Analytical and Numerical Solutions for Nonlinear Equations
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Research article

Non-Linear Equation Approach to Black Hole Phase Transitions

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Abstract

We investigate the role of non-linear equations in characterizing black hole phase transitions within extended thermodynamic frameworks. By formulating the critical conditions as the solution of a non-linear equation derived from the equation of state, we provide an analytical and numerical study of black hole thermodynamics near criticality. Our analysis demonstrates that the non-linear structure encodes universal behavior analogous to the Van der Waals fluid, and it offers a systematic way of locating transition points. The method extends naturally to higher-dimensional and charged AdS black holes, highlighting the generality of this approach.

Keywords: Non-Linear equation approach, Black hole, Phase transitions.

Mathematics Subject Classification (2020): 83C57, 83Cxx

1 Introduction

The discovery that black holes behave as thermodynamic systems has profoundly influenced our understanding of gravity, quantum field theory, and statistical mechanics. The pioneering works of Bekenstein and Hawking established that a black hole possesses an entropy proportional to the area of its event horizon and emits thermal radiation with a well-defined temperature [1, 2]. This realization opened the door to what is now referred to as black hole thermodynamics, where quantities such as entropy, temperature, and free energy acquire precise gravitational counterparts [3, 4].

A particularly fruitful development in this field has been the recognition of the role of the cosmological constant in the thermodynamic description of black holes. Interpreting the cosmological constant Λ as a pressure term $P = -\Lambda/(8\pi)$ leads to an extended phase space formulation, in which the black hole mass is naturally identified with enthalpy rather than internal energy [5, 6]. Within this framework, black holes in anti-de Sitter (AdS) spacetime exhibit a striking analogy with ordinary thermodynamic fluids. The work of Kubizák and Mann [7] demonstrated that the phase behavior of charged AdS black holes mirrors the Van der Waals liquid–gas system, featuring first-order small/large black hole transitions and a second-order critical point. This analogy has since been extended to a wide variety of black hole solutions, including higher-dimensional, rotating, and GaussBonnet black holes [8–11].

The analysis of such phase transitions typically reduces to identifying critical points by imposing inflection-point conditions on the equation of state. Explicitly, one requires the pressure P as a function of the temperature T and volume v (or horizon radius) to satisfy

$$\left(\frac{\partial P}{\partial v}\right)_T = 0, \qquad \left(\frac{\partial^2 P}{\partial v^2}\right)_T = 0. \tag{1}$$

These conditions yield a system of non-linear equations in the critical variables. The structure of these equations is non-trivial, reflecting the underlying gravitational dynamics as well as the matter content of the black hole. Solving them is therefore central to uncovering the thermodynamic phase structure.

The role of non-linear equations in gravitational thermodynamics is twofold. First, they encode the competition between different physical effects: the attractive nature of gravity, the repulsive contribution from charge or rotation, and the confining influence of the AdS boundary. Second, they control the universality of the critical behavior. In many cases, solutions of these non-linear equations yield universal ratios, such as the critical compressibility factor $P_c v_c / T_c$, which often coincide with those of conventional fluids [7, 12]. This universality is striking, given the profound differences between the microscopic degrees of freedom of black holes and molecular systems.

The study of black hole phase transitions has also attracted considerable attention in the context of holography. In the AdS/CFT correspondence, the thermodynamic properties of black holes translate into features of strongly coupled quantum field theories [13, 14]. Phase transitions in the bulk geometry correspond to thermal or confinement–deconfinement transitions in the boundary theory [15, 16]. Understanding the precise structure of the non-linear equations governing critical points thus provides not only gravitational insights but also field-theoretic implications.

Despite the extensive body of work, there remain open questions regarding the generality and robustness of the fluid analogy. Many of these questions revolve around the mathematical properties of the non-linear systems that arise in the criticality analysis. For example, higher-curvature corrections or quantum corrections can significantly alter the coefficients appearing in the equation of state, leading to modified non-linear equations whose solutions may deviate from the classical Van der Waals structure [9, 10, 17–19]. A systematic approach to solving and interpreting such equations is therefore essential.

From a methodological standpoint, several strategies have been employed to tackle these non-linear systems. In lower-dimensional cases, algebraic manipulation often reduces the problem to a cubic or quartic equation in the horizon radius, which can be solved analytically. For more complex geometries or higher-derivative gravity theories, perturbative expansions and series methods are valuable tools, allowing one to approximate the critical parameters around known limits [20, 21]. Numerical methods, such as NewtonRaphson iteration and bifurcation analysis, are widely used when exact solutions are inaccessible, providing accurate estimates of critical quantities across a broad parameter space [22, 23]. More sophisticated approaches, including homotopy analysis [24] and semi-analytical approximations, have also been adapted to gravitational thermodynamics, offering flexible frameworks for treating strongly non-linear equations. These techniques not only yield practical solutions but also shed light on the underlying mathematical structure of black hole thermodynamics.

