (0,2)$ in $d = 1$; only constants in the GN inequality change [22]. (ii) For attractive $g < 0$, failure of boundedness from below when $|g|$ is large produces a fold/termination of stationary branches, consistent with our numerics. Closely related thresholds appear in whole-line NLS and in trapped GrossPitaevskii theory [36,37].

If $U \in C^k$, elliptic regularity on the ODE implies $\psi \in C^{k+2}$, and ψ is real-analytic when U is analytic. For confining $U(a) \gtrsim a^p$ with $p > 0$, Agmon-type estimates give super-Gaussian decay governed by U at large a [22, Ch. 8]. The DeWitt boundary condition and positivity imply monotonic increase from the boundary: $\psi'(0) > 0$, while for sufficiently convex U the ground state has a single interior maximum and decays thereafter. On the half-line, uniqueness of the ground state can be shown for a broad class of convex U by ODE phase-plane arguments.

Let (ω_0^2, ψ_0) be the simple ground-state eigenpair of the linear problem

$$\big(-\partial_a^2 + U(a)\big)\psi = \omega^2\psi, \qquad \psi \in \mathscr{H},\ \ \|\psi\|_2 = 1.$$

Define $\mathscr{F}(\psi,\omega^2,g) := \big(-\partial_a^2 + U + g|\psi|^2 - \omega^2\big)\psi$ together with the phase-fixing constraint $\langle\psi,\psi_0\rangle = 1$. The Fréchet derivative $D_{(\psi,\omega^2)}\mathscr{F}|_{(\psi_0,\omega_0^2,0)}$ is invertible on the orthogonal complement of ψ_0, hence by the implicit-function theorem there exist $\delta > 0$ and smooth maps $g \mapsto (\psi(g), \omega^2(g))$ for $|g| < \delta$ with

$$\omega^2(g) = \omega_0^2 + g\,\mu_1 + \mathscr{O}(g^2), \qquad \mu_1 = \int_0^\infty |\psi_0(a)|^4\,da,$$

and $\psi(g) = \psi_0 + g\,\eta_1 + \mathscr{O}(g^2)$ with η_1 solving the linearized inhomogeneous equation orthogonal to ψ_0 (LyapunovSchmidt reduction) [38, 39]. This perturbation theory matches our numerical continuation.

Higher (nodal) stationary states can be obtained by constrained minimization on Nehari or nodal Nehari manifolds and by mountain-pass techniques [40–42]. Confinement guarantees the PalaisSmale condition, so standard critical-point theory applies. On the half-line, excited states alternate in parity under reflection across $a = 0$ when extended to $\mathbb{R}$ with either odd (Dirichlet) or even (Neumann) continuation, which is useful for numerical initialization.

For $g < 0$, the quartic term lowers the energy. The GN control used above shows boundedness from below only for $g > -g_{\rm crit}$, with

$$g_{\rm crit} \asymp \inf_{\psi\in\mathscr{H},\|\psi\|_2=1} \frac{\mathfrak{q}[\psi]}{\|\psi\|_4^4}.$$

As $g \downarrow -g_{\rm crit}$ along the ground-state branch, the Jacobian in Newton continuation develops a small singular value and a saddlenode (fold) occurs in (g,ω^2), in agreement with the energetic picture and with GP theory in traps [36,37].

While our focus is stationary states, the time-dependent (relational) flow

$$i\partial_\phi \psi = \left(-\partial_a^2 + U + g|\psi|^2\right)\psi, \quad a \in (0,\infty),\ \psi(0,\phi) = 0,$$

conserves mass and energy. For whole-line NLS, the VakhitovKolokolov slope condition $d\|\psi\|_2^2/d\omega < 0$ (at fixed $g < 0$) correlates with spectral stability of solitary waves [43–45]. Within our limited, confined half-line framework, a related *energetic* criterion employs $d\omega^2/dg$ along stationary paths. Our numerical results confirm the anticipated stability shift at the fold for $g < 0$ and consistent stability for $g > 0$, without engaging in a complete spectral proof.

The analysis extends with minor modifications to: (i) Robin boundary conditions $\psi'(0) = \kappa\psi(0)$, $\kappa \in \mathbb{R}$ (changing only constants in trace inequalities); (ii) nonlocal nonlinearities $g(K*|\psi|^2)\psi$ with $K \geq 0$ integrable (by HardyLittlewoodSobolev and compactness from confinement); and (iii) multiwell confining potentials, for which ground states remain unique and positive while excited-state multiplicity increases (standard PerronFrobenius arguments for the linear limit combined with continuation).

4 Numerical Methods and Computational Framework

This section details the discretization strategies and solver algorithms employed to compute stationary solutions of the half-line nonlinear eigenvalue problem with the DeWitt boundary condition. Our computational approach combines Chebyshev spectral collocation on a truncated domain with careful enforcement of boundary conditions and quadrature-consistent normalization, an energy-minimization scheme based on imaginary-time evolution to obtain ground states robustly, and a NewtonKrylov continuation method with pseudo-arclength parameterization to track solution branches in the nonlinearity strength g, pass through fold points for attractive interactions, and assess local conditioning. Our choices follow standard best practices in spectral methods and nonlinear continuation [31–33, 46–49] and are specifically adapted to the half-line geometry and the DeWitt boundary condition at the origin. All numerical computations and symbolic verifications are performed using Maple 2024, which provides a robust platform for combining high-precision numerical analysis with symbolic validation of analytical results.

We begin by truncating the semi-infinite physical domain to a finite interval $a \in [0,L]$ with $L \gg 1$ chosen sufficiently large so that the computed state decays to machine noise at $a = L$. To map the computational domain to the canonical spectral interval $[-1,1]$, we employ the linear transformation

$$a = a(x) = \tfrac{L}{2}(x+1), \qquad x \in [-1,1]. \tag{15}$$

This linear map maintains a high density of collocation points near $a = 0$, which is beneficial because the ground state rises from the boundary according to the DeWitt condition and often peaks at modest values of a, requiring good resolution in this region to capture the initial monotonic increase and the location of the maximum. For states exhibiting very extended tails or when exploring large nonlinearity strengths where the wavefunction spreads significantly, we optionally use an algebraic map $a = \ell\frac{1+x}{1-x}$ with a transparent boundary condition at $x \to 1^-$ that allows the tail to extend to infinity in principle. However, in practice the linear map proved sufficient for all computations reported in this work, as the exponential or super-Gaussian decay of solutions in confining potentials ensures negligible amplitude beyond moderate values of L.

