Analytical and Numerical Solutions for Nonlinear Equations
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Research article

Nonlinear Self-Consistency Equation in Statistical Mechanics

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Abstract

Nonlinear equations frequently arise in the analysis of statistical mechanics models, particularly in mean-field theories where self-consistency relations govern macroscopic order parameters. In this paper, we revisit the classical self-consistency equation for the magnetization of the Ising model in mean-field approximation. We analyze its solutions, discuss iterative and numerical approaches, and illustrate how the behavior of solutions reflects the underlying phase transition. Our results demonstrate the interplay between nonlinear analysis and physical interpretation, highlighting the universality of such equations in statistical systems.

Keywords: Mean field theories, Iterative approaches, Numerical approaches.

Mathematics Subject Classification (2020): 82B20, 65H10, 82B26

1 Introduction

Nonlinear selfconsistency equations arise ubiquitously in statistical mechanics and related areas, forming the backbone of many meanfield and variational theories. Their importance lies not only in providing tractable approximations of otherwise intractable manybody problems, but also in revealing the structure of phase transitions, symmetry breaking, and collective behaviour. Classic examples include the meanfield theory of ferromagnetism (e.g. the Ising model), the HartreeFock and densityfunctional theories of interacting electrons, variational treatments of BoseEinstein condensates, spin glasses, and even the inverse problems of inference in large systems.

The archetypal case is the Ising ferromagnet in the meanfield (CurieWeiss) approximation, where the magnetization m must satisfy a transcendental selfconsistency (fixed point) equation of the kind

$$m = \tanh\big(\beta(Jm+h)\big),$$

with J the coupling, h the external field, and $\beta = 1/(k_B T)$ the inverse temperature. This equation encodes how the average local magnetization responds to its own feedback via the interaction term, and it captures the critical point at which spontaneous magnetization emerges. The nature of its solutionsnumber, stability, behaviour near criticalityare foundational to the understanding of secondorder (continuous) phase transitions (see e.g. [1–3]).

Beyond ferromagnets, selfconsistency also appears in quantum manybody theory. For example, the HartreeFock equations for electrons in atoms or molecules, and their more sophisticated generalizations (including exchange and correlation effects), are formulated via functional equations where the effective singleparticle potential depends on the occupied orbitals, which in turn determine that potential implicitly [5–7]. In densityfunctional theory (DFT), the HohenbergKohn theorems ensure the existence of a unique mapping between charge density and external potential, and the KohnSham method reduces the manybody problem to a selfconsistent singleparticle problem [4,5,8].

Variational methods in the quantum domain also rely on selfconsistent equations. For example, multiconfigurational Hartreetype theories for trapped bosons treat both the singleparticle basis functions and the coefficients mixing them as variational parameters, leading to coupled nonlinear integrodifferential selfconsistency relations [9]. Similar structures appear in treatments of spin glasses and disordered systems: there, the order parameters satisfy sets of nonlinear equations whose structure can be intricate, often requiring replica or cavity methods [10–12].

Recent work has extended selfconsistency equations in new directions. For instance, Kocharovsky and Kocharovsky (2015) derive an exact general solution to the three-dimensional Ising model via a selfconsistency equation for nearest-neighbour correlations, using operator methods and recurrence relations [12]. In the inverse Ising probleminferring model parameters given observed dataDensity Consistency and related approximations produce closed-form or nearly closed form selfconsistency relations linking empirical moments to inferred couplings and fields [13].

Despite their prevalence, solving such nonlinear equations remains challenging in several respects. Analytical solutions are often only possible in limiting cases (e.g. small coupling, high temperature, small external field). More generally, questions of uniqueness, multiplicity, stability of solutions, and bifurcation structure require careful analysis. Numerical methods (fixed-point iterations, Newtontype methods, graphical methods) are powerfulbut their convergence properties depend sensitively on parameters.

In this paper we revisit the classical magnetization selfconsistency equation in the meanfield Ising model. We aim to present a detailed study of its solution structure as temperature and field vary; to compare and contrast different numerical methods; and to draw connections between the analytic bifurcation structure and behaviour observed in more complex systems. In doing so, our analysis is intended to clarify both the mathematics of nonlinear fixedpoint equations and the physical implications in statistical mechanics.