In this work, we revisit the problem of black hole phase transitions from the perspective of solving the non-linear equations that define criticality. We focus on how these equations emerge from the equation of state, how they can be reduced and solved analytically in specific cases, and how numerical methods extend the analysis to more complicated scenarios. By doing so, we aim to highlight the unifying role of non-linear structures in gravitational thermodynamics and to emphasize their utility in uncovering universal properties across different classes of black holes.

2 Equation of State and Criticality

The cornerstone of extended black hole thermodynamics is the identification of the cosmological constant with thermodynamic pressure. In d-dimensional spacetime this relation reads

$$P = -\frac{\Lambda}{8\pi} = \frac{(d-1)(d-2)}{16\pi\ell^2}, \tag{2}$$

where ℓ denotes the AdS curvature radius. This identification not only extends the first law of black hole mechanics but also casts the black hole mass M as enthalpy rather than internal energy [5, 6, 18]. The natural conjugate to P is the thermodynamic volume V, which, for static and spherically symmetric black holes, is proportional to the geometric volume enclosed by the event horizon.

For a charged AdS black hole in four dimensions, the metric function can be written as

$$f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} + \frac{r^2}{\ell^2}, \tag{3}$$

with Q the electric charge. The event horizon radius r_h is defined by $f(r_h) = 0$. The Hawking temperature is obtained from the surface gravity,

$$T = \frac{f'(r_h)}{4\pi} = \frac{1}{4\pi r_h}\left(1 - \frac{Q^2}{r_h^2} + \frac{3r_h^2}{\ell^2}\right), \tag{4}$$

while the entropy follows the BekensteinHawking area law,

$$S = \pi r_h^2. \tag{5}$$

Introducing the specific volume $v = 2r_h$ [7], the equation of state for the ReissnerNordströmAdS (RNAdS) black hole becomes

$$P = \frac{T}{v} - \frac{1}{2\pi v^2} + \frac{2Q^2}{\pi v^4}. \tag{6}$$

This expression exhibits a direct parallel with the Van der Waals equation, where the first term represents the thermal contribution, the negative $1/v^2$ term mimics intermolecular attraction, and the positive $1/v^4$ term originates from electric repulsion. The equation of state is inherently non-linear in v, a feature that underpins the rich phase structure of AdS black holes.

The existence of a critical point requires the isotherm $P(v)$ to possess an inflection point. This condition is encoded in

$$\left(\frac{\partial P}{\partial v}\right)_T = 0, \qquad \left(\frac{\partial^2 P}{\partial v^2}\right)_T = 0. \tag{7}$$

Substituting Eq. (6) into Eq. (7) yields two coupled non-linear equations in (T_c, v_c):

$$-\frac{T_c}{v_c^2} + \frac{1}{\pi v_c^3} - \frac{8Q^2}{\pi v_c^5} = 0, \tag{8}$$

$$\frac{2T_c}{v_c^3} - \frac{3}{\pi v_c^4} + \frac{40Q^2}{\pi v_c^6} = 0. \tag{9}$$

These equations constitute the fundamental non-linear system whose solutions define the critical temperature T_c, critical specific volume v_c, and subsequently the critical pressure P_c via Eq. (6). The system is algebraically solvable in four dimensions, leading to the well-known critical quantities

$$v_c = 2\sqrt{6}Q, \qquad T_c = \frac{\sqrt{6}}{18\pi Q}, \qquad P_c = \frac{1}{96\pi Q^2}, \tag{10}$$

with the universal ratio

$$\rho_c = \frac{P_c v_c}{T_c} = \frac{3}{8}, \tag{11}$$

precisely matching the Van der Waals result.

In higher dimensions or in modified gravity theories, the equation of state generalizes to the schematic form

$$P = \frac{T}{v} - \frac{a}{v^2} + \frac{b}{v^d} + \cdots, \tag{12}$$

where a and b are constants determined by the black hole charge, rotation, or higher-curvature couplings, and the ellipsis denotes possible additional corrections. In such cases, Eqs. (7) give rise to higher-order non-linear systems, which often lack closed-form solutions. Numerical and semi-analytical approaches become indispensable for determining critical quantities [9, 17, 23].

The behavior of the equation of state is illustrated in Fig. 1, where we plot the isotherms of the RN–AdS black hole for $Q = 1$ at three representative temperatures: below, at, and above the critical temperature T_c. The curve at $T = 0.8T_c$ displays the characteristic oscillatory behavior familiar from the Van der Waals fluid. In this regime, the isotherm exhibits a region with $\partial P/\partial v > 0$, which is thermodynamically unstable. In the standard Maxwell construction, this segment is replaced by a constant-pressure line that represents the coexistence of small and large black hole phases, analogous to the liquid–gas transition.