We discretize the solution using Chebyshev spectral collocation by placing unknowns at the ChebyshevGaussLobatto points $x_j = \cos(\pi j/N)$ for $j = 0,\dots,N$, which cluster near the endpoints of $[-1,1]$ and provide exponential convergence for smooth solutions. We employ the standard Chebyshev first and second differentiation matrices D and $D^{(2)}$ on $[-1,1]$, rescaling them to $[0,L]$ via the chain rule to account for the coordinate transformation. The implementation in Maple 2024 utilizes built-in routines for generating Chebyshev nodes and differentiation matrices with high-precision arithmetic, ensuring numerical stability even at large polynomial orders. The DeWitt boundary condition $\psi(0) = 0$ is imposed strongly by eliminating the row and column corresponding to $j = 0$ when assembling the discrete operator, effectively removing this degree of freedom from the algebraic system. At the right boundary $a = L$ we impose $\psi(L) = 0$ for stationary states, which is consistent with the observed exponential or super-Gaussian decay of solutions under confining potentials U and ensures that the tail contribution to integrated quantities is negligible.

To evaluate norms and the energy functional with spectral accuracy, we use ClenshawCurtis quadrature with weights w_j on $[-1,1]$, appropriately rescaled by $L/2$ on $[0,L]$ to account for the Jacobian of the coordinate transformation [31, 50]. Specifically, the discrete L^2

norm and kinetic energy are approximated as

$$\|\psi\|_2^2 \approx \frac{L}{2}\sum_{j=0}^{N} w_j |\psi_j|^2, \qquad \int_0^L |\psi'|^2\, da \approx \frac{2}{L}\sum_{j=0}^{N} w_j \left[(D\psi)_j\right]^2, \tag{16}$$

with analogous expressions for the potential energy and nonlinear interaction terms. The cubic nonlinearity $g|\psi|^2\psi$ is evaluated pointwise in physical space at the collocation nodes. For robustness against aliasing errors that can arise from products of functions represented by Chebyshev polynomials, we over-resolve by increasing N and, in tests where the solution develops sharper features or steeper gradients, we apply a discrete cosine transform round-trip with Chebyshev padding before truncation. This padding strategy is the Chebyshev analogue of the $2/3$ dealiasing rule used in Fourier spectral methods [32, 46] and ensures that high-frequency aliasing artifacts do not pollute the solution. Maple 2024's efficient fast Fourier transform capabilities facilitate these operations with minimal computational overhead.

Let $\mathbf{u} \in \mathbb{R}^n$ collect the interior nodal values ψ_j for $j = 1, \ldots, N-1$, excluding the boundary points where Dirichlet conditions are imposed. The discrete linear operator corresponding to $-\partial_a^2 + U(a)$ is

$$\mathbf{L} = \frac{4}{L^2}\mathbf{D}^{(2)} + \mathrm{diag}\big(U(a_j)\big), \tag{17}$$

where $\mathbf{D}^{(2)}$ denotes the interior block of the Chebyshev second-derivative matrix after removal of boundary rows and columns, and the factor $4/L^2$ arises from the chain rule applied to the coordinate transformation. The nonlinear residual for the eigenpair $(\mathbf{u}, \lambda)$ with $\lambda = \omega^2$ is

$$\mathbf{R}(\mathbf{u}, \lambda; g) = \big(\mathbf{L} + g\,\mathrm{diag}(\mathbf{u} \odot \mathbf{u})\big)\mathbf{u} - \lambda\,\mathbf{u}, \tag{18}$$

where $\odot$ denotes the Hadamard or elementwise product. The normalization constraint ensuring unit L^2 norm is discretized as

$$c(\mathbf{u}) = \frac{L}{2}\mathbf{u}^\top \mathbf{W}\mathbf{u} - 1 = 0, \tag{19}$$

where $\mathbf{W}$ is the diagonal matrix of ClenshawCurtis weights restricted to interior nodes. We monitor both the maximum norm $\|\mathbf{R}\|_\infty$ and the weighted L^2 norm $\|\mathbf{R}\|_2$ of the residual as primary convergence diagnostics, requiring them to fall below 10^{-10} before accepting a solution.

A robust computational path to the ground state, particularly when good initial guesses are unavailable, is to minimize the discrete energy functional under the normalization constraint using a gradient flow in fictitious time τ. We implement the projected gradient descent scheme

$$\partial_\tau \mathbf{u} = -\frac{\delta \mathscr{E}_h}{\delta \mathbf{u}} + \mu(\tau)\,\mathbf{u}, \tag{20}$$

where the discrete variational derivative is $\delta\mathscr{E}_h/\delta\mathbf{u} = (\mathbf{L} + g\,\mathrm{diag}(\mathbf{u}\odot\mathbf{u}))\mathbf{u}$, and the Lagrange multiplier $\mu(\tau)$ is chosen at each time step to enforce the constraint $c(\mathbf{u}(\tau)) \equiv 0$, yielding a projected gradient flow on the unit sphere in the weighted L^2 norm. Time stepping employs adaptive semi-implicit Euler or stabilized exponential time differencing ETD1 schemes for the linear part, combined with explicit treatment of the nonlinearity and periodic renormalization to prevent drift from the constraint manifold. Both schemes are standard for computing GrossPitaevskii (GP) ground states in atomic physics [51–53] and converge reliably to the global energy minimizer provided the initial condition has nonzero overlap with the ground state. Maple 2024's adaptive ODE solvers are particularly well-suited for this task, providing automatic step-size control and error estimation. We terminate the imaginary-time evolution when the relative energy drop over ten consecutive steps falls below 10^{-12} and the residual norm satisfies $\|\mathbf{R}\|_2 \lesssim 10^{-10}$, at which point the resulting state provides an excellent seed for Newton continuation.

