Analytical and Numerical Solutions for Nonlinear Equations
©Available online at https://ansne.du.ac.ir/
Online ISSN: 3060–785X
2025, Volume 10, Issue 1, pp. 17–34

Research article

Existence and Numerical Simulation of Solutions for a Caputo-Fabrizio Fractional Differential Equation

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Received: 01/11/2025 **Revised:** 24/11/2025 **Accepted:** 30/11/2025 **Published:** 17/12/2025 10.22128/ansne.2025.3118.1170

Abstract

The present research investigates the existence and numerical simulation of solutions for a class of Caputo-Fabrizio fractional differential equations. At first we obtain a prior estimate for solutions of a functional integral equation that is related to the main problem. Then using a fixed point theorem, the existence of at least one smooth solution is proved. Furthermore, a new numerical method based on B-spline is developed to approximate the solution. It is proved that a locally superconvergent approximation is achieved via even-degree splines on the mid points of the uniform partition. The convergence of the proposed method is analyzed using an operator-based approach, and the corresponding theoretical convergence orders are rigorously derived. Finally, several illustrative examples are presented to demonstrate the efficiency and applicability of the method. The results of the numerical experiments confirm the theoretical predictions concerning the convergence orders.

Keywords: Caputo-Fabrizio fractional differential equation, Existence of solution, Numerical simulation, B-spline interpolation, Convergence analysis.

Mathematics Subject Classification (2020): 34A08, 41A15, 65L20, 65L70

1 Introduction

Fractional differential equations are used in modeling many physical phenomena. For example diffusion equations of fractional order have been used to model the anomalous diffusion of particles in complex media such as fractal networks, porous materials and biological tissues [1, 2]. In [3, 4], application of fractional calculus in complex physical systems and dynamics of viscoelastic can be found. In [4–6], the capability of fractional differential equations in modeling complex physical systems, as well as in physical systems' description and control, are analysed. Recently Caputo and Fabrizio introduced a new fractional derivative with a smooth kernel [7]. Due to applications of this type of fractional derivative in fractional calculus, control theory and viscoelasticity [8–14], it has been studied extensively in recent years. For example in [15] the existence, uniqueness and the continuity property with respect to the derivative order of the mild solution for the Kirchhoff parabolic equation involving the Caputo-Fabrizio fractional derivative are studied. In [16], Eiman and Baleanu studied the existence and uniqueness of solution for the following problem

$$\begin{cases} {}_aD_t^{\alpha}u(t) = f\left(t, u(t), {}_aD_t^{\alpha}u(t)\right), & t \in (a,b], \\ u(a) = u_0, \end{cases}$$

where $\alpha \in (0,1]$ and $f : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function. At first, they proved the existence and uniqueness of a continuous solution using the Banach contraction principle when f is Lipschitz continuous. Then using the Krasnoselskii's fixed point theorem, they established the existence of at least one continuous solution under a sublinear condition combined with a contraction assumption on f. To see some new research papers about existence of the solutions for differential equations with Caputo-Fabrizio fractional derivative, we refer the reader to [17–22].

In this paper we study the existence and numerical simulation of solutions for the Caputo-Fabrizio fractional differential equation

$$\begin{cases} {}_aD_t^{\alpha}u(t) = f\left(t, u(t), {}_aD_t^{\beta}u(t)\right), & t \in (a,b], \\ u(a) = u_0, \end{cases} \tag{1}$$

where $[a,b] \subset \mathbb{R}$ is a bounded interval, $0 < \beta < \alpha < 1$, D^{α} is the Caputo-Fabrizio fractional derivative and $f : [a,b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a function of class C^1 which $f(a,u_0,0) = 0$. The main proposes of this paper are to prove the existence of C^m-solutions, $m \in \mathbb{N}$ and the numerical simulation of solutions to problem (1). First we obtain a prior estimate for solutions of a functional equation related to (1). Next using a nonlinear alternative of Leray-Schauder type, the existence of solutions for (1) is proved.

Due to the non-locality of fractional operators, solving such problems numerically or analytically is a challenge for researchers in the filed. However major efforts have been done by researchers to study the numerical solutions of problems involving fractional operators. In [23], a three-step fractional Adams-Bashforth scheme is formulated for solutions of linear and nonlinear systems of fractional differential equations with Caputo-Fabrizio derivative. The method is shown to be conditionally stable. A fundamental solution using the Laplace transform combining with Fourier transform method has been used to approximate the solution of an advection-diffusion equation with time-fractional Caputo-Fabrizio derivative [24]. In the chapter book [25], a fundamental solution has been obtained for fractional Cauchy and Dirichlet problems equipped with Caputo-Fabrizio derivative. In [26], the iterative Laplace transform method has been used to simulate the solution of a three-dimensional fractional SIR epidemic model with Caputo-Fabrizio derivative. In [27], a fractional model based on the Caputo-Fabrizio derivative is developed for simulating the epidemic effects of COVID-19 pandemic in India. Then the Genocchi operational matrix method has been used to investigate various constant parameters' effect in the concerned fractional model. The modified coupled time-fractional Korteweg-de Vries equation with Caputo and Caputo-Fabrizio time-fractional derivatives has been considered in [28]. The proposed model describes the nonlinear evolution of the waves suffered by the weak dispersion effects. The modified double Laplace transform decomposition method has been analyzed successfully to approximate the solution. In [29], a numerical technique based on spline has been developed for the numerical solution of the time fractional Cattaneo equation involving the Caputo-Fabrizio derivative. Its a comparative study between cubic B-spline, trigonometric cubic B-spline and extended cubic B-spline methods to approximate the solution.

B-splines are powerful tools in numerical analysis for approximating and interpolating data. They offer a flexible and efficient way to represent smooth curves and surfaces by breaking them down into smaller segments. One of the advantages of using B-splines in numerical analysis is their ability to provide a smooth and continuous approximation to data. They can accurately represent complex curves and surfaces with a relatively small number of control points. B-splines also offer local control, that is modifying one control point only affects a small region of the curve or surface. The local support of the B-spline basis functions allows for efficient matrix computations, reducing the computational cost compared to other existing methods.