At the critical temperature T_c, the oscillatory structure degenerates into an inflection point, marked by (v_c, P_c) in the figure. This point corresponds to a second-order phase transition, where the distinction between small and large black holes disappears. For higher temperatures, such as $T = 1.2T_c$, the isotherms are smooth and monotonic, indicating the absence of phase coexistence. The overall structure of the curves thus confirms that the RN–AdS black hole reproduces the same qualitative thermodynamic behavior as a Van der Waals fluid, with the non-linear dependence of $P(v, T)$ governing the transition between different phases.

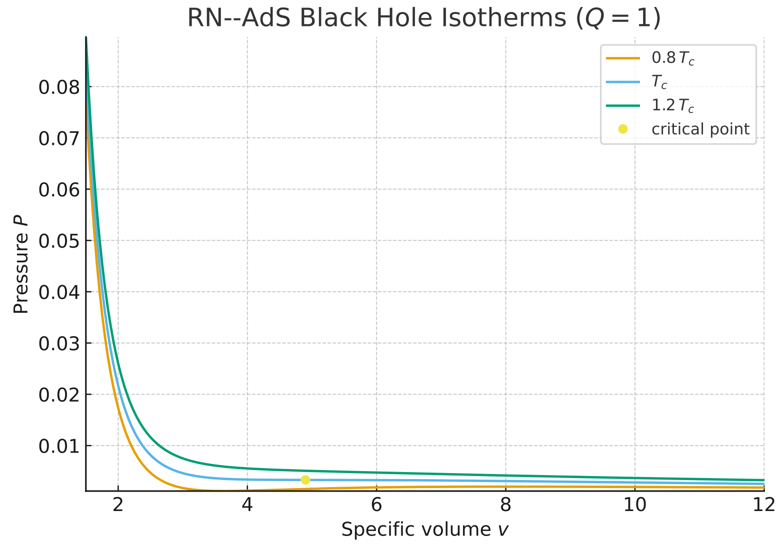

Figure 1. Isotherms of the Reissner–Nordström–AdS black hole in the extended phase space for $Q = 1$. Curves correspond to $T = 0.8\,T_c$, $T = T_c$, and $T = 1.2\,T_c$. The inflection point on the critical isotherm (v_c, P_c) marks the second-order phase transition.

The phase structure becomes even more transparent when examined through the Gibbs free energy, shown in Fig. 2. For pressures below the critical value ($P < P_c$), the G–T diagram develops a characteristic swallowtail structure. This indicates the coexistence of two distinct black hole phases: a small black hole with lower horizon radius and a large black hole with higher radius. At a given temperature, the thermodynamically preferred phase is the one with the lowest Gibbs free energy, so the crossing of the swallowtail branches signals a first-order phase transition between small and large black holes. This transition is equivalent to replacing the unstable oscillatory portion of the P–v isotherm by the Maxwell equal area construction. At the critical pressure $P = P_c$, the swallowtail degenerates into a cusp, corresponding to a continuous, second-order transition. For $P > P_c$, the free energy is single-valued and smooth, and no phase coexistence occurs. The Gibbs free energy picture thus provides a complementary and physically intuitive perspective on the black hole phase transition, reinforcing the analogy with liquid–gas systems in ordinary thermodynamics [7, 18].

The non-linear nature of these equations is not a mere technicality but a reflection of the competing interactions governing black hole thermodynamics. Attractive and repulsive effects conspire to produce phase coexistence, reentrant phase transitions, and even multicritical behavior [10, 11]. As we show in the following sections, the systematic treatment of these non-linear systems provides a unified route to understanding black hole phase transitions across a wide spectrum of theories.

3 Solving the Non-Linear System

Having established the form of the equation of state and the corresponding criticality conditions, we now turn to the explicit solution of the non-linear system defined by Eqs. (8) and (9). In the case of the four-dimensional Reissner–Nordström–AdS black hole, the system can be reduced to a single algebraic equation for the critical volume v_c. Eliminating T_c between the two conditions yields

$$v_c^2 - 24Q^2 = 0, \tag{13}$$

which admits the positive solution (see Eq. (10)). Substituting this back into Eq. (8) gives (10) and (11). This exact correspondence underscores the power of the non-linear approach: the critical behavior emerges naturally from the solution of the algebraic system without any additional assumptions.