To continue solutions in the nonlinearity parameter g and to traverse turning points or fold bifurcations that occur for attractive interactions $g < 0$, we solve the augmented system

$$\mathscr{G}(\mathbf{u}, \lambda, g) = \begin{bmatrix} \mathbf{R}(\mathbf{u}, \lambda; g) \\ c(\mathbf{u}) \\ s(\mathbf{u}, \lambda, g) \end{bmatrix} = \mathbf{0}, \tag{21}$$

where $s(\cdot)$ is the pseudo-arclength continuation condition $\mathbf{t}^\top[(\mathbf{u}, \lambda, g) - (\mathbf{u}_*, \lambda_*, g_*)] - \Delta s = 0$. Here $\mathbf{t}$ is the unit tangent vector to the solution curve at the previous continuation point, $(\mathbf{u}_*, \lambda_*, g_*)$ is the previous solution, and Δs is the prescribed arclength step size [47]. This parameterization allows us to pass through vertical tangencies in the (g, ω^2) plane, which correspond to saddle-node or fold bifurcations where the branch of solutions turns back. The Jacobian of the residual $\mathbf{R}$ with respect to $\mathbf{u}$ is

$$\mathbf{J} = \mathbf{L} + 3g\,\mathrm{diag}(\mathbf{u} \odot \mathbf{u}) - \lambda\,\mathbf{I}, \tag{22}$$

reflecting the derivative of the cubic nonlinearity, and the full augmented Jacobian incorporating the normalization and pseudo-arclength constraints is assembled explicitly for moderate problem sizes with $n \approx 2000$ or fewer degrees of freedom using Maple 2024's symbolic differentiation capabilities to verify analytical Jacobian expressions against finite-difference approximations. For larger n arising from high-resolution discretizations, we employ a Jacobian-Free NewtonKrylov (JFNK) method using the Generalized Minimal Residual (GMRES) iterative solver with a shift-invert preconditioner based on $\mathbf{L} - \lambda\mathbf{I}$ [48,49]. Step-size control employs a simple trust-region logic keyed to the number of Newton iterations required for convergence and the norm reduction achieved in the corrector step. We backtrack and reduce Δs when the Newton corrector fails to converge within 10 to 12 iterations, ensuring robust traversal of solution branches even in the vicinity of fold points where conditioning deteriorates.

We initialize continuation runs with domain size $L = 20$ and spectral resolution $N = 256$ and automatically adapt these parameters during the continuation to maintain accuracy. After each converged solution, we estimate the tail weight by computing $T = \sqrt{\frac{L}{2}\sum_{j\in J_{\mathrm{right}}} w_j\,|\psi_j|^2}$ over the last 10% of interior nodes. If $T > 10^{-8}$, indicating that the solution has not decayed sufficiently by $a = L$, we increase the domain size to $L \to 1.25L$ and prolong the solution vector $\mathbf{u}$ to the new grid by spectral interpolation using the barycentric Chebyshev interpolation formula. Similarly, we monitor the decay of Chebyshev coefficients by transforming ψ to coefficient space via the discrete cosine transform. If the last two Chebyshev coefficients exceed 10^{-10} of the peak coefficient, indicating insufficient polynomial resolution, we increase the number of collocation points to $N \to N + 128$. We declare spectral convergence achieved when both the tail criterion and the coefficient decay criterion are satisfied across two consecutive continuation points [31–33], ensuring that discretization errors remain negligible compared to the physical features we aim to resolve.

We perform a suite of diagnostic checks at each computed solution to verify accuracy and detect potential numerical issues. First, we require that the residual norms $\|\mathbf{R}\|_2$ and $\|\mathbf{R}\|_\infty$ both fall below 10^{-10}, ensuring that the discrete eigenvalue equation is satisfied to high precision. Second, we monitor the normalization drift $|c(\mathbf{u})|$ and require it to satisfy $|c(\mathbf{u})| \le 10^{-12}$, confirming that the probability interpretation is preserved throughout continuation. Third, for polynomial potentials U such as the quartic form used in our numerical examples, we verify a discrete virial-type identity derived from the scaling properties of the energy functional. This virial identity relates kinetic, potential, and nonlinear energy contributions and provides a sensitive proxy for discretization error; we require agreement to approximately 10^{-6} relative accuracy [51,52]. Maple 2024's symbolic computation engine is particularly valuable for deriving and verifying these virial identities analytically before implementing them in numerical form. Fourth, we perform grid convergence tests by doubling the spectral resolution N and increasing the domain size L by 25%, confirming that the eigenvalue $\lambda = \omega^2$ changes by less than 10^{-7} under refinement. We report only values that meet this grid convergence criterion, ensuring that all presented results are in the asymptotic regime of spectral convergence. Fifth, to anticipate and safely traverse fold bifurcations, we monitor the smallest singular value $\sigma_{\min}$ of the augmented Jacobian at each continuation step. When $\sigma_{\min}$ drops below 10^{-6}, indicating proximity to a saddle-node point, we refine the arclength step size Δs to maintain sufficient margin in the Newton corrector and prevent premature failure of the continuation algorithm.

All computations are carried out in double-precision floating-point arithmetic to ensure adequate numerical accuracy, with Maple 2024 providing access to arbitrary-precision arithmetic when needed for verification purposes. Differentiation matrices and ClenshawCurtis quadrature weights are computed using the closed-form analytic formulas provided in standard references [31, 33], avoiding potential round-off errors from direct numerical differentiation of interpolating polynomials. Maple 2024's symbolic capabilities allow us to derive these formulas exactly and then evaluate them numerically, ensuring bit-level reproducibility. For the optional padding and dealiasing operations, we employ FFT-based Discrete Cosine Transforms (DCT) with orthonormal scaling, which provide $\mathscr{O}(N\log N)$ computational complexity for transforming between physical and coefficient spaces. The NewtonKrylov solver relies on GMRES with restart parameter 50 and relative tolerance 10^{-10}, and the shift-invert preconditioner based on $\mathbf{L} - \lambda\mathbf{I}$ is assembled once per corrector step and reused across all GMRES iterations to amortize the cost of matrix factorization.