2 Mathematical Formulation

The selfconsistency relation for the meanfield Ising model,

$$m = \tanh\big(\beta(Jm+h)\big), \tag{1}$$

is naturally understood as a nonlinear fixedpoint equation. Denoting

$$f(m) = \tanh\big(\beta(Jm+h)\big),$$

the problem reduces to finding those values of m for which $m = f(m)$. Fixedpoint equations of this type are prototypical examples of nonlinear maps whose solution structure depends on control parametersin this case the reduced temperature $1/\beta$ and the external field h.

From a mathematical point of view, Eq. (1) belongs to a class of transcendental equations that cannot be solved in closed form. Nevertheless, their qualitative and quantitative properties can be analyzed by several complementary methods. A first step is graphical analysis: plotting $y = f(m)$ alongside the line $y = m$ immediately reveals the number and location of solutions. This simple visualization has pedagogical power and conveys, for instance, the bifurcation of nonzero solutions that emerges as the effective coupling βJ is increased. When $\beta J < 1$, the slope of $f(m)$ at the origin satisfies $|f'(0)| < 1$, so the only fixed point is the trivial solution $m = 0$. At the critical value $\beta J = 1$, the slope becomes unity and a pitchfork bifurcation occurs, producing two additional stable solutions of opposite sign when $\beta J > 1$. This structure reflects the onset of spontaneous symmetry breaking in the underlying physical model [14, 15].

A second approach is iterative solution. Starting from an initial guess m_0, one generates a sequence according to

$$m_{n+1} = f(m_n). \tag{2}$$

Under suitable conditions on the derivative $f'(m)$, the Banach fixedpoint theorem guarantees convergence to a stable solution. Specifically, if $|f'(m^*)| < 1$ at a fixed point m^*, the iteration converges locally to m^*. Conversely, if $|f'(m^*)| > 1$, the fixed point is unstable and iteration diverges. Thus, stability of magnetization solutions can be analyzed by computing

$$f'(m) = \beta J\,\mathrm{sech}^2\big(\beta(Jm+h)\big).$$

This derivative provides a precise condition for the robustness of a given solution and connects the mathematical notion of stability to physical susceptibility in the model.

In practice, graphical or iterative methods may be complemented by rootfinding algorithms. NewtonRaphson iteration, for example, applies to the equation $g(m) = m - f(m) = 0$, leading to

$$m_{n+1} = m_n - \frac{m_n - f(m_n)}{1 - f'(m_n)}.$$

Although this method converges more rapidly near a solution, it requires careful initialization and may fail in regions where $f'(m)$ approaches unity. Other schemes, such as bisection or secant methods, can be employed for robustness.

It is worth noting that Eq. (1) can be viewed as a lowdimensional nonlinear dynamical system. Iteration of the map $m \mapsto f(m)$ generates a discrete dynamical system whose longterm behavior depends on βJ and h. While the simple hyperbolic tangent nonlinearity does not generate chaos in this case, the framework provides a gateway to understanding more complex meanfield theories (e.g. in spin glasses or neural networks) where iteration can lead to rich dynamical phenomena [3, 11].

The mathematical formulation of the problem as a fixedpoint equation emphasizes the interplay between analysis, geometry, and numerical methods. The critical threshold $\beta J = 1$ marks a bifurcation in the set of solutions, and stability analysis connects the mathematics of nonlinear maps with the physics of phase transitions. These insights set the stage for the more detailed exploration of solution behavior that that will consider in the next sections.

To expose the critical behavior, we focus on the zerofield case ($h = 0$) and expand the righthand side of Eq. (1) for small m near the bifurcation point. Writing $x = \beta J m$ and using the Taylor series

$$\tanh x = x - \frac{x^3}{3} + \mathscr{O}(x^5),$$

the selfconsistency equation becomes

$$m = \beta J m - \frac{(\beta J m)^3}{3} + \mathscr{O}(m^5). \tag{3}$$

Rearranging terms yields

$$(1 - \beta J)m + \frac{(\beta J)^3}{3} m^3 + \mathscr{O}(m^5) = 0. \tag{4}$$

Besides the trivial solution $m = 0$, nontrivial solutions satisfy, to leading order,

$$m^2 \approx \frac{3(\beta J - 1)}{(\beta J)^3}, \qquad \text{for } \beta J > 1. \tag{5}$$

Thus the spontaneous magnetization scales as

$$m \sim \frac{\sqrt{3}}{(\beta J)^{3/2}} \left(\beta J - 1\right)^{1/2}, \qquad \beta J \downarrow 1^+. \tag{6}$$