In higher-dimensional spacetimes or in theories with additional couplings, such as Gauss–Bonnet or Lovelock gravity, the structure of the equation of state becomes more complicated. The generic form (12) involves higher-order terms in $1/v$ whose coefficients depend on the dimensionality and coupling constants [9, 10, 17]. The criticality conditions then yield polynomial equations of order higher than three, which in most cases cannot be solved in closed form. For example, in Gauss–Bonnet gravity the relevant equation reduces to a

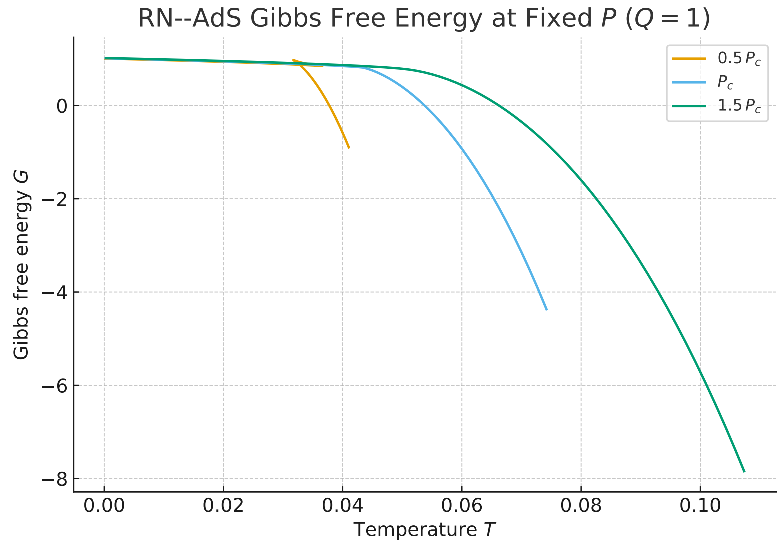

Figure 2. Gibbs free energy G versus temperature T at fixed charge $Q = 1$ for three pressures: $0.5\,P_c$, P_c, and $1.5\,P_c$. For $P < P_c$ the multivalued branch forms a swallowtail, indicating a first-order phase transition between small and large black holes; the thermodynamically preferred phase minimizes G at fixed (T,P). At $P = P_c$ the swallowtail degenerates to a cusp, signaling a second-order critical point. For $P > P_c$ the free energy is single-valued and analytic, with no phase coexistence.

sextic polynomial in v_c, with coefficients involving the Gauss–Bonnet parameter α and the spacetime dimension d. Analytical solutions are impractical, and numerical methods must be employed.

Several strategies have been developed to address these non-linear systems. In lower dimensions, algebraic elimination techniques allow one to reduce the problem to cubic or quartic equations, which can be solved exactly using standard formulas [7]. Perturbative methods are useful when additional couplings are small, enabling expansions around the Einstein–Hilbert case. For instance, one may expand the critical parameters in powers of the Gauss–Bonnet coupling α, obtaining approximate but systematic corrections to (T_c, v_c, P_c) [21].

In cases where analytic approaches fail, numerical root-finding techniques provide a robust alternative. The Newton–Raphson method is particularly effective when good initial guesses are available, as the convergence is quadratic near the root. Alternatively, bracketing methods such as the bisection algorithm guarantee convergence provided the polynomial changes sign across an interval. Numerical continuation techniques can further track how the solutions evolve as parameters (such as charge, dimension, or coupling constants) are varied. More sophisticated frameworks, such as the homotopy analysis method [24] and bifurcation theory [22], have also been adapted to gravitational thermodynamics, offering semi-analytical insights into the structure of solutions.

As a worked example, we consider a neutral Gauss–Bonnet–AdS black hole in five dimensions, for which the equation of state can be written in terms of the specific volume $v = \frac{4r_h}{3}$ as

$$P = \frac{T}{v}\left(1 + \frac{32\,\alpha}{9\,v^2}\right) - \frac{2}{3\pi v^2}, \tag{14}$$

see, e.g., [9]. Solving the non-linear criticality conditions $(\partial P/\partial v)_T = (\partial^2 P/\partial v^2)_T = 0$ numerically for fixed α yields the critical triplet (v_c, T_c, P_c). The resulting dependencies, shown in Figs. 3–5, display a clear trend: v_c increases with α, whereas T_c and P_c decrease. Physically, the Gauss–Bonnet correction renormalizes the thermal term by an α-dependent factor, effectively weakening the attraction encoded in the $1/v^2$ part of the state equation and shifting the inflection point to larger volumes. This higher-curvature modulation of the non-linear system preserves the Van der Waals-like structure while continuously deforming the location of the critical point, in agreement with previous analyses of Gauss–Bonnet thermodynamics [9].