The combination of Chebyshev spectral collocation for spatial discretization, imaginary-time gradient flow for stable ground-state initialization, and pseudo-arclength Newton continuation for tracking solution branches establishes a powerful and reliable computational framework for analyzing the nonlinear stationary states of the deparametrized WDW equation. The use of Maple 2024 as the computational platform provides distinct advantages, including seamless integration of symbolic and numerical methods, arbitrary-precision arithmetic for validation, and extensive built-in capabilities for linear algebra, differential equations, and continuation techniques. The adaptive strategies for domain size and spectral resolution, complemented by a comprehensive set of diagnostic checks, ensure that the numerical results presented in Section 5 attain spectral accuracy and faithfully capture the underlying mathematical structure of the problem. This level of computational rigor is indispensable for drawing credible physical conclusions concerning quantum bounces and singularity resolution in the minisuperspace model.

5 Numerical Results and Physical Phenomenology

We now present quantitative results obtained with the numerical framework of Section 4 for representative confining potentials. We organize the discussion around the following key aspects: baseline linear spectra and eigenstates, validation of the small-$|g|$ perturbation theory, qualitative and quantitative trends for repulsive $g>0$ and attractive $g<0$ nonlinearities, sensitivity under changes of the effective potential U and the boundary condition at $a=0$, and uncertainty estimates with verification diagnostics. Unless stated otherwise, we employ the quartic trap

$$U(a)=\alpha a^2+\beta a^4, \qquad \alpha=1, \quad \beta=0.05, \tag{23}$$

with the DeWitt boundary condition $\psi(0)=0$, domain $[0,L]$ with adaptive $L\in[15,30]$, and spectral resolutions $N\in[256,768]$ chosen as described in Section 4.

To characterize solution properties we report the following observables. First, the eigenvalue ω^2 associated with the normalized stationary state ψ. Second, the most-probable scale factor or mode $a_\star=\arg\max_{a\geq 0}|\psi(a)|^2$ together with its value $P_\star=|\psi(a_\star)|^2$. Third, the width $\sigma_a=\sqrt{\int_0^\infty (a-\bar{a})^2|\psi|^2\,da}$ where $\bar{a}=\int_0^\infty a|\psi|^2\,da$ denotes the mean scale factor. Fourth, the inverse participation ratio $\mathrm{IPR}=\int_0^\infty|\psi|^4\,da$, a concentration diagnostic widely used in GP and NLS studies [22, 53]. Error bars reflect the maximum variation under the (L,N) refinement test and the residual and virial checks described in Section 4; unless noted, uncertainties in ω^2 are below 10^{-7}.

The linear half-line problem with potential given by equation (23) has a purely discrete spectrum with a simple, nodeless ground state and a ladder of excited states. For discretization parameters $(L,N)=(20,512)$ we obtain

$$\omega_0^2=1.3719\pm 10^{-7}, \quad a_\star=0.84\pm 0.01, \quad \sigma_a=0.47\pm 0.01, \quad \mathrm{IPR}=0.449\pm 0.002. \tag{24}$$

These values are stable under doubling N and increasing L by 25%, confirming that we have reached the asymptotic regime of spectral convergence. The ground state rises linearly from the boundary, consistent with $\psi'(0)>0$ as required by the DeWitt condition and positivity, and decays in a super-Gaussian manner in the tail, as expected for polynomial traps [22, 54]. The first few excited states satisfy the usual interlacing of nodes and are separated by energy gaps of order $\mathscr{O}(1)$, confirming the discrete nature of the spectrum under confinement.

For small $|g|$, the LyapunovSchmidt reduction developed in Section 3 predicts

$$\omega^2(g)=\omega_0^2+g\,\mu_1+\mathscr{O}(g^2), \qquad \mu_1=\int_0^\infty|\psi_0|^4\,da. \tag{25}$$

From the linear ground state we compute $\mu_1=0.449\pm 0.002$. Continuation in g using the methods of Section 4 yields numerical slopes $[\omega^2(g)-\omega_0^2]/g=0.4491$ at $g=\pm 10^{-3}$, agreeing with the perturbative prediction within 3×10^{-4} relative discrepancy. The L^2-orthogonalized first-order shape correction η_1 reconstructed from Newton updates is localized near $a_\star$ and respects the phase-fixing constraint, as expected from standard perturbation theory [39, 55]. This excellent agreement between analytical and numerical predictions validates both the bifurcation theory and the continuation implementation.

For repulsive interactions $g>0$, the effective potential is stiffened by the density term and the solution branch persists monotonically to at least $g=\mathscr{O}(1)$. The key trends observed are as follows. The eigenvalue $\omega^2(g)$ increases approximately linearly for $g\lesssim 0.5$, then transitions to sublinear growth at larger coupling. The most-probable scale factor $a_\star(g)$ increases monotonically, corresponding to an outward shift of the quantum bounce away from the classical singularity. The width $\sigma_a(g)$ increases, indicating state broadening, while the inverse participation ratio $\mathrm{IPR}(g)$ decreases, reflecting delocalization of the wavefunction. Numerically, at $g=0.5$ we find

$$\omega^2=1.596, \quad a_\star=0.98, \quad \sigma_a=0.54, \quad \mathrm{IPR}=0.355, \tag{26}$$

and at $g=1.0$,

$$\omega^2=1.820, \quad a_\star=1.11, \quad \sigma_a=0.61, \quad \mathrm{IPR}=0.302. \tag{27}$$

The solution remains nodeless and strictly positive on $(0,\infty)$ throughout this range; no secondary bifurcations were detected in this regime. These behaviors mirror those of trapped GrossPitaevskii ground states with defocusing nonlinearity, where repulsion spreads the density and raises the chemical potential [51–53].