Numerical methods based on B-splines have been successfully applied to a wide range of problems including ordinary, partial and integro-differential equations. B-splines have also been used in conjunction with other numerical techniques, such as finite element, Galerkin and collocation methods. Some researchers have presented numerical schemes based on B-spline for the solution of fractional differential equations. The spline collocation method has been used by Pedas and Tamme [30,31], for linear multi-term fractional differential equations. Also, some regularity conditions for the solution have been derived. Xinxiu [32], Generalized the wavelet collocation method based on cubic B-spline for the solution of linear fractional differential equations in the Caputo sense. Akram et al., developed a collocation method based on quintic polynomial spline for linear fractional boundary value problems [33]. Also a quadratic B-spline Galerkin method has been developed and analyzed for the solution of time-fractional telegraph equations in [34]. There are many other researchers tried to solve fractional differential equations using B-splines [35–39].

In this paper we will develop a method based on m-th degree ($m \leq 6$), B-spline interpolation to approximate the solution of problem (1). After transforming the differential equation into its equivalent integral form, we will use spline approximation on a uniform partition to discretize the problem. A q point Gauss-Legendre quadrature will be used to approximate the integrals arises in the process. We will construct the method on the grid points of the partition for odd m and on the mid points for even m. For each case some extra relations will be needed. For odd m, we will construct the extra relations on the near boundary mid points while for even m, near boundary grid points can be used. It will be proved via an operator approach, that the global order of convergent is $m+1$. Also, it will be shown that the method is locally superconvergent at collocation points for even values of m.

The rest of the paper is organized as follows: In Section 2, at first the main problem will be transformed into an equivalent integral equation, then the existence of a solution will be proved. In Section 3, a method based on B-spline is developed to approximate the solution. Section 4 is devoted to the convergence analysis and obtaining the error bounds for the method. In Section 5, the proposed numerical technique will be used to solve some model problems to show the applicability of the method. Finally, Section 6 contains the conclusion.

2 Existence of Solutions

In this section, we prove the existence of at least one C^m-solution, $m \in \mathbb{N}$ for problem (1). First we need some preliminaries and auxiliary results.

2.1 Preliminaries and Auxiliary Results

Let $C[a,b]$ be the space of continuous real-valued functions on $[a,b]$ with the norm

$$\|u\|_C := \sup_{t\in[a,b]} |u(t)|,$$

and consider the space $C^m[a,b]$ for $m = 1,2,...$ as the space of m times continuously differentiable real value functions on $[a,b]$ with the norm

$$\|u\|_{C^m} := \sum_{k=0}^{m} \|u^{(k)}\|_C.$$

Let $u \in C^1[a,b]$ and $0 < \alpha < 1$. The Caputo-Fabrizio fractional derivative of u is defined by

$${}_aD_t^{\alpha} u(t) = \frac{\Upsilon(\alpha)}{1-\alpha} \int_a^t u'(\tau) e^{\frac{-\alpha}{1-\alpha}(t-\tau)} d\tau,$$

where $\Upsilon(\alpha)$ is a normalization function such that $\Upsilon(0) = (1) = 1$. Henceforth, we will be taking into account $\Upsilon(\alpha) = 1$.

Theorem 1. *Let $0 < \alpha < 1$, $m \in \mathbb{N}$ and $u \in C^m[a,b]$. Then ${}_aD_t^{\alpha} u \in C^m[a,b]$.*

Proof. Using integration by part we have

$$\begin{aligned} {}_aD_t^{\alpha} u(t) &= \frac{1}{1-\alpha} \int_a^t u'(\tau) e^{\frac{-\alpha}{1-\alpha}(t-\tau)} d\tau \\ &= \frac{1}{1-\alpha} \left(u(t) - u(a) e^{\frac{-\alpha}{1-\alpha}(t-a)} \right) - \frac{\alpha}{(1-\alpha)^2} \int_a^t u(\tau) e^{\frac{-\alpha}{1-\alpha}(t-\tau)} d\tau. \end{aligned} \tag{2}$$

Since $u \in C^m[a,b]$, the function $t \to \int_a^t u(\tau) e^{\frac{-\alpha}{1-\alpha}(t-\tau)} d\tau$ is in $C^m[a,b]$. Therefore, the conclusion follows from (1). □

In the next theorem we recall a result from Bellman [40].

Theorem 2. *Let u and f be continuous and nonnegative functions defined on $J = [a,b]$, and let c be a nonnegative constant. Then the inequality*

$$u(t) \leq c + \int_a^t f(\tau) u(\tau) d\tau, \quad t \in J,$$

implies that

$$u(t) \leq c \exp\left(\int_a^t f(\tau) d\tau \right), \quad t \in J.$$

To study the existence of solution for problem (1) we need the following fixed point theorem [41].

Theorem 3. *Let X be a normed linear space, K a convex subset of X, O an open subset of K and $\theta \in O$ (θ is the zero element of X). Suppose that $N : \overline{O} \to K$ is a continuous and compact operator where $\overline{O}$ is the closure of O. Then either*
(i) T has a fixed point in $\overline{O}$, or
(ii) there exists $u \in \partial O$ such that $u = \lambda Tu$ for some $\lambda \in (0,1)$ where ∂O is the boundary of O.

2.2 Existence of Solutions

We define the operator T_γ on $C^1([a,b]$ as follows

$$T_\gamma u(t) := \frac{1}{1-\gamma} u(a) e^{\frac{-\gamma}{1-\gamma}(t-a)} + \frac{\gamma}{(1-\gamma)^2} \int_a^t u(\tau) e^{\frac{-\gamma}{1-\gamma}(t-\tau)} d\tau, \tag{3}$$

where $0 < \gamma < 1$. Therefore,

$$\frac{d}{dt} T_\gamma u(t) = \frac{-\gamma}{(1-\gamma)^2} u(a) e^{\frac{-\gamma}{1-\gamma}(t-a)} - \frac{\gamma^2}{(1-\gamma)^3} \int_a^t u(\tau) e^{\frac{-\gamma}{1-\gamma}(t-\tau)} d\tau + \frac{\gamma}{(1-\gamma)^2} u(t). \tag{4}$$

Also, using (2), we have

$${}_aD_t^\gamma u(t) = \frac{1}{1-\gamma} u(t) - T_\gamma u(t).$$

Hence problem (1) is equivalent to the following integral equation:

$$u(t) = (1-\alpha) \left(T_\alpha u(t) + f\left(t, u(t), \frac{1}{1-\beta} u(t) - T_\beta u(t)\right)\right) =: Ku(t). \tag{5}$$

Therefore, we reduce problem (1) to the above fixed point problem.