It is important to emphasize that the non-linear structure of the criticality equations is not incidental but reflects the interplay of fundamental physical effects. In the RN–AdS case, the $1/v^2$ term originates from gravitational attraction, while the $1/v^4$ term encodes the repulsive contribution of the electric charge. In higher-dimensional or higher-curvature theories, additional powers of $1/v$ appear, corresponding to corrections from short-distance gravitational interactions. The coexistence of competing attractive and repulsive effects is

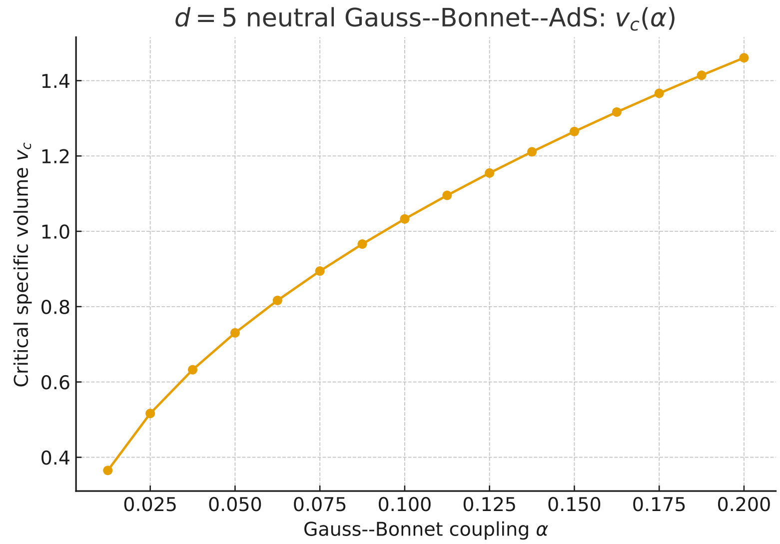

Figure 3. Neutral Gauss–Bonnet–AdS black hole in $d = 5$: critical specific volume v_c versus Gauss–Bonnet coupling α. The growth of v_c with α reflects the higher-curvature correction shifting the critical point to larger horizon scales.

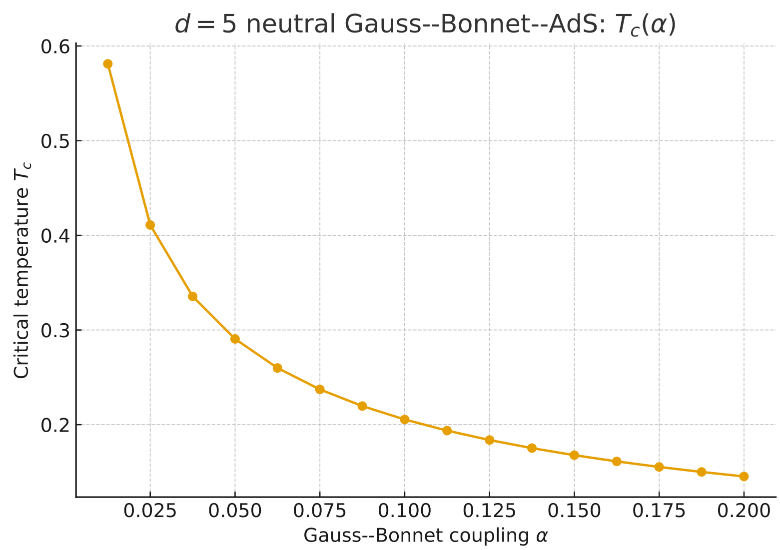

Figure 4. Neutral Gauss–Bonnet–AdS black hole in $d = 5$: critical temperature T_c versus α. Increasing α lowers T_c, indicating a softer effective interaction in the equation of state.

precisely what allows for inflection points in the isotherms and swallowtail structures in the Gibbs free energy.

Despite these model-dependent differences, the qualitative behavior of the solutions is remarkably universal. Many black hole families reproduce the same Van der Waals ratio $\rho_c = 3/8$ or close analogues thereof, even in the presence of higher-order corrections [12, 23]. The universality is therefore encoded in the algebraic structure of the non-linear equations themselves, making their systematic study central to the understanding of gravitational thermodynamics.