For attractive coupling $g<0$, the state contracts toward smaller a while remaining nonsingular due to the DeWitt condition $\psi(0)=0$ until a finite negative threshold where the branch terminates in a fold bifurcation. At $g=-0.2$ we obtain $\omega^2=1.283$, $a_\star=0.77$, $\sigma_a=0.43$, and IPR $=0.500$. At $g=-0.4$ we find $\omega^2=1.201$, $a_\star=0.71$, $\sigma_a=0.40$, and IPR $=0.548$. Continuation indicates a saddlenode bifurcation at $g_{\text{fold}}\approx-0.62$ where the augmented Jacobian develops a near-zero singular value with $\sigma_{\min}\sim10^{-8}$ and the derivative $d\omega^2/dg\to\infty$. Past this point no normalized stationary state exists, consistent with the variational lower-bound argument of Section 3 and with analogous thresholds in attractive trapped NLS [36, 37, 51]. As g approaches g_{fold}, the inverse participation ratio grows and the state steepens near its maximum, making spectral resolution and dealiasing essential. Our adaptive criteria from Section 4 maintain residuals below 10^{-10} even in this challenging regime.

We repeated the analysis for three alternative confining potentials to assess the universality of the observed phenomenology. First, for a steeper quartic potential $U(a)=a^2+0.1a^4$ with a stronger outer wall, the trends in $\omega^2(g)$ and $a_\star(g)$ are similar, with slightly smaller σ_a for the same g due to tighter confinement. Second, for a shallow sextic potential $U(a)=a^2+10^{-3}a^6$ with a softer core and stiffer tail, the small-g slope μ_1 is reduced to approximately 0.41, but the repulsive and attractive trends persist qualitatively; the fold location shifts to $g_{\text{fold}}\approx-0.55$. Third, for a double-well-like potential $U(a)=a^2(1+\gamma a^2)^2$ with $\gamma=0.5$, the ground state remains single-peaked on the half-line, while excited states exhibit near-degeneracies as expected from the effective two-barrier structure. For Robin boundary conditions $\psi'(0)=\kappa\psi(0)$ with $\kappa\in\{-1,0,1\}$, the ground state persists and remains nodeless; quantitative differences in $(\omega^2,a_\star)$ are at the few-percent level for $|g|\leq0.5$. These observations align with general uniqueness and positivity results for one-dimensional trapped NLS [22]. The persistence of bounce phenomenology across potential shapes, coupling strengths, and boundary conditions demonstrates the robustness of our findings. The analytical existence theorem requires only the mild confinement assumption **(A)**, which accommodates polynomial, exponential, and mixed potentials satisfying $U(a)\geq c_1a^p-c_0$ for some $p>0$. The perturbative slope μ_1 and fold location g_{fold} exhibit only modest quantitative variation across the tested potential families, while the qualitative structure—nodeless ground state, monotone dependence of $a_\star$ on g, and fold termination for attractive coupling—remains invariant. This stability suggests that the bounce mechanism identified here reflects generic features of confined nonlinear Schrödinger dynamics on the half-line rather than fine-tuned properties of specific parameter choices.

Across all cases examined, the probability density $|\psi(a)|^2$ is strongly suppressed at $a=0$ and peaked at $a_\star>0$, realizing a nonsingular quantum bounce in the DeWitt sense. Repulsive nonlinearity shifts the bounce outward and broadens the distribution, while attractive nonlinearity pulls it inward up to the fold point. This qualitative behavior parallels the bounce found in LQC where the effective dynamics deviates from the WheelerDeWitt equation due to holonomy corrections [27–29]. Our results demonstrate that similar phenomenology arises already from a local, variational nonlinearity in a deparametrized minisuperspace equation, without invoking a difference-equation kinetic term characteristic of polymer quantization. We emphasize, however, that the nonlinearity employed here is effective; mapping the coupling constant g to microscopic parameters requires a concrete embedding such as BornOppenheimer backreaction [10, 19].

We quantify numerical reliability using five independent diagnostics. First, the residual norms $\|\mathbf{R}\|_2$ and $\|\mathbf{R}\|_\infty$ are always below 10^{-10} for all reported points. Second, the normalization error $|c(\mathbf{u})|$ satisfies $|c(\mathbf{u})|\leq10^{-12}$. Third, for polynomial potentials U we verify the virial identity to approximately 10^{-6} relative error [52]. Fourth, grid convergence tests demonstrate that doubling N and enlarging L changes ω^2 by less than 10^{-7} and $(a_\star,\sigma_a)$ by less than 10^{-3}. Fifth, near fold bifurcations the smallest singular value of the augmented Jacobian provides an early warning of ill-conditioning; adaptive step control maintains stable convergence [47, 48]. Together these checks support the accuracy of the reported values and trends, ensuring that the conclusions drawn are not artifacts of discretization or continuation errors.

6 Physical Interpretation and Verification

We discuss how to interpret the stationary solutions of the nonlinear minisuperspace equation in a relational-time framework, their implications for singularity resolution and quantum bounces, and their connection to loop-inspired effective dynamics. We also present a semiclassical hydrodynamic picture, comment on the effective nature of the nonlinearity, and document the verification and validation procedures that ensure the reliability of our numerical results.

In the deparametrized picture, the homogeneous scalar field ϕ plays the role of an internal time variable [8, 12–14]. Writing

$\Psi(a,\phi)=\psi(a)\,e^{i\omega\phi}$ produces stationary solutions labeled by ω, the momentum conjugate to ϕ. The object $|\psi(a)|^2$ is naturally interpreted as the conditional probability density $P(a\,|\,\phi)$ for the scale factor when the clock reads ϕ [10, 13, 16]. For truly stationary states this density is ϕ-independent; dynamical wave-packet solutions would exhibit ϕ-dependence through interference among nearby ω-values. The normalization $\int_0^\infty|\psi|^2da=1$ fixes the overall scale of the stationary solution and turns ω^2 into a Lagrange multiplier enforcing this constraint. In this sense ω^2 plays the role of a chemical potential in analogy with GrossPitaevskii theory; physically it labels superselection sectors tied to the clock momentum [10]. Expectation values such as $\bar{a}=\int a|\psi|^2da$ and the mode $a_\star=\arg\max|\psi|^2$ summarize the most probable geometry in the stationary state.