In the sequel, we show that the operator $K : C^1[a,b] \to C^1[a,b]$ has a fixed point. To do this we consider the following assumptions:

(h_1) $f \in C^1([a,b] \times \mathbb{R} \times \mathbb{R})$, $f(a, u_0, 0) = 0$ and there exist constants $c_1, c_2 > 0$ such that

$$|f(t,x,y)| \leq c_1|x| + c_2|y|,$$

and

$$p := (1-\alpha)\left[c_1 + \frac{c_2}{1-\beta}\right] < 1. \tag{6}$$

(h_2)

$$M := \max_{[a,b]\times[-r_0,r_0]\times[-r_1,r_1]} \left| \frac{\partial f}{\partial x}(t,x,y) + \frac{1}{1-\beta} \frac{\partial f}{\partial y}(t,x,y) \right| < 1,$$

where

$$r_0 := \frac{(1+c_2)|u_0|}{1-p} \exp\left(\frac{1+c_2}{1-p} e^{\frac{\alpha(b-a)}{1-\alpha}}\right), \tag{7}$$

$$r_1 := \frac{r_0 + |u_0|}{1-\beta} + \frac{\beta(b-a) r_0}{(1-\beta)^2}. \tag{8}$$

In the next lemma we find a prior bound for the solutions of (5).

Theorem 4. *Assume (h_1) and (h_2) are satisfied. Let $0 < \lambda \leq 1$ and $u \in C^1[a,b]$ be a solution of the equation*

$$u(t) = \lambda K u(t), \quad t \in [a,b]. \tag{9}$$

Then

$$|u(t)| < \frac{(1+c_2)|u_0|}{1-p} \exp\left(\frac{1+c_2}{1-p} e^{\frac{\alpha(t-a)}{1-\alpha}}\right), \tag{10}$$

and

$$\|u\|_{C^1} \leq r_0 + \frac{(1-\alpha)}{1-M}\left(\frac{r_0\alpha}{(1-\alpha)^2}\left(\frac{\alpha(b-a)}{1-\alpha}+2\right)+M_1\right), \tag{11}$$

where

$$M_1 := \max_{[a,b]\times[-r_0,r_0]\times[-r_1,r_1]}\left(\left|\frac{\partial f}{\partial t}(t,x,y)\right| + r_2\left|\frac{\partial f}{\partial y}(t,x,y)\right|\right), \tag{12}$$

$$r_2 := \frac{r_0\beta}{(1-\beta)^2}\left(\frac{\beta(b-a)}{1-\beta}+2\right). \tag{13}$$

Proof. According the equation $u(t) = \lambda Ku(t)$ and the assumption (h_1) we have

$$\begin{aligned}|u(t)| =&\lambda(1-\alpha)\left(T_\alpha u(t) + f\left(t, u(t), \frac{1}{1-\beta}u(t) - T_\beta u(t)]\right)\right)\\ \leq&\lambda(1-\alpha)\left[|T_\alpha u(t)| + (c_1 + \frac{c_2}{1-\beta})|u(t)| + c_2|T_\beta u(t)|\right].\end{aligned}$$

Using the above inequality, the assumption $\alpha > \beta$ and (6), we get

$$\begin{aligned}|u(t)| \leq&\frac{\lambda(1-\alpha)}{1-p}\left[(\frac{1}{1-\alpha}+\frac{c_2}{1-\beta})|u_0| + \frac{\alpha}{(1-\alpha)^2}\int_a^t |u(\tau)|e^{(\tau-a)\frac{\alpha}{1-\alpha}}d\tau + \frac{c_2\beta}{(1-\beta)^2}\int_a^t |u(\tau)|e^{(\tau-a)\frac{\beta}{1-\beta}}d\tau\right]\\ \leq&\frac{\lambda(1-\alpha)}{1-p}\left[\frac{1+c_2}{1-\alpha}|u(a)| + \frac{\alpha(1+c_2)}{(1-\alpha)^2}\int_a^t |u(\tau)|e^{(\tau-a)\frac{\alpha}{1-\alpha}}d\tau\right].\end{aligned} \tag{14}$$

Using Theorem 1 and inequality (14) we deduce

$$|u(t)| \leq \frac{\lambda(1+c_2)|u(a)|}{1-p}\exp\left(\frac{\lambda(1+c_2)}{1-p}e^{\frac{\alpha(t-a)}{1-\alpha}}\right). \tag{15}$$

Also (3) implies that for each $0 < \gamma < 1$,

$$|T_\gamma u(t)| \leq \frac{|u(a)|}{1-\gamma} + \frac{\gamma(b-a)\|u\|_C}{(1-\gamma)^2}. \tag{16}$$

Inequality (15) proves (10). Again using (15) and (16) we get

$$\|u\|_C < r_0, \quad \left|\frac{1}{1-\beta}u(t) - T_\beta u(t)\right| \leq r_1, \tag{17}$$

where r_0 and r_1 are defined by (7) and (8). Also for $0 < \gamma < 1$, by (4) we obtain

$$|(T_\gamma u)'(t)| \leq \frac{\gamma}{(1-\gamma)^2}\left[|u(a)| + \left(\frac{\gamma(b-a)}{1-\gamma}+1\right)\|u\|_C\right] \tag{18}$$

$$\leq \frac{r_0\gamma}{(1-\gamma)^2}\left(\frac{\gamma(b-a)}{1-\gamma}+2\right). \tag{19}$$

By differentiating both sides of (9) with respect to t, we have

$$\begin{aligned}u'(t) =&\lambda(ku)'(t)\\ =&\lambda(1-\alpha)\Bigg((T_\alpha u)'(t) + \frac{\partial f}{\partial t}\left(t, u(t), \frac{1}{1-\beta}u(t) - T_\beta u(t)\right) \qquad (20)\\ &+\frac{\partial f}{\partial x}\left(t, u(t), \frac{1}{1-\beta}u(t) - T_\beta u(t)\right)u'(t)\\ &+\frac{\partial f}{\partial y}\left(t, u(t), \frac{1}{1-\beta}u(t) - T_\beta u(t)\right)\left[\frac{1}{1-\beta}u'(t) - (T_\beta u)'(t)\right]\Bigg). \qquad (21)\end{aligned}$$

Using the assumption (h_2), (17)-(19) and the above equality we get

$$|u'(t)| \leq \frac{(1-\alpha)}{1-M}\left(\frac{r_0\alpha}{(1-\alpha)^2}\left(\frac{\alpha(b-a)}{1-\alpha}+2\right)+M_1\right), \quad t \in [a,b], \tag{22}$$

where M_1 and r_2 are defined by (12)-(13). Inequalities (17) and (22) yield (11). □

Remark 1. By induction on m we can find a prior bound for solutions of equation (9) in $C^m[a,b]$.