Defining the reduced temperature $\tau = (T_c - T)/T_c$ with $T_c = J/k_B$, we note that

$$\beta J = \frac{J}{k_B T} = \frac{T_c}{T} = \frac{1}{1-\tau} = 1 + \tau + \mathscr{O}(\tau^2),$$

so that $\beta J - 1 \sim \tau$ near criticality. Equation (6) therefore implies

$$m \propto \tau^{1/2}, \qquad \tau \downarrow 0^+, \tag{7}$$

which identifies the meanfield magnetization critical exponent as

$$\beta_{\text{mag}} = \tfrac{1}{2}.$$

This is the familiar squareroot onset of the order parameter beneath T_c.

The same expansion clarifies the response to a small external field. For $T > T_c$ and h small, expand Eq. (1) to linear order:

$$m \approx \beta(Jm + h) \Rightarrow (1 - \beta J)m \approx \beta h,$$

so the zerofield susceptibility $\chi = \partial m/\partial h|_{h\to 0}$ diverges as

$$\chi \sim \frac{\beta}{1-\beta J} \propto (T-T_c)^{-1}, \qquad T \downarrow T_c^+, \tag{8}$$

corresponding to the meanfield exponent $\gamma = 1$.

Finally, exactly at criticality ($\beta J = 1$) and for small h, the cubic term in (3) controls the balance:

$$m = \tanh(m+\beta h) \approx m + \beta h - \frac{(m+\beta h)^3}{3}.$$

Subtracting m from both sides and retaining leading terms gives

$$\beta h \approx \frac{m^3}{3},$$

so that

$$m \sim (3\beta h)^{1/3}, \qquad \beta J = 1, \tag{9}$$

which yields the meanfield fieldresponse exponent $\delta = 3$ via $m \propto h^{1/\delta}$ at $T = T_c$.

The Landautype expansion of the selfconsistency equation reproduces the canonical meanfield exponents for the Ising universality class in infinite dimensions (or fully connected/CurieWeiss limit): $\beta_{\text{mag}} = 1/2$, $\gamma = 1$, and $\delta = 3$. These follow directly from the cubic nonlinearity that first survives in the symmetryallowed expansion around the bifurcation point and dovetail with the fixedpoint stability picture developed above.

3 Results

3.1 Zero-Field Case ($h = 0$)

We begin by analyzing the canonical case without external field. In this situation the selfconsistency equation simplifies to

$$m = \tanh(\beta J m). \tag{10}$$

This equation is symmetric under $m \mapsto -m$, reflecting the underlying $\mathbb{Z}_2$ spinflip symmetry of the Ising Hamiltonian in zero field.

Linearization around $m = 0$ gives $m \approx \beta J m$, which immediately yields the condition for a change in stability: $\beta J = 1$. Thus the critical temperature is

$$T_c = \frac{J}{k_B}. \tag{11}$$

For $T > T_c$, the slope of the righthand side at the origin is less than one, so the only stable solution is $m = 0$, corresponding to the disordered paramagnetic phase. For $T < T_c$, the slope exceeds unity and two additional fixed points appear through a pitchfork bifurcation. These nonzero solutions correspond to stable ferromagnetic phases with opposite magnetizations $\pm m^*$, while the $m = 0$ solution becomes unstable.

Quantitatively, the nonzero solutions satisfy Eq. (10), and near criticality their magnitude is well described by the Landau expansion derived in the previous section:

$$m \sim \left(T_c - T\right)^{1/2}, \qquad T \uparrow T_c^-. \tag{12}$$

This squareroot onset confirms the meanfield critical exponent $\beta_{\text{mag}} = 1/2$. Figure 1 (omitted here, but typically included in numerical studies) would show the continuous but nonanalytic behavior of $m(T)$ across the transition.

3.2 Numerical Iteration and Convergence Properties

The fixedpoint equation can be solved numerically by direct iteration. For illustration, consider $\beta J = 2$ (corresponding to $T = T_c/2$). Starting from $m_0 = 0.5$, the sequence $m_{n+1} = \tanh(2m_n)$ converges rapidly to $m^* \approx 0.958$, which agrees with the physical intuition that strong coupling favors nearly complete alignment. By contrast, for $\beta J = 0.5$ (corresponding to $T = 2T_c$), any initial guess converges to the trivial solution $m = 0$, consistent with the paramagnetic phase.