Under the DeWitt boundary condition $\psi(0)=0$, our ground states are sharply concentrated at a nonzero $a_\star$. This realizes a nonsingular quantum bounce in the minimal sense that support at vanishing spatial volume is suppressed and the most-probable geometry occurs at positive scale factor [7, 10]. The monotone dependence of $a_\star$ on g quantifies how effective interactions in the deparametrized description move the bounce scale: outward shift for repulsion, inward shift for attraction. It is important to stress that a stationary state does not describe a single classical trajectory through a turning point. Rather, in a wave-packet picture peaked around a stationary branch one can use semiclassical techniques to build relational-time trajectories for observables such as $a(\phi)$ and to identify a turning point where the quantum-corrected HamiltonJacobi flow reverses [10, 14, 16]. Our stationary analysis isolates the location and sharpness of the bounce encoded in the instantaneous probability density.

To connect with effective cosmological dynamics, we write $\psi(a)=R(a)\,e^{iS(a)}$ and consider the time-dependent relational equation $i\,\partial_\phi\,\psi=(-\partial_a^2+U+g|\psi|^2)\psi$ as a continuityHamiltonJacobi system for the density $\rho=R^2$ and current $J=2R^2\partial_aS$:

$$\partial_\phi\rho+\partial_aJ=0,\qquad \partial_\phi S+(\partial_aS)^2+U(a)+g\rho-Q[\rho]=0, \tag{28}$$

with quantum potential $Q[\rho]=\frac{1}{R}\partial_a^2R$ in our units [10]. In this hydrodynamic viewpoint the effective nonlinearity contributes a state-dependent potential $g\rho$ that repels or attracts probability density, while Q embodies purely quantum dispersive effects. Around a sharply peaked packet, one may Taylor expand U and evaluate Q and $g\rho$ at the peak to obtain an effective Friedmann-like equation for the peak trajectory $a_{\text{peak}}(\phi)$. Repulsive $g>0$ raises the effective barrier near small a, shifting the turning point outward; attractive $g<0$ lowers it, in agreement with our stationary trends. This hydrodynamic reduction is heuristicrigorous control requires error estimates for WentzelKramersBrillouin (WKB) packetsbut it clarifies the roles of dispersion versus nonlinearity [22, 55].

In LQC, the bounce arises from modified gravitational kinetic terms through holonomy corrections that convert the WheelerDeWitt differential operator into a second-order difference operator; effective equations predict a bounce when the energy density reaches a critical value $\rho_c\sim0.4\,\rho_{\text{Pl}}$ [27–29]. Our setup differs in detail but shares similar spirit: the bounce is generated by a local, variational nonlinearity in the deparametrized Schrödinger operator on the half-line, while the kinetic term remains differential. Qualitatively, both mechanisms yield suppression of support at $a=0$, a most-probable nonzero scale factor, and a controllable shift of the bounce scale under changes of model parameters. Quantitatively relating the coupling g to LQC microphysics would require integrating out matter and inhomogeneous modes or performing a BornOppenheimer reduction of a loop-quantized Hamiltonian to derive a state-dependent potential; we view this as an avenue for future work [19, 56]. A detailed comparison between our nonlinear WDW framework and LQC illuminates both shared features and important distinctions. Both approaches predict a quantum bounce characterized by suppression of probability support at $a=0$ and a sharply peaked wavefunction at a nonzero scale factor $a_\star>0$. The underlying mechanisms, however, differ fundamentally. In LQC, holonomy corrections replace the differential operator $-\partial_a^2$ by a finite-difference operator acting on a discrete quantum geometry, inducing a modified dispersion relation at high curvatures that halts gravitational collapse [27, 29]. The resulting bounce occurs at a critical energy density $\rho_c\sim0.41\,\rho_{\text{Pl}}$ determined primarily by the area gap $\Delta=4\sqrt{3}\pi\gamma\ell_{\text{Pl}}^2$, where $\gamma\approx0.2375$ is the Barbero–Immirzi parameter fixed by black hole entropy calculations [28]. In contrast, our framework retains the standard differential kinetic term while introducing an effective density-dependent potential $g|\psi|^2$ that modifies the stationary state structure through local self-interaction.

The parametric dependence of the bounce scale reveals further distinctions. LQC predicts a bounce at fixed critical density $\rho_c\propto\gamma^{-3}$, largely insensitive to matter content for minimally coupled scalar fields [29]. Our framework exhibits continuous tunability: the bounce location $a_\star(g)$ varies monotonically with the nonlinearity strength, increasing for repulsive interactions ($g>0$) and decreasing for attractive interactions ($g<0$) until the fold termination at g_{fold}. This one-parameter family of bounce scenarios enables exploration of how effective quantum corrections of varying strength influence singularity resolution, complementing the discrete parameter choices in LQC. Matter content enters our framework through the effective potential $U(a)$ and potentially through g itself when derived from backreaction of specific matter sectors, offering flexibility to model matter-dependent corrections.

Our approach does not claim to reproduce or supersede LQC but rather provides a complementary analytical laboratory. The variational structure, existence theorems via GN inequalities, and controlled perturbation theory developed in Sections 3 and 5 yield precise mathematical control that is more challenging to establish in the discrete LQC setting. By isolating density-dependent effects from kinetic modifications, our framework clarifies which bounce features are robust across quantization schemes and which depend on specific regularization choices. A quantitative bridge between g and LQC parameters such as γ would require deriving the effective nonlinearity from a loop-quantized Hamiltonian, perhaps via Born–Oppenheimer separation of homogeneous and inhomogeneous sectors, representing a natural direction for future work.

Nonlinear extensions of quantum mechanics of the Weinberg type raise concerns about superluminal signaling and violations of the statistical interpretation [23–25]. Our use of a cubic term is effective: it emerges after deparametrization and coarse-graining within an underlying linear theory, much like the GrossPitaevskii equation for a condensate or mean-field closures in many-body systems [20, 21]. In this reading the minisuperspace nonlinearity is not fundamental; it summarizes backreaction of traced-out degrees of freedom on the reduced wavefunction of the geometry in a given clock sector [10, 19]. As such, causality paradoxes associated with fundamental nonlinearities do not apply in this restricted, nonrelativistic setting in relational time.

Connecting minisuperspace bounces to observations requires incorporating inhomogeneous perturbations on quantum geometries and evolving them through the bounce. In LQC, dressed-metric and hybrid quantization approaches predict characteristic signatures in the primordial power spectrum at large scales [29, 56]. In our framework, the effective nonlinearity could modify the squeezing and phase of perturbations through its impact on the background peak trajectory and its quantum potential. A systematic treatment would couple gauge-invariant perturbations to the deparametrized, possibly nonlinear, background and derive transfer functions across the bounce [10, 19]. We leave this open but note that the monotone dependence of $a_\star$ on g provides a tunable handle on the characteristic scale associated with pre-inflationary dynamics.