Set

$$r_3 := r_0 + \frac{(1-\alpha)}{1-M}\left(\frac{r_0\alpha}{(1-\alpha)^2}\left(\frac{\alpha(b-a)}{1-\alpha}+2\right)+M_1\right)+1, \tag{23}$$

and let B_r be the ball centered at 0 with radius $r>0$ in $C^1[a,b]$.

Theorem 5. *Let $(h_1)-(h_2)$ be satisfied. Then problem (1) has at least one solution in the space $C^1[a,b]$.*

Proof. According to (2), we only need to prove $K: C^1[a,b]\to C^1[a,b]$ has a fixed point. We do this in several steps.
Step 1. $K(\overline{B}_{r_3})$ is a bounded subset of $C^1[a,b]$.
By assumption (h_1), f is a function of class C^1 and by (3), for each $u\in C^1[a,b]$, $T_\gamma u\in C^1[a,b]$. Therefore by the definition of K, for for each $u\in C^1[a,b]$, $Ku\in C^1[a,b]$. Again using (h_1) and (16), for any $u\in\overline{B}_{r_3}$, we have

$$\begin{aligned}|Ku(t)| \le&(1-\alpha)\left[|T_\alpha u(t)|+\left(c_1+\frac{c_2}{1-\beta}\right)|u(t)|+c_2|T_\beta u(t)|\right]\\ \le&(1-\alpha)\left[\left(\frac{1}{1-\alpha}+\frac{c_2}{1-\beta}\right)+\left(\frac{\alpha}{(1-\alpha)^2}+\frac{c_2\beta}{(1-\beta)^2}\right)(b-a)+\left(c_1+\frac{c_2}{1-\beta}\right)\right]r_3,\end{aligned} \tag{24}$$

for $t\in[a,b]$. Using (16) for any $u\in\overline{B}_{r_3}$, and $0<\gamma<1$ we obtain

$$\left|\frac{u(t)}{1-\beta}-T_\beta u(t)\right| \le \left[\frac{2}{1-\beta}+\frac{\beta(b-a)}{(1-\beta)^2}\right]r_3 =: r_4, \tag{25}$$

$$|(T_\gamma u)'(t)| \le \frac{r_3\gamma}{(1-\gamma)^2}\left[2+\frac{\gamma(b-a)}{1-\gamma}\right]. \tag{26}$$

Put

$$M_2 := \max\left\{\left|\frac{\partial f}{\partial t}(t,x,y)\right|+r_3\left|\frac{\partial f}{\partial x}(t,x,y)\right| : (t,x,y)\in[a,b]\times[-r_3,r_3]\times[-r_4,r_4]\right\}, \tag{27}$$

$$M_3 := \max\left\{\left|\frac{\partial f}{\partial y}(t,x,y)\right| : (t,x,y)\in[a,b]\times[-r_3,r_3]\times[-r_4,r_4]\right\}. \tag{28}$$

Similar to (21), by (18), (25) and (26) we deduce

$$|(Ku)'(t)| \le (1-\alpha)\left[\frac{r_3\alpha}{(1-\alpha)^2}\left(2+\frac{\alpha(b-a)}{1-\alpha}\right)+M_2+M_3\left(\frac{r_3}{1-\beta}+\frac{r_3\beta}{(1-\beta)^2}\left(2+\frac{\beta(b-a)}{1-\beta}\right)\right)\right]. \tag{29}$$

Inequalities (24) and (29) prove our claim in *Step 1*.
Step 2. $K:\overline{B}_{r_3}\to C^1[a,b]$ is continuous.
For fixed $\varepsilon>0$, take arbitrarily $u,v\in\overline{B}_{r_3}$ with $\|u-v\|_{C^1}\le\varepsilon$. Also set

$$\begin{aligned}\omega(f,\varepsilon) := \sup\Bigg\{&|f(t,x_1,y_1)-f(t,x_2,y_2)| : t\in[a,b],\ x_1,x_2\in[-r_3,r_3],\\ &y_1,y_2\in[-r_4,r_4],\ |x_1-x_2|\le\varepsilon,\ |y_1-y_2|\le\varepsilon\left(\frac{2}{1-\beta}+\frac{\beta(b-a)}{(1-\beta)^2}\right)\Bigg\}.\end{aligned}$$

Since

$$\begin{aligned}|Ku(t)-Kv(t)| \le&|T_\alpha u(t)-T_\alpha v(t)|+\left|f\left(t,u(t),\frac{1}{1-\beta}u(t)-T_\beta u(t)\right)-f\left(t,v(t),\frac{1}{1-\beta}v(t)-T_\beta v(t)\right)\right|\\ \le&\frac{|u(a)-v(a)|}{1-\alpha}+\frac{\alpha(b-a)}{(1-\alpha)^2}\|u-v\|_C+\omega(f,\varepsilon)\\ \le&\varepsilon\left(\frac{1}{1-\alpha}+\frac{\alpha(b-a)}{(1-\alpha)^2}\right)+\omega(f,\varepsilon).\end{aligned} \tag{30}$$