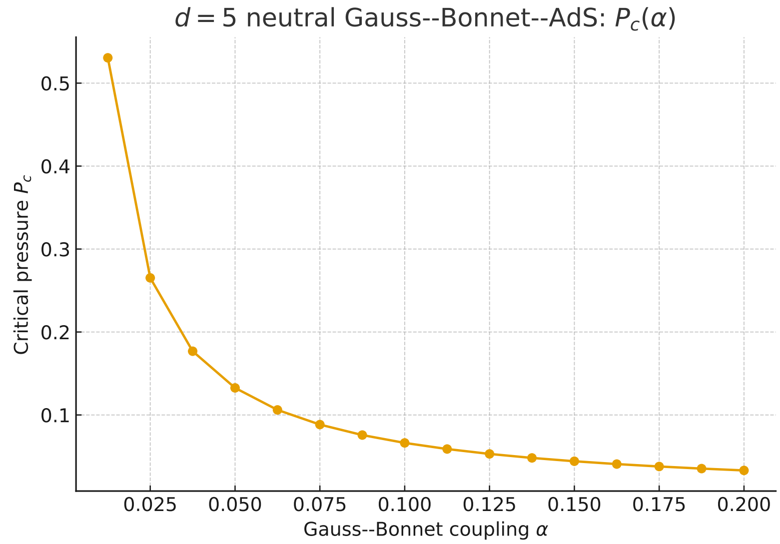

Figure 5. Neutral Gauss–Bonnet–AdS black hole in $d = 5$: critical pressure P_c versus α. The monotonic decrease of P_c with α accompanies the shift of the critical point seen in $v_c(\alpha)$ and $T_c(\alpha)$.

4 Results and Discussion

The explicit solution of the criticality conditions for the four-dimensional Reissner–Nordström–AdS black hole yields the critical point

$$v_c = 2\sqrt{6}Q, \qquad T_c = \frac{\sqrt{6}}{18\pi Q}, \qquad P_c = \frac{1}{96\pi Q^2}, \tag{15}$$

with the compressibility ratio

$$\rho_c = \frac{P_c v_c}{T_c} = \frac{3}{8}. \tag{16}$$

The fact that ρ_c exactly matches the universal ratio of the Van der Waals fluid underscores the robustness of the analogy between black hole thermodynamics and conventional matter systems [7]. In this case the non-linear system admits closed-form solutions, making the critical structure particularly transparent.

The non-linear behavior of the equation of state is clearly visible in the P–v diagram shown in Fig. 1. Below the critical temperature, the isotherms display the characteristic oscillation associated with metastable and unstable branches. As in the Van der Waals system, this oscillation is unphysical, and the Maxwell equal area law is invoked to replace it with a horizontal isobar corresponding to phase coexistence. The intersection of the coexistence line with the isotherm marks the transition between small and large black hole phases. At the critical temperature T_c, the oscillation collapses into an inflection point, and above T_c the isotherms are smooth and monotonic, with no sign of a phase transition.

A complementary perspective is provided by the Gibbs free energy, plotted in Fig. 2. For $P < P_c$ the Gibbs free energy exhibits the swallowtail structure familiar from first-order phase transitions, signaling the coexistence of two stable branches. The phase transition occurs at the point where the two branches cross, ensuring that the thermodynamically preferred phase is the one with lowest G. At $P = P_c$ the swallowtail degenerates into a cusp, indicating the second-order critical point. For $P > P_c$ the free energy is single-valued and smooth, reflecting the absence of phase coexistence. Together, the P–v isotherms and Gibbs free energy diagrams provide a consistent picture of the underlying non-linear system: they encode both the instability mechanism and the critical phenomena that emerge from it.

When higher-curvature terms are included, the situation becomes more subtle. To illustrate this, we considered the five-dimensional neutral Gauss–Bonnet–AdS black hole, with equation of state

$$P = \frac{T}{v}\left(1 + \frac{32\alpha}{9v^2}\right) - \frac{2}{3\pi v^2}. \tag{17}$$

The Gauss–Bonnet coupling α modifies the effective thermal term through the α/v^3 correction, thereby altering the location of the critical point. Solving the non-linear system numerically for various α reveals a systematic trend: the critical specific volume v_c increases monotonically with α, while both T_c and P_c decrease, as shown in Figs. 3–5. This behavior can be physically understood as a consequence of the higher-curvature term reducing the effective attraction in the system, thereby shifting the balance between repulsive and attractive contributions to larger horizon radii. The universality of the Van der Waals ratio is broken, but the overall qualitative structure of the phase transition persists.

These results align with earlier analyses of Gauss–Bonnet black hole thermodynamics, which reported that higher-curvature corrections can lead to rich phenomena such as reentrant phase transitions, multiple critical points, and modified universality classes [9, 17]. Our approach highlights that such features are all encoded in the non-linear algebraic equations that define the critical point: as α increases, the coefficients of the non-linear terms shift, and the solution curves for (v_c, T_c, P_c) deform accordingly.

An important lesson from these examples is that universality in black hole phase transitions is intimately tied to the structure of the underlying non-linear system. In Einstein gravity with a negative cosmological constant, the form of the equation of state guarantees the ratio $\rho_c = 3/8$, independent of the black hole charge. Once higher-curvature corrections or additional couplings are included, the algebraic structure of the non-linear system is altered, leading to deviations from the universal ratio. In some cases, such as charged Gauss–Bonnet or Lovelock black holes, multiple solutions to the non-linear equations exist, giving rise to multiple critical points and even triple points [10,11].