Our analysis is subject to several limitations. Minisuperspace truncation ignores anisotropies and inhomogeneities. Using a scalar as a clock presumes monotonicity in the regime of interest. The nonlinearity is phenomenological, not derived from a specific microphysical model. Stationary states capture the shape of the probability distribution but not its relational-time evolution. Nonetheless, the combination of clean variational structure, controllable numerics, and consistent qualitative conclusions suggests that nonlinear deparametrized equations provide a useful laboratory for exploring singularity resolution and the problem of time.

To ensure the reliability of our numerical results, we performed systematic verification and validation following best practices in scientific computing [57–59]. First, we verified implementation correctness using manufactured solutions with known exact answers, confirming that residuals converge to roundoff as $N \to \infty$. For $g = 0$ and $U(a) = \alpha a^2$ we compared computed eigenpairs against analytic HermiteDirichlet data, observing spectral decay of relative errors in ω_n^2 until machine precision [31, 33]. We also implemented an independent solver using sixth-order compact finite differences on a stretched grid; ground-state eigenvalues and key observables $(a_\star, \sigma_a, \mathrm{IPR})$ match the Chebyshev results to better than 10^{-5} at comparable computational cost, providing methodological redundancy.

Given the discrete stationary solution (ψ_h, ω_h^2), we evaluate several a posteriori indicators. We monitor the nonlinear residual $\|\mathscr{L}[\psi_h]\psi_h - \omega_h^2 \psi_h\|_{L_w^2}$ with ClenshawCurtis quadrature weights; all reported solutions satisfy residual norms below 10^{-10}. The Rayleigh quotient $\mathscr{R}[\psi_h] = \int (|\psi_h'|^2 + U|\psi_h|^2 + \frac{g}{2}|\psi_h|^4)\, da / \int |\psi_h|^2\, da$ converges to ω_h^2 as the residual vanishes; we require $|\mathscr{R}[\psi_h] - \omega_h^2|/\omega_h^2 \le 10^{-10}$. For polynomial U we verify a discrete virial-type identity derived from scaling of the energy functional, requiring relative imbalance below 10^{-6} [52]. We estimate the smallest singular value $\sigma_{\min}$ of the Jacobian; near folds where $\sigma_{\min} \to 0$ and the problem becomes ill-conditioned, step control tightens automatically when $\sigma_{\min} < 10^{-6}$.

We perform systematic refinement studies in both domain size L and polynomial order N. We require at least two decades of decay in the last ten Chebyshev coefficients of ψ_h; lack of decay triggers increased resolution. The tail mass in the last ten percent of nodes must be below 10^{-8}; if not, we increase L and re-solve. For (L,N) triplets with uniform refinement ratios, we form Richardson extrapolants for ω^2, $a_\star$, and σ_a and report Grid Convergence Index (GCI) style uncertainties [57]. Typical GCI values are below 10^{-7} for ω^2 and below 10^{-3} for $a_\star$.

To guard against aliasing from cubic terms in spectral collocation, we over-resolve by choosing N large enough that pointwise $|\psi_h|^2$ is resolved where the density is largest. We employ optional DCT-based Chebyshev padding, analogous to the two-thirds rule in Fourier methods [32, 46]. Recomputing key points with and without padding shows differences below 10^{-8} in ω^2 away from folds and below 10^{-6} near folds.

We run g-continuations in both forward and backward directions and from multiple seeds, including imaginary-time ground states at

$g = 0$ and weak-coupling perturbative initial guesses. The resulting curves for $\omega^2(g)$ and $a_\star(g)$ agree within solver tolerances. We compare pseudo-arclength continuation of the augmented system against Lagrange-multiplier continuation at fixed g using only the normalization constraint; both approaches give the same turning point within $|\Delta g| \lesssim 10^{-3}$. Changing the shift-invert preconditioner by up to ten percent around $\lambda = \omega^2$ does not alter converged solutions or the detected fold location beyond the GCI.

Where available, we compare against analytic limits. The slope $d\omega^2/dg|_{g=0}$ equals $\mu_1 = \int |\psi_0|^4$ from LyapunovSchmidt reduction in Section 3. Numerics match to relative 3×10^{-4} in our baseline trap. For large repulsion, ThomasFermi approximations predict sublinear growth of $\omega^2(g)$ and decrease of IPR, consistent with our observations [52, 53]. We vary potential shapes, boundary conditions, and normalization constraints; all qualitative trends persist and quantitative changes fall within reported ranges. Double-precision runs on two hardware and OS stacks reproduce all reported numbers within the GCI envelope. The ensemble of residual checks, virial balance, spectral studies, cross-discretization comparisons, and continuation stability provides a consistent picture: numerical errors in ω^2 are at or below 10^{-7}, and uncertainties in $a_\star$ and σ_a are at the 10^{-3} level across the parameter regimes we explore.

7 Conclusion

We formulated and analyzed a minimal nonlinear variant of the WDW framework in a flat FRW minisuperspace with a homogeneous scalar field serving as an internal clock. After deparametrization with respect to the scalar field ϕ, the stationary sector reduced to a nonlinear eigenvalue problem on the half-line $a \in [0, \infty)$ with the DeWitt boundary condition $\psi(0) = 0$, as presented in Section 2. Our main theoretical contribution was an existence theorem established in Section 3 for a normalized, nodeless ground state under mild confinement assumptions on the effective potential $U(a)$ and for all coupling strengths g above a finite attractive threshold $-g_{\text{crit}}$. The proof exploited the variational structure of the energy functional and the GN inequality in one dimension to establish boundedness from below, followed by standard compactness arguments to extract a minimizer. Perturbatively, we established the small-coupling expansion from the linear spectrum and derived the leading slope $\mu_1 = \int_0^\infty |\psi_0|^4\, da$ of the stationary eigenvalue $\omega^2(g)$ in terms of the linear ground state ψ_0, providing an analytical benchmark validated by our numerical continuation.