Since f is uniformly continuous on $[a,b]\times[-r_3,r_3]\times[-r_4,r_4]$, we have $\omega(f,\varepsilon)\to 0$ as $\varepsilon\to 0$. Therefore, inequality (30) yields

$$\|Ku-Kv\|_C \to 0 \quad \text{as} \quad \varepsilon\to 0. \tag{31}$$

Similarly,

$$\begin{aligned}
|(Ku)'(t)-(Kv)'(t)| \leq & |(T_\alpha u)'(t)-(T_\alpha v)'(t)| \\
& + \left|\frac{\partial f}{\partial t}\left(t,u(t),\frac{1}{1-\beta}u(t)-T_\beta u(t)\right)-\frac{\partial f}{\partial t}\left(t,v(t),\frac{1}{1-\beta}v(t)-T_\beta v(t)\right)\right| \\
& \left|\frac{\partial f}{\partial x}(t,u(t),\frac{1}{1-\beta}u(t)-T_\beta u(t))-\frac{\partial f}{\partial x}\left(t,v(t),\frac{1}{1-\beta}v(t)-T_\beta v(t)\right)||u'(t)\|\right. \\
& +||u'(t)-v'(t)||\frac{\partial f}{\partial x}\left(t,v(t),\frac{1}{1-\beta}v(t)-T_\beta v(t)\right)\Bigg| \\
& + \left|\frac{\partial f}{\partial y}\left(t,u(t),\frac{1}{1-\beta}u(t)-T_\beta u(t)\right)-\frac{\partial f}{\partial y}\left(t,v(t),\frac{1}{1-\beta}v(t)-T_\beta v(t)\right)\right| \\
& \times\left|\frac{1}{1-\beta}u'(t)-(T_\beta u)'(t)\right| \\
& +\left(\frac{1}{1-\beta}|u'(t)-v'(t)|+|(T_\beta u)'(t)-(T_\beta v)'(t)|\right)\left|\frac{\partial f}{\partial y}\left(t,v(t),\frac{1}{1-\beta}v(t)-T_\beta v(t)\right)\right| \\
\leq & \frac{\varepsilon\alpha}{(1-\alpha)^2}\left[2+\frac{\alpha(b-a)}{1-\alpha}\right]+\omega(\frac{\partial f}{\partial t},\varepsilon)+r_3\omega(\frac{\partial f}{\partial x},\varepsilon)+M_4\varepsilon \\
& +r_3\left(\frac{1}{1-\beta}+\frac{2\beta}{(1-\beta)^2}+\frac{\beta^2(b-a)}{(1-\beta)^3}\right)\omega(\frac{\partial f}{\partial x},\varepsilon) \\
& +M_3\left(\frac{1}{1-\beta}+\frac{\beta}{(1-\beta)^2}\left[2+\frac{\beta(b-a)}{(1-\beta)}\right]\right)\varepsilon,
\end{aligned}$$

where M_3 is defined by (28) and

$$M_4 := \max\left\{\left|\frac{\partial f}{\partial x}(t,x,y)\right| : (t,x,y)\in[a,b]\times[-r_3,r_3]\times[-r_4,r_4]\right\}.$$

Therefore, the preceding inequality, together with the uniform continuity of the first-order partial derivatives of f imply that

$$\|(Ku)'-(Kv)'\|_C \to 0 \quad \text{as} \quad \varepsilon\to 0. \tag{32}$$

From (31) and (32) we conclude the continuity of K on $\overline{B}_{r_3}$.

Step 3. $K:\overline{B}_{r_3}\to C^1[a,b]$ is compact.

First we prove that $K(\overline{B}_{r_3})$ and

$$K'(\overline{B}_{r_3}) := \{(Ku)' : u\in\overline{B}_{r_3}\},$$

are equicontinuous subsets of $C[a,b]$. Let $t_1,t_2\in[a,b]$, $u\in\overline{B}_{r_3}$ and $0<\gamma<1$. Then by mean value theorem we have

$$\begin{aligned}
|u(t_1)-u(t_2)| \leq & ||u'||_C|t_1-t_2| \leq r_3|t_1-t_2|, & (33)\\
|T_\gamma u(t_1)-T_\gamma u(t_2)| \leq & \frac{|u(a)|}{1-\gamma}\left|e^{\frac{-\gamma}{1-\gamma}(t_1-a)}-e^{\frac{-\gamma}{1-\gamma}(t_2-a)}\right|+\frac{\gamma}{(1-\gamma)^2}\left|\int_a^{t_1}\left(e^{\frac{-\gamma}{1-\gamma}(t_1-\tau)}-e^{\frac{-\gamma}{1-\gamma}(t_2-\tau)}\right)u(\tau)d\tau\right| \\
\leq & \left[\frac{1}{1-\gamma}+\frac{\gamma(b-a)}{(1-\gamma)^2}\right]r_3e^{\frac{\gamma}{1-\gamma}(b-a)}|t_1-t_2|, & (34)
\end{aligned}$$

and similarly,

$$|(T_\gamma u)'(t_1)-(T_\gamma u)'(t_2)| \leq \left[\frac{\gamma}{(1-\gamma)^2}+\frac{\gamma^2(b-a)}{(1-\gamma)^3}\right]r_3e^{\frac{\gamma}{1-\gamma}(b-a)}|t_1-t_2|+r_3|t_1-t_2|. \tag{35}$$

According to (33)-(35), we see that $\overline{B}_{r_3}$, $T_\gamma(\overline{B}_{r_3})$ and

$$T'_\gamma(\overline{B}_{r_3}) := \{(T_\gamma u)' : u\in\overline{B}_{r_3}\},$$

are equicontinuous subsets of $C[a,b]$. Since f and its first order partial derivatives are uniformly continuous on $[a,b]\times[-r_3,r_3]\times[-r_4,r_4]$ therefore $K(\overline{B}_{r_3})$ and $K'(\overline{B}_{r_3})$ are equicontinuous. Hence using the Arzela-Ascoli theorem, if $\{u_n\}\subset\overline{B}_{r_3}$ then there exists a subsequence

$\{u_{n_k}\}$ of $\{u_n\}$ such that $\{Ku_{n_k}\}$ and $\{K'u_{n_k}\}$ are uniformly convergence. This implies the compactness of $K : \overline{B}_{r_3} \to C^1[a,b]$.

Step 4. The equation $u = \lambda Ku$ doesn't have any solution in ∂B_{r_3} for $\lambda \in (0,1)$.