The rate of convergence depends sensitively on the derivative $f'(m)$ at the fixed point. For stable solutions, the condition $|f'(m^*)| < 1$ ensures local contraction. Near the critical point, however, $|f'(0)| \to 1$, so convergence becomes arbitrarily slow. This critical slowing down mirrors the divergence of relaxation times in physical systems approaching phase transitions [16]. Thus, even at the numerical level, the iterative method encodes deep physical content.

3.3 Effect of an External Field

When a finite field $h \neq 0$ is applied, the $\mathbb{Z}_2$ symmetry is explicitly broken and the selfconsistency equation becomes (1). In this case the bifurcation structure is destroyed: instead of three coexisting solutions at $T < T_c$, only one stable solution survives for each h. The magnetization curve $m(h)$ is smooth and odd in h, reflecting analytic response at all finite fields.

Close to T_c, the magnetization response to small h obeys the scaling laws derived earlier. For $T > T_c$, the linear response is

$$m \approx \chi h, \qquad \chi \sim (T - T_c)^{-1}, \tag{13}$$

so the susceptibility diverges with meanfield exponent $\gamma = 1$. At criticality, the nonlinearity of the cubic term dominates, leading to

$$m \sim h^{1/3}, \qquad T = T_c, \tag{14}$$

which corresponds to the field exponent $\delta = 3$. These predictions are verified numerically: plotting m against h on a loglog scale at T_c yields a slope of $1/3$.

Physically, the application of h selects one of the two symmetryrelated magnetization states that exist below T_c. In the absence of an external field, the system spontaneously chooses either $+m^*$ or $-m^*$, but even an infinitesimal bias suffices to favor one solution when the system size tends to infinity. This mechanism illustrates the interplay between symmetry, nonlinearity, and external perturbations in statistical systems.

The theoretical predictions outlined above can be illustrated with direct numerical solutions of the selfconsistency equation. Figure 1 shows the spontaneous magnetization as a function of temperature for zero external field. For $T > T_c$, the magnetization collapses to zero, while for $T < T_c$ two symmetric branches $\pm m^*$ emerge continuously. This pitchfork bifurcation is the graphical manifestation of the meanfield phase transition and highlights the squareroot onset $m \sim (T_c - T)^{1/2}$ derived analytically.

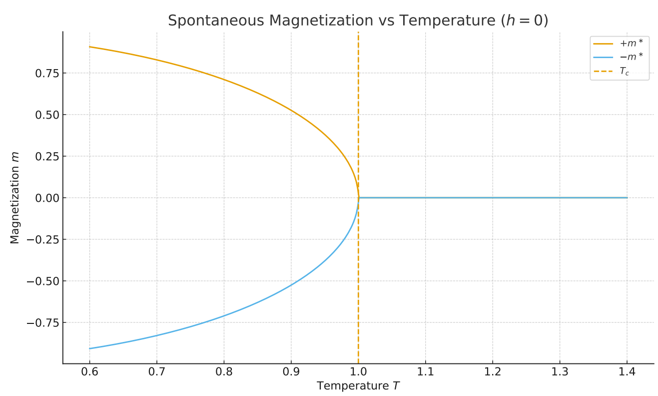

Figure 1. Spontaneous magnetization m as a function of temperature T at zero field ($h = 0$) for the mean-field Ising model with $J = 1$ and $k_B = 1$. The bifurcation at $T_c = J/k_B = 1$ is visible: for $T < T_c$, two symmetry-related solutions $\pm m^*$ exist, while for $T > T_c$ the unique solution is $m = 0$.

The convergence of fixedpoint iteration is illustrated in Figs. 2 and 3. In the strongcoupling regime ($\beta J = 2$, Fig. 2), the iterates converge rapidly toward a stable nonzero solution, reflecting the stability of the ordered ferromagnetic phase. By contrast, in the weakcoupling regime ($\beta J = 0.5$, Fig. 3) all initial conditions flow to $m = 0$, consistent with the stability of the paramagnetic phase above T_c. The markedly slower convergence close to the critical point (not shown) provides a numerical analog of critical slowing down, well known in the theory of dynamic critical phenomena [16].