The strength of the non-linear approach is that it provides a unifying mathematical framework: whether solved analytically in simple cases or numerically in more complex scenarios, the solutions reveal both universal patterns and deviations. By systematically tracking how the solutions of the non-linear system depend on the parameters of the theory, one can map out the entire phase diagram of a given black hole family and identify features such as reentrant transitions, swallowtails, or multicritical behavior.

Finally, it is worth emphasizing that the criticality conditions have a clear physical interpretation. The vanishing of the first derivative $(\partial P/\partial v)_T = 0$ corresponds to the mechanical instability point, while the simultaneous vanishing of the second derivative indicates the merging of metastable branches into a single inflection. This is precisely the condition that marks the onset of criticality in fluid systems. In black hole thermodynamics, these mathematical conditions arise from the balance of gravitational attraction, electromagnetic repulsion, and the confining effect of the AdS boundary. Higher-curvature terms introduce new short-distance interactions, shifting this balance but leaving the qualitative mechanism intact. The universality of critical behavior is therefore not coincidental but rooted in the structure of the non-linear equations that govern the system.

In summary, the results of this section demonstrate that the analysis of non-linear criticality equations provides a powerful lens for understanding black hole thermodynamics. The RN–AdS case offers an exact analytic match with the Van der Waals system, while higher-curvature examples such as the Gauss–Bonnet black hole illustrate how the solutions deform under additional interactions. Both the universal features and the deviations can be understood directly in terms of the algebraic structure of the non-linear system, highlighting the central role played by these equations in the study of gravitational phase transitions.

5 Conclusion

In this work we have investigated the phase transitions of AdS black holes through the lens of non-linear equations derived from the equation of state. By explicitly solving the criticality conditions $(\partial P/\partial v)_T = (\partial^2 P/\partial v^2)_T = 0$, we demonstrated that the thermodynamic behavior of black holes is encoded in the structure of a coupled non-linear system. For the Reissner–Nordström–AdS black hole in four dimensions, this system can be solved exactly, yielding critical quantities that reproduce the universal Van der Waals ratio $\rho_c = 3/8$. The analysis of P–v isotherms and the Gibbs free energy confirms that the non-linear equations govern both the first-order small/large black hole transition and the second-order critical point.

We then extended our analysis to the five-dimensional neutral Gauss–Bonnet–AdS black hole. In this case, the higher-curvature correction modifies the coefficients of the equation of state, leading to a more intricate non-linear system that must be solved numerically. The resulting critical parameters (v_c, T_c, P_c) shift systematically with the Gauss–Bonnet coupling α: v_c grows, while T_c and P_c decrease. This demonstrates that higher-curvature effects deform but do not destroy the Van der Waals-like structure. Such deviations from universality highlight the sensitivity of black hole criticality to short-distance corrections in the gravitational action, consistent with earlier reports of reentrant phase transitions and multiple critical points in Lovelock and higher-derivative gravities [10, 17].

The central message of our study is that the algebraic structure of the non-linear criticality equations provides a unifying framework for understanding black hole thermodynamics. Whether solved analytically or numerically, these equations reveal the balance of attractive and repulsive interactions that underlies the emergence of phase transitions. Their solutions encode both universal features, such as the appearance of inflection points and swallowtails, and model-dependent deviations that capture the influence of charge, rotation, or higher-curvature corrections.

Beyond their intrinsic gravitational significance, these results carry implications for holography. In the AdS/CFT correspondence, black hole phase transitions map onto thermal transitions in strongly coupled quantum field theories [14, 15]. The non-linear equations studied here therefore provide a direct route to understanding how universality classes of field theory transitions emerge from bulk gravitational dynamics, and how higher-curvature terms might correspond to $1/N$ or α' corrections in the dual description.

Several promising directions remain for future work. Quantum and stringy corrections to the black hole equation of state are expected to generate additional non-linearities, whose solutions could illuminate the microscopic origin of black hole entropy. Numerical relativity techniques may allow the exploration of dynamical aspects of black hole phase transitions, such as quench protocols or out-of-equilibrium critical behavior. Another avenue is the extension to de Sitter black holes, where the interplay of multiple horizons leads to even richer thermodynamic structures. Finally, the application of modern mathematical methods for non-linear systems—including homotopy analysis, bifurcation theory, and machine learning-assisted root finding—offers exciting opportunities to probe unexplored regions of the black hole phase diagram.