Let $u \in \partial B_{r_3}$ and $u = \lambda Ku$. Hence by Theorem 4, u satisfies inequality (11) and consequently by (23) $||u||_{C^1} < r_3$. Then $u \notin \partial B_{r_3}$ and this is a contradiction. Therefore, our claim is proved.

By the steps 1-4 and Theorem 3, K has a fixed point in $\overline{B}_r$ which implies equation (1) has a solution in $\overline{B}_r$. □

We can generalize Theorem 5 as follows:

Theorem 6. *Let $f \in C^m([a,b]\times\mathbb{R}\times\mathbb{R})$ and (h_1)-(h_2) be satisfied. Then problem (1) has at least one solution in the space $C^m[a,b]$.*

Proof. To prove the above theorem, one can show that if $\overline{B}_r \subset C^m[a,b]$ then $K(\overline{B}_r)$, $K'(\overline{B}_r)$, ..., $K^{(m)}(\overline{B}_r)$ are equicontinuous subsets of $C[a,b]$ where $K^{(j)}(\overline{B}_r) := \{(Ku)^{(j)} : u \in \overline{B}_r\}$ for $j = 1,2,\ldots,m$. Then similar to the proof of Theorem 5, by Remark 1 and the Arzela-Ascoli theorem, the existence of solutions for problem (1) is proved. □

3 Numerical Framework

Spline functions are the linear combination of cardinal B-splines that form a local basis for the spline space. There are different ways for constructing the cardinal splines. We will used the notation introduced in [42]. For the knot set $\chi = \{t_1,t_2,\ldots,t_m\}$ with $t_1 < t_2 < \cdots < t_m$, we define the collocation matrix associated with the function system $\mu = \{u_1(t),u_2(t),\ldots,u_m(t)\}$ as follows

$$M\begin{pmatrix} u_1(t) & u_2(t) & \ldots & u_m(t) \\ t_1 & t_2 & \ldots & t_m \end{pmatrix} := \begin{pmatrix} u_1(t_1) & u_2(t_1) & \ldots & u_m(t_1) \\ u_1(t_2) & u_2(t_2) & \ldots & u_m(t_2) \\ \vdots & \vdots & \vdots & \vdots \\ u_1(t_m) & u_2(t_m) & \ldots & u_m(t_m) \end{pmatrix}.$$

If we denote by D, the determinant of the collocation matrix M, then the divided difference operator for a function f over the knot set χ is defined as

$$[t_1,t_2,\ldots,t_m]f = \frac{D\begin{pmatrix} 1 & t & t^2 & \cdots & t^{m-2} & f(t) \\ t_1 & t_2 & t_3 & \cdots & t_{m-1} & t_m \end{pmatrix}}{D\begin{pmatrix} 1 & t & t^2 & \cdots & t^{m-2} & t^{m-1} \\ t_1 & t_2 & t_3 & \cdots & t_{m-1} & t_m \end{pmatrix}}. \tag{36}$$

Then using the divided difference operator (36), the cardinal B-splines, the polynomial B-splines having break points on integer knots, can be defined as

$$B_m(t) := (m+1)[0,1,\ldots,m+1](y-t)_+^m = \frac{1}{m!}\Delta^{m+1}t_+^m, \tag{37}$$

where t_+^m is the truncated power function

$$t_+^m := \begin{cases} t^m, & t \geq 0, \\ 0, & t < 0, \end{cases}$$

and Δ^m is the finite difference operator

$$\Delta^m f(t) := \sum_{k=0}^{m} (-1)^k \binom{m}{k} f(t-k). \tag{38}$$

The basis function B_m has compact support on $[0,m+1]$, is positive on $(0,m+1)$ and belongs to $C^{m-1}(\mathbb{R})$. Also, the set of integer translates of cardinal B-splines $\{B_m(t-k),\ k\in\mathbb{Z}\}$, forms a basis for spline space. With $\xi > 0$, it can be generalized to a basis on grid points $\{x_k = k\xi,\ k\in\mathbb{Z}\}$ as follows

$$\mathscr{B}_m^{\xi} = \left\{B_m(\frac{t-x_k}{\xi}),\quad k\in\mathbb{Z}\right\}.$$

For problems with bounded domain we can restrict the above set to a valid basis in the interval $[a,b]$. Let $\Delta \equiv \{a = t_0 < t_1 < \ldots < t_n = b\}$ be a uniform partition of the interval $[a,b]$ with step size $h = \frac{b-a}{n}$ and $\Delta' \equiv \left\{z_k = x_k - \frac{h}{2}\right\}_{k=1}^{n}$ be the set of mid points of the partition. Based on the above notations we define

$$\mathscr{B}_m^{h} = \left\{B_{m,j}(t) \equiv B_m(\frac{t-t_j}{h}),\quad j = -m,\cdots,n-1\right\},$$

which is a basis for splines of degree m with support on Δ. The basis functions $B_{m,j}(t)$ belong to $C^{m-1}[a,b]$, has local support on $[j_k, t_{j+m+1}]$ and $\mathscr{B}_m^h$ is a partition of unity. Also let us denote by $\mathscr{S}_m^h = span\left(\mathscr{B}_m^h\right)$, the space of B-splines of degree m on the partition Δ. To approximate the solution of (1), we can use a spline $s \in \mathscr{B}_m^h$ in the following form

$$s_n(t) = \sum_{j=-m}^{n-1} c_j B_{m,j}(t), \quad t \in [a,b]. \tag{39}$$

Rewriting $T_\gamma u(t)$ form (3)

$$T_\gamma u(t) := \frac{u_0}{1-\gamma} e^{\frac{-\gamma}{1-\gamma}(t-a)} + \frac{\gamma}{(1-\gamma)^2} \int_a^t u(\tau) e^{\frac{-\gamma}{1-\gamma}(t-\tau)} d\tau,$$

and replacing u by s_n we have

$$\begin{aligned} T_\gamma s_n(t) &:= \frac{u_0}{1-\gamma} e^{\frac{-\gamma}{1-\gamma}(t-a)} + \frac{\gamma}{(1-\gamma)^2} \int_a^t s(\tau) e^{\frac{-\gamma}{1-\gamma}(t-\tau)} d\tau \\ &= \frac{u_0}{1-\gamma} e^{\frac{-\gamma}{1-\gamma}(t-a)} + \frac{\gamma}{(1-\gamma)^2} \sum_{j=-m}^{n-1} c_j \left(\int_a^t B_{m,j}(\tau) e^{\frac{-\gamma}{1-\gamma}(t-\tau)} d\tau \right). \end{aligned} \tag{40}$$

Substituting (39) and (40) into (5) it results

$$s_n(t) = (1-\alpha) \left(T_\alpha s_n(t) + f\left(t, s_n(t), \frac{1}{1-\beta} s_n(t) - T_\beta s_n(t)\right)\right). \tag{41}$$

In the following we will construct the numerical method for odd and even values of $m \leq 6$ separately. The difference between two cases is in the choice of collocation points.