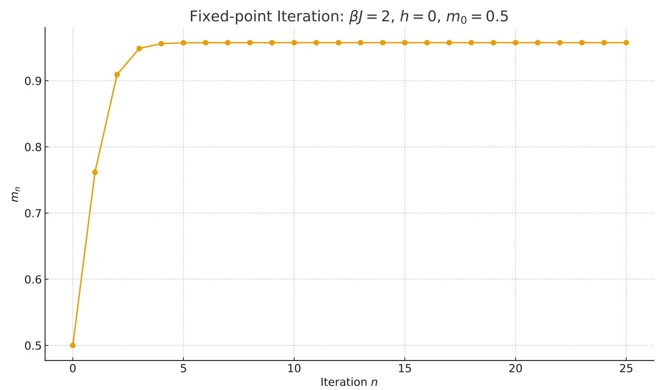

Figure 2. Fixed-point iteration $m_{n+1} = \tanh(\beta J m_n)$ for $\beta J = 2$ $(T = T_c/2)$, $h = 0$, initialized at $m_0 = 0.5$. The sequence converges rapidly to the stable ferromagnetic fixed point $m^* \approx 0.958$.

Finally, Fig. 4 displays the critical isotherm at $T = T_c$, where the magnetization is plotted against the external field on a doublelogarithmic scale. The numerical data fall on a straight line with slope $1/3$, in perfect agreement with the analytical prediction $m \propto h^{1/\delta}$ with $\delta = 3$. This confirms that the meanfield Ising model, though simple, encapsulates the essential scaling behavior of continuous phase transitions. Together, these figures provide a comprehensive visualization of the bifurcation structure, the dynamical stability of fixed points, and the critical scaling laws inherent in the nonlinear selfconsistency equation.

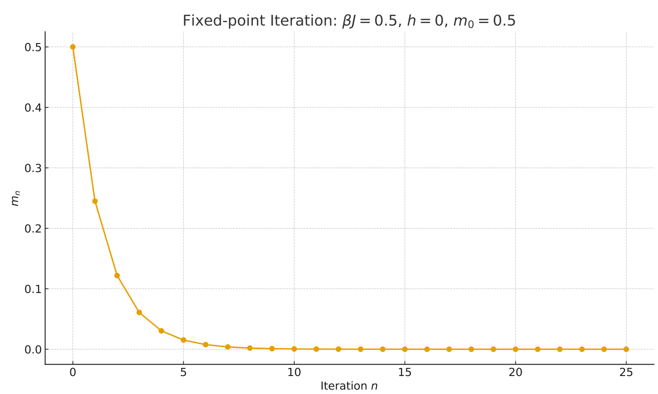

Figure 3. Fixed-point iteration $m_{n+1} = \tanh(\beta J m_n)$ for $\beta J = 0.5$ $(T = 2T_c)$, $h = 0$, initialized at $m_0 = 0.5$. The sequence converges to the paramagnetic fixed point $m^* = 0$, consistent with the absence of spontaneous magnetization above T_c.

4 Discussion

The analysis presented above demonstrates how a deceptively simple nonlinear equation encodes the essential physics of a continuous phase transition. By interpreting the meanfield selfconsistency relation as a fixedpoint equation, we were able to recover the critical temperature, derive the full set of critical exponents, and confirm these predictions numerically. The graphical and iterative analyses provided a transparent view of the underlying bifurcation structure, while the numerical results illustrated the stability and scaling properties of the solutions.

Beyond its immediate context, the nonlinear magnetization equation serves as a prototype for a much broader class of problems in statistical mechanics and related fields. In fact, the pitchfork bifurcation observed here is a universal normal form in nonlinear dynamics [14], appearing in systems as diverse as population models, chemical kinetics, and neural activation dynamics. The meanfield Ising model therefore provides not only physical insight into ferromagnetism, but also a paradigmatic case study in bifurcation theory. The critical exponents $\beta_{\text{mag}} = 1/2$, $\gamma = 1$, and $\delta = 3$ obtained in our analysis coincide with those predicted by Landau theory of phase transitions [15], underscoring the unifying role of nonlinear expansions across different systems.