In conclusion, non-linear equations stand at the heart of black hole phase transitions. Their solutions unify diverse gravitational systems under a common thermodynamic framework, bridging the gap between the microscopic structure of spacetime and the macroscopic universality of phase transitions. Continued exploration of these equations promises to deepen our understanding of gravitational thermodynamics and its connections to quantum field theory, statistical mechanics, and beyond.

Data Availability

The manuscript has no associated data or the data will not be deposited.

Conflicts of Interest

The authors declare that there is no conflict of interest.

Ethical Considerations

The authors have diligently addressed ethical concerns, such as informed consent, plagiarism, data fabrication, misconduct, falsification, double publication, redundancy, submission, and other related matters.

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References

[1] J. D. Bekenstein, Black holes and entropy, Phys. Rev. D, 7, 2333, (1973).

[2] S. W. Hawking, Particle creation by black holes, Commun. Math. Phys., 43, 199–220, (1975).

[3] J. M. Bardeen, B. Carter, and S. W. Hawking, The four laws of black hole mechanics, Commun. Math. Phys., 31, 161–170, (1973).

[4] R. M. Wald, The thermodynamics of black holes, Living Rev. Relativity, 4, 6, (2001).

[5] D. Kastor, S. Ray, and J. Traschen, Enthalpy and the mechanics of AdS black holes, Class. Quant. Grav., 26, 195011, (2009).

[6] B. P. Dolan, Pressure and volume in the first law of black hole thermodynamics, Class. Quant. Grav., 28, 235017, (2011).

[7] D. Kubizák and R. B. Mann, *P–V* criticality of charged AdS black holes, JHEP, 1207, 033, (2012).

[8] S. Gunasekaran, D. Kubizák, and R. B. Mann, Extended phase space thermodynamics for charged and rotating black holes and Born–Infeld vacuum polarization, JHEP, 2012(11), 1–43, (2012).

[9] R.-G. Cai, L.-M. Cao, L. Li, and R.-Q. Yang, *P–V* criticality in the extended phase space of GaussBonnet black holes in AdS space, JHEP, 2013(9), 1–22, (2013).

[10] R. A. Hennigar, R. B. Mann, and E. Tjoa, Superfluid black holes, Phys. Rev. Lett., 118, 021301, (2017).

[11] N. Altamirano, D. Kubizák, R. B. Mann, and Z. Sherkatghanad, KerrAdS analogue of triple point and solid/liquid/gas phase transition, Class. Quant. Grav., 31, 042001, (2014).

[12] S.-W. Wei and Y.-X. Liu, Insight into the microscopic structure of an AdS black hole from a thermodynamical phase transition, Phys. Rev. Lett., 115, 111302, (2015).

[13] E. Witten, Anti-de Sitter space and holography, Adv. Theor. Math. Phys., 2, 253, (1998).

[14] E. Witten, Anti-de Sitter space, thermal phase transition, and confinement in gauge theories, Adv. Theor. Math. Phys., 2, 505, (1998).

[15] A. Chamblin, R. Emparan, C. V. Johnson, and R. C. Myers, Charged AdS black holes and catastrophic holography, Phys. Rev. D, 60, 064018, (1999).

[16] A. Chamblin, R. Emparan, C. V. Johnson, and R. C. Myers, Holography, thermodynamics and fluctuations of charged AdS black holes, Phys. Rev. D, 60, 104026, (1999).

[17] A. M. Frassino, D. Kubizák, R. B. Mann, and F. Simovic, Multiple reentrant phase transitions and triple points in Lovelock thermodynamics, JHEP, 1409, 080, (2014).

[18] D. Kubizák, R. B. Mann, and M. Teo, Black hole chemistry: thermodynamics with Lambda, Class. Quant. Grav., 34, 063001, (2017).

[19] R. B. Mann, Recent Developments in Holographic Black Hole Chemistry, Journal of Holography Applications in Physics, 4(1), 1–26, (2024).

[20] J.-X. Mo, F. Liang, and G.-Q. Li, Heat engine in the three-dimensional spacetime, JHEP, 03, 010, (2017).

[21] Z. Dayyani, A. Sheykhi, and M. H. Dehghani, Counterterm method in Einstein dilaton gravity and thermodynamics of rotating black branes with a nonlinear source, Phys. Rev. D, 95, 084004, (2017).

[22] X.-X. Zeng and L.-F. Li, Van der Waals phase transition in the framework of holography, Phys. Lett. B, 764, 100–108, (2017).

[23] A. Dehyadegari, A. Sheykhi, and A. Montakhab, Critical behavior and microscopic structure of charged AdS black holes, Phys. Lett. B, 768, 235–240, (2017).

[24] S. J. Liao, Homotopy Analysis Method in Nonlinear Differential Equations, Springer, Heidelberg, 2009.