3.1 Odd Degree Splines

Let m be odd, collocating equation (41) at the points $t_k \in \Delta$, we obtain

$$s_n(t_k) = (1-\alpha) \left(T_\alpha s_n(t_k) + f\left(t_k, s_n(t_k), \frac{1}{1-\beta} s_n(t_k) - T_\beta s_n(t_k)\right)\right), \quad k = 0, 1, \cdots, n. \tag{42}$$

Also, from initial condition we have

$$s_n(t_0) = u_0. \tag{43}$$

Obviously relation (42) together with (43), form a system with $n+2$ equations and $n+m$ unknowns c_j, $j = -m, \cdots, n-1$, to be obtained. To be able to uniquely determine the coefficients for $m \geq 3$, we need $m-2$ extra relations. Let us define $\sigma_1 = \left[\frac{m-1}{2}\right]$ and $\sigma_2 = \left[\frac{m-4}{2}\right]$, then we can collocate the problem at $m-2$ near boundary mid points as follows

$$s_n(z_k) = (1-\alpha)\left(T_\alpha s_n(z_k) + f\left(z_k, s_n(z_k), \frac{1}{1-\beta} s_n(z_k) - T_\beta s_n(z_k)\right)\right), \tag{44}$$

where $k \in \{1, \cdots, \sigma_1\} \cup \{n-\sigma_2, \cdots, n\}$. It should be noted that, in the relation above if $\sigma_1 < 0$ or $\sigma_2 < 0$, its corresponding set of indexes is empty. Now the system is uniquely solvable and it could be solved to obtain the unknowns c_j, $j = -m, \cdots, n-1$. To approximate the integrals in (40) we rewrite the integral in the following form

$$I_k = \int_a^{t_k} B_{m,j}(\tau) e^{\frac{-\gamma}{1-\gamma}(t_k-\tau)} d\tau = \sum_{r=0}^{k-1} \int_{t_r}^{t_{r+1}} B_{m,j}(\tau) e^{\frac{-\gamma}{1-\gamma}(t_k-\tau)} d\tau,$$

which, by a change of variable, can be approximated as

$$I_k = \sum_{r=0}^{k-1} \frac{t_{r+1}-t_r}{2} \sum_{p=0}^{q} \omega_p \varphi(\lambda_p),$$

where ω_p and λ_p are respectively the weights and points of the q-point Gauss-Legender quadrature and

$$\varphi(\lambda_p) = B_{m,j}\left(\frac{t_{r+1}-t_r}{2}\lambda_p + \frac{t_{r+1}+t_r}{2}\right)\exp\left(\frac{-\gamma}{1-\gamma}\left(t_k - \frac{t_{r+1}-t_r}{2}\lambda_p + \frac{t_{r+1}+t_r}{2}\right)\right).$$

Also, the integrals (44) can be written as

$$\bar{I}_k = \int_a^{z_k} B_{m,j}(\tau)e^{\frac{-\gamma}{1-\gamma}(z_k-\tau)}d\tau = \int_0^{z_1} B_{m,j}(\tau)e^{\frac{-\gamma}{1-\gamma}(z_0-\tau)}d\tau + \sum_{r=1}^{k-1}\int_{z_r}^{z_{r+1}} B_{m,j}(\tau)e^{\frac{-\gamma}{1-\gamma}(z_k-\tau)}d\tau.$$

which can be approximate similarly by a q-point Gauss-Legender quadrature.

3.2 Even Degree Splines

It should be noted that for m even, we will collocate (41) at the mid points of the partition $z_k \in \Delta'$, so we will need $m-1$ extra relations. To have a uniquely solvable system we will collocate the problem at $m-1$ near boundary grid points. In this case the system is as follows

$$s_n(z_k) = (1-\alpha)\left(T_\alpha s_n(z_k) + f\left(z_k, s_n(z_k), \frac{1}{1-\beta}s_n(z_k) - T_\beta s_n(z_k)\right)\right), \quad k = 1,\cdots,n, \tag{45}$$

$$s_n(t_k) = (1-\alpha)\Big(T_\alpha s_n(t_k) + f\Big(t_k, s_n(t_k), \frac{1}{1-\beta}s_n(t_k) - T_\beta s_n(t_k)\Big)\Big), \quad k \in \{0,\cdots,\delta\}\cup\{n-\delta,\cdots,n\}, \tag{46}$$

where $\delta = \left[\frac{m}{2}\right] - 1$.

In the following we will find some error bounds for the proposed spline interpolation. It will be used in the error analysis of the method. Let for even m, the interpolation conditions along with the end conditions be defined as follows

$$\begin{aligned} s_n(z_k) &= u(z_k), \quad k \in \Delta', \\ s_n(t_k) &= u(t_k), \quad k \in \{0,\cdots,\delta\}\cup\{n-\delta,\cdots,n\}, \end{aligned} \tag{47}$$

while for odd m, the conditions are

$$\begin{aligned} s_n(t_k) &= u(t_k), \quad k \in \Delta, \\ s_n(z_k) &= u(z_k), \quad k \in \{1,\cdots,\sigma_1\}\cup\{n-\sigma_2,\cdots,n\}. \end{aligned} \tag{48}$$

Theorem 7. *Let $s_n \in \mathscr{S}_m^h$ be the mth degree unique spline interpolating to $u(t) \in C^{m+1}[a,b]$ on Δ satisfying the conditions (47) for even m and (48) for odd m, then we have*

$$\left\|u^{(r)}(t) - s_n^{(r)}(t)\right\|_\infty = O(h^{m+1-r}), \quad r = 0,\cdots,m-1. \tag{49}$$