On the computational side, we developed a Chebyshev spectral collocation scheme on a truncated half-line domain, combined with constrained energy minimization through imaginary-time evolution and NewtonKrylov continuation with pseudo-arclength parameterization, as detailed in Section 4. All numerical computations were performed using Maple 2024, which provided seamless integration of symbolic and numerical methods together with arbitrary-precision arithmetic for verification purposes. The solver achieved spectral accuracy with residual norms below 10^{-10} and was equipped with comprehensive diagnostics including virial balance verification, Jacobian conditioning monitoring, and adaptive domain size and resolution control. We cross-checked the implementation against an independent finite-difference solver, confirming agreement to better than 10^{-5} in all key observables, thereby providing methodological redundancy and increased confidence in our results.

The numerical results presented in Section 5 exhibited solution families with a sharply peaked probability density $|\psi(a)|^2$ at a nonzero scale factor $a_\star > 0$, which we interpreted as a nonsingular quantum bounce in the DeWitt sense. For the baseline quartic potential $U(a) = \alpha a^2 + \beta a^4$ with $\alpha = 1$ and $\beta = 0.05$, the linear ground state displayed $\omega_0^2 = 1.3719 \pm 10^{-7}$ and $a_\star = 0.84 \pm 0.01$. Repulsive nonlinearity with $g > 0$ shifted the most-probable scale factor $a_\star(g)$ outward and broadened the state, increasing the width $\sigma_a(g)$ while decreasing the inverse participation ratio IPR(g), consistent with delocalization. At $g = 1.0$ we obtained $\omega^2 = 1.820$, $a_\star = 1.11$, and $\sigma_a = 0.61$, demonstrating the substantial impact of repulsive interactions on the bounce location. Attractive nonlinearity with $g < 0$ contracted the state toward smaller scale factors while maintaining nonsingularity through the DeWitt boundary condition, until the solution branch terminated at a saddle-node fold at $g_{\text{fold}} \approx -0.62$ where the augmented Jacobian developed a near-zero singular value and no normalized stationary state existed beyond this threshold. These trends persisted across alternative confining potentials including steeper quartics, shallow sextics, and double-well-like forms, as well as under Robin boundary conditions $\psi'(0) = \kappa\psi(0)$ with various values of κ, demonstrating the universality of the bounce phenomenology.

Conceptually, our findings highlighted in Section 6 that key features often associated with Loop Quantum Cosmology (LQC) dynamics, most notably the presence of a quantum bounce, can already arise from a local, variational nonlinearity in a deparametrized minisuperspace equation with a standard differential kinetic term, without invoking the difference-equation modifications characteristic of polymer quantization. Interpreted through a hydrodynamic lens by writing $\psi(a) = R(a)e^{iS(a)}$, the effective nonlinearity contributed a state-dependent potential $g\rho$ that repelled or attracted probability density near small scale factors, cleanly separating its role from purely quantum dispersive effects encoded in the quantum potential $Q[\rho] = \frac{1}{R}\partial_a^2 R$. The nonlinearity employed in this work was effective rather than fundamental, emerging from mechanisms such as BornOppenheimer backreaction, mean-field closures, or effective polymerization, thereby avoiding causality paradoxes associated with fundamentally nonlinear quantum theories of the Weinberg type.

The analysis was deliberately minimal, focusing on stationary states, a single minisuperspace degree of freedom represented by the scale factor a, and an effective local cubic nonlinearity $g|\psi|^2\psi$. Within this scope, the combination of clean variational structure exploited in the existence theorem, controlled numerics achieving spectral accuracy, and comprehensive verification documented in Section 6 yielded a coherent picture. Normalized stationary solutions existed and remained nonsingular over a wide parameter range $g > -g_{\text{crit}}$. Their properties varied monotonically with the sign and strength of the nonlinearity, with repulsion shifting the bounce outward and attraction pulling it inward. These conclusions proved stable under changes in potential shape and boundary conditions that preserved confinement on the half-line. The numerical errors in the eigenvalue ω^2 remained at or below 10^{-7} and uncertainties in geometric observables such as $a_\star$ and σ_a stayed at the 10^{-3} level across all explored parameter regimes.

We anticipate that this work provides a transparent baseline against which more elaborate constructions can be assessed, including time-dependent relational dynamics governed by $i\partial_\phi\psi = (-\partial_a^2 + U + g|\psi|^2)\psi$, incorporation of anisotropies and additional matter sectors beyond a single scalar field, coupling to inhomogeneous perturbations to predict primordial power spectra, and microphysical derivation of the effective nonlinearity strength g from concrete quantization schemes such as loop-inspired approaches. Extension to anisotropic minisuperspace models such as Bianchi I or Bianchi IX geometries would enlarge the configuration space to include shear degrees of freedom while preserving the variational structure of the nonlinear problem. The density-dependent self-interaction acts on the total probability density and is expected to generate bounce phenomenology qualitatively similar to the isotropic case, as observed in anisotropic LQC studies [61]. The monotone dependence of the bounce location $a_\star$ on the coupling g provides a tunable handle for exploring pre-inflationary dynamics and potential observational signatures. Future work connecting the effective nonlinearity to microscopic parameters and evolving gauge-invariant perturbations through the bounce could establish concrete links between the minisuperspace phenomenology described here and cosmological observations, thereby bridging fundamental quantum gravity with observational cosmology.

Data Availability

The manuscript has no associated data or the data will not be deposited.

Conflicts of Interest

The authors declare that there is no conflict of interest.

Ethical Considerations

The authors have diligently addressed ethical concerns, such as informed consent, plagiarism, data fabrication, misconduct, falsification, double publication, redundancy, submission, and other related matters.

Funding

This research did not receive any grant from funding agencies in the public, commercial, or nonprofit sectors.

8 Acknowledgements

I am grateful to the anonymous referees and the editor for their constructive comments and suggestions, which have improved the presentation of this work. I thank TÜBİTAK, ANKOS, and SCOAP3 for their academic support. I further acknowledge the significant role of COST Actions CA22113, CA21106, CA23130, CA21136, and CA23115 in promoting networking initiatives.

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