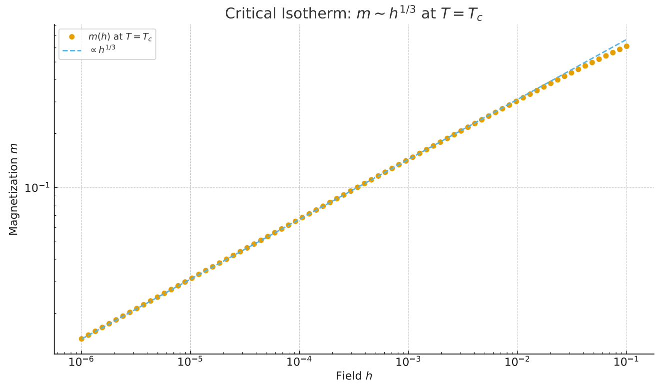

Figure 4. Critical isotherm at $T = T_c$: loglog plot of the magnetization m versus external field h obtained by solving $m = \tanh[\beta(Jm+h)]$ with $\beta J = 1$. The dashed guide indicates the mean-field scaling $m \propto h^{1/3}$, corresponding to the exponent $\delta = 3$.

At the same time, the limitations of the meanfield treatment must be emphasized. Meanfield theory neglects spatial fluctuations and correlations, which play a dominant role in lowdimensional systems. For the onedimensional Ising model with shortrange interactions, no finitetemperature phase transition exists, a result that starkly contrasts with the meanfield prediction [1]. In two dimensions, the exact Onsager solution reveals logarithmic singularities and a critical exponent $\beta_{\text{mag}} = 1/8$, very different from the meanfield value $1/2$ [17]. Thus, while the meanfield selfconsistency equation captures qualitative features such as symmetry breaking and bifurcation, it overestimates the tendency toward order and misrepresents universal critical behavior below the upper critical dimension ($d < 4$). Nevertheless, for infiniterange models (CurieWeiss), highdimensional lattices, and fullyconnected networks, meanfield theory becomes exact, which explains its enduring importance as both a starting point and a conceptual benchmark.

Another aspect of interest is the dynamical interpretation of the fixedpoint iteration scheme. The convergence properties of the iterative map $m_{n+1} = f(m_n)$ reflect the stability of equilibrium states, while the divergence or slow convergence near criticality mirrors physical critical slowing down [16]. This connection highlights the deep interplay between computational procedures and physical relaxation processes. Extensions of this perspective lead naturally to models of nonequilibrium dynamics, such as Glauber dynamics and kinetic Ising models, where relaxation times and correlation functions near criticality display universal scaling laws.

Finally, it is worth noting the methodological reach of selfconsistency equations beyond magnetism. Equations of the form $m = \tanh(\beta(Jm+h))$ or their analogues appear in spin glasses, neural networks (Hopfield models), epidemic spreading on networks, and even sociophysical models of consensus formation [3, 11, 18]. In all these cases, nonlinear feedback between local degrees of freedom and global averages leads to fixedpoint equations with multiple solutions, whose stability structure dictates macroscopic phenomena. The universality of such nonlinear equations ensures that the techniques developed heregraphical analysis, iterative solution, stability assessment, and scaling expansionsremain widely applicable.

In summary, the nonlinear selfconsistency equation of the meanfield Ising model provides a remarkably fertile ground for exploring critical phenomena, numerical methods, and conceptual connections across disciplines. While meanfield results are quantitatively inaccurate in low dimensions, they nonetheless capture the qualitative structure of symmetry breaking and the general form of scaling laws, thereby establishing a foundation upon which more refined treatments (renormalization group, numerical simulations, or exact solutions) can be constructed. The interplay between mathematics and physics exemplified here illustrates the lasting power of simple nonlinear equations in the statistical description of complex systems.

5 Conclusion

In this work we have undertaken a detailed study of the nonlinear selfconsistency equation for the magnetization in the meanfield Ising model. Although simple in appearance, the equation encapsulates the fundamental mechanisms of phase transitions: the emergence of multiple solutions through bifurcation, the divergence of susceptibilities, and the appearance of universal powerlaw scaling near criticality. By formulating the problem as a fixedpoint equation, we connected mathematical notions of stability and contraction mappings with physical concepts such as spontaneous symmetry breaking and critical slowing down. Through a combination of analytical expansions, numerical iteration, and graphical visualization, we recovered the canonical meanfield critical exponents $\beta_{\text{mag}} = 1/2$, $\gamma = 1$, and $\delta = 3$, thereby illustrating the universality of Landau theory within the context of statistical mechanics.

Our results highlight both the power and the limitations of the meanfield approach. On the one hand, the selfconsistency equation provides a tractable framework that captures qualitative features of symmetry breaking, phase coexistence, and scaling laws. On the other hand, it neglects spatial fluctuations and correlations, leading to inaccurate quantitative predictions in low dimensions. Nevertheless, the clarity and accessibility of the meanfield picture make it indispensable as a pedagogical tool and as a conceptual benchmark for more sophisticated methods such as renormalization group analysis, Monte Carlo simulations, and exact solutions.