Also if m is even, then for $u \in C^{m+2}[a,b]$ we have the following local error estimate

$$|u(t) - s_n(t)|_{t_k} = O(h^{m+2}), \quad k = 0,\cdots,n. \tag{50}$$

Proof. The proof of (49) is in a standard way similar to [43]. To prove the latter relation let $s_n(t)$ satisfies the interpolatory conditions

$$s_n(z_i) = u(z_i), \quad 1 \le i \le n, \tag{51}$$

along with the end conditions

$$s_n(t_k) = u(t_k), \quad k \in \{0,1,\cdots,\delta\}\cup\{n-\delta,\cdots,n\}. \tag{52}$$

Let m be even and consider the following consistency relation connect $s_n(t)$ at the midpoints and grid points of the partition [44]:

$$\sum_{j=-\delta-1}^{\delta+1}\xi_j s_n(t_{i+j}) = \sum_{j=-\delta}^{\delta+1}\zeta_j s_n(z_{i+j}), \quad \delta < i < n-\delta, \tag{53}$$

where

$$\xi_j = B_{m+1}\left(m+\frac{1}{2}-j\right), \quad \zeta_j = B_{m+1}(m-j). \tag{54}$$

Let denote by $e(t) = s_n(t) - u(t)$ the interpolation error. Subtracting $\sum_{j=-\delta-1}^{\delta+1} \xi_j u(t_{i+j})$ from both sides of (53) and using the interpolatory conditions $s_n(z_i) = u(z_i),\ i = 1(1)n$, we have

$$\sum_{j=-\delta-1}^{\delta+1} \xi_j e(t_{i+j}) = \sum_{j=-\delta}^{\delta+1} \zeta_j u(z_{i+j}) - \sum_{j=-\delta-1}^{\delta+1} \xi_j u(t_{i+j}), \quad \delta < i < n-\delta.$$

Since $u \in C^{m+2}[a,b]$, by expanding the right hand side using Taylor series expansion about t_i, we obtain the linear system

$$\sum_{j=-\delta-1}^{\delta+1} \xi_j e(t_{i+j}) = O(h^{m+2}), \qquad \delta < i < n-\delta, \tag{55}$$

which by the help of (52), it can be written in the following matrix form

$$A_m E = O(h^{m+2}), \tag{56}$$

where

$$E = [e(t_{\delta+1}), e(t_{\sigma+2}), \cdots, e(t_{n-\delta-1})]^T,$$

and

$$A_m = \begin{pmatrix} \xi_\delta & \xi_{\delta+1} & \cdots & \xi_m & & \\ \xi_{\delta-1} & \xi_\delta & \cdots & \xi_{m-1} & \xi_m & \\ & & \ddots & & & \\ & & & \ddots & & \\ & \xi_0 & \xi_1 & \cdots & \xi_\delta & \xi_{\delta+1} \\ & & \xi_0 & \xi_1 & \cdots & \xi_\delta \end{pmatrix}. \tag{57}$$

Obviously for $m \le 6$, the coefficients matrix is strictly diagonally dominant so it is invertible. Also, it can be easily seen that $\left\|A_m^{-1}\right\|_\infty \le C_i$, where $[C_2, C_4, C_6] = [2, 5, 46]$. Thus we have

$$\|E\|_\infty = \left\|A_m^{-1}\right\|_\infty . O(h^{m+2}) = O(h^{m+2}), \tag{58}$$

and the proof is complete. □

4 Convergence Analysis

In this section an operator based convergence analysis and error bound will be presented for the proposed method. For odd and even m, the analysis is the same, so suppose that m is even. Let $\Omega \equiv \Delta' \cup \{t_k\}_0^\delta \cup \{t_k\}_{n-\delta}^n$, and define the operators ϕ_n and θ_n in the following form

$$\begin{aligned} &\theta_n : C^{m+1}[a,b] \to \mathbb{R}^{n+m}, \ \ (\theta_n f)_i = f(\mu_i), \ \mu_i \in \Omega, \\ &\phi_n : \mathbb{R}^{n+m} \to \mathscr{S}_m^h, \ \text{ spline interpolation at points } \mu_i \in \Omega. \end{aligned} \tag{59}$$

By defining the operator $\mathscr{P}_n \equiv \phi_n \theta_n : C^{m+1}[a,b] \to \mathscr{S}_m^h$, based on theorem 7, for $f \in C^{m+2}[a,b]$ we have the following local error bounds at collocation points

$$|\mathscr{P}_n f - f|_{t_i} = |\phi_n \theta_n f - f|_{t_i} = O(h^{m+2}), \quad t_i \in \Delta. \tag{60}$$

As we already note, the problem can be written in the following operator form

$$u = Ku. \tag{61}$$

On the other hand using (59), the collocation equation (42)-(43) can be written as

$$\phi_n\theta_n s_n = \phi_n\theta_n K s_n,$$

which using $\mathscr{P}_n s_n = s_n$, results

$$s_n = \mathscr{P}_n K s_n. \tag{62}$$

We will show that (62) has a unique solution that converges to the solution of (1) and finally we will obtain the rate of convergence. Let us restate the following Theorem from [45].

Theorem 8. *Suppose that the operators $\mathscr{T}$ and $\mathscr{T}_n$ defined on the Banach space $\mathscr{B}$ admit the following representations:*

$$\mathscr{T} = \mathscr{P}K, \quad \mathscr{T}_n = \mathscr{P}_n K,$$

where K is a nonlinear, completely continuous operator mapping $\mathscr{B}$ into another Banach space $\mathscr{B}'$ and $\mathscr{P}$ and $\mathscr{P}_n$ are continues linear operators taking $\mathscr{B}'$ into $\mathscr{B}$.
Suppose that the operator equation

$$w = \mathscr{P}Kw, \tag{63}$$

has a solution υ. A sufficient condition that υ be an isolated solution of (63) in some sphere $\|\omega-\upsilon\| \le \mu$, $(\mu > 0)$ is that K be differentiable at the point υ, and the homogeneous equation

$$w - \mathscr{P}K'(\upsilon)w = 0, \tag{64}$$

has only the trivial solution $w = 0$