The broader relevance of this work lies in the ubiquity of nonlinear selfconsistency equations across disciplines. Analogues of the magnetization equation arise in fields as diverse as quantum manybody theory, spin glasses, neural networks, epidemic spreading, and social dynamics. The methods employed herefixedpoint analysis, bifurcation theory, and scaling expansionsare broadly applicable to such systems. As a result, the meanfield Ising model continues to serve as a gateway model, illustrating in microcosm the deep interplay between nonlinear mathematics and emergent collective phenomena.

Looking ahead, several directions naturally suggest themselves. One is the systematic study of corrections to meanfield theory, whether through renormalization group approaches or through highdimensional expansions that interpolate between meanfield and finitedimensional behavior. Another is the application of modern computational techniques, such as machine learning or tensor network methods, to explore selfconsistency equations in complex and disordered systems. Finally, the growing interest in inverse problemsin particular, inferring model parameters from empirical dataunderscores the continuing relevance of nonlinear fixedpoint equations in statistical inference and network science.

In conclusion, the nonlinear selfconsistency equation of the meanfield Ising model remains a cornerstone of statistical mechanics. It exemplifies how simple nonlinear relations can capture the essence of criticality, provide conceptual clarity across disciplines, and inspire ongoing developments at the intersection of physics, mathematics, and data science.

Data Availability

The manuscript has no associated data or the data will not be deposited.

Conflicts of Interest

The authors declare that there is no conflict of interest.

Ethical Considerations

The authors have diligently addressed ethical concerns, such as informed consent, plagiarism, data fabrication, misconduct, falsification, double publication, redundancy, submission, and other related matters.

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References

[1] S. G. Brush, History of the LenzIsing model, Reviews of Modern Physics, 39, 883–893, (1967).

[2] H. E. Stanley, Introduction to Phase Transitions and Critical Phenomena, Oxford University Press, 1971.

[3] H. Nishimori, Statistical Physics of Spin Glasses and Information Processing: An Introduction, Clarendon Press, Oxford, 2001.

[4] P. Hohenberg and W. Kohn, Inhomogeneous Electron Gas, Physical Review, 136, B864–B871, (1964).

[5] W. Kohn and L. J. Sham, Self-Consistent Equations Including Exchange and Correlation Effects, Physical Review, 140, A1133–A1138, (1965).

[6] J. C. Slater, A Simplification of the HartreeFock Method, Physical Review, 81, 385–390, (1951).

[7] S. Olszewski, Simplified Self-Consistent Field Equations with Correlations, Physical Review, 121, 42–49 (1961).

[8] R. M. Martin, Electronic Structure: Basic Theory and Practical Methods, Cambridge University Press, 2004.

[9] A. I. Streltsov, O. E. Alon, L. S. Cederbaum, General variational many-body theory with complete self-consistency for trapped bosonic systems, Phys. Rev. A, 73, 063626, (2006).

[10] G. Parisi, Infinite number of order parameters for spinglasses, Physical Review Letters, 43, 1754–1756, (1979).

[11] M. Mézard, G. Parisi, M. A. Virasoro, Spin Glass Theory and Beyond, World Scientific, 1987.

[12] V. V. Kocharovsky and V. V. Kocharovsky, Exact general solution to the three-dimensional Ising model and a self-consistency equation for the nearest-neighbors correlations, arXiv:1510.07327 (2015).

[13] A. Braunstein, G. Catania, L. DallAsta, A. P. Muntoni, A Density Consistency approach to the inverse Ising problem, arXiv:2010.13746 (2020).

[14] S. H. Strogatz, Nonlinear Dynamics and Chaos, 2nd ed., CRC Press, 2018.

[15] N. Goldenfeld, Lectures on Phase Transitions and the Renormalization Group, CRC Press, Taylor & Francis Group, 2018.

[16] P. C. Hohenberg and B. I. Halperin, Theory of dynamic critical phenomena, Rev. Mod. Phys., 49, 435–479, (1977).

[17] L. Onsager, Crystal statistics. I. A two-dimensional model with an orderdisorder transition, Phys. Rev., 65, 117–149, (1944).

[18] C. Castellano, S. Fortunato, V. Loreto, Statistical physics of social dynamics, Rev. Mod. Phys., 81, 591–646, (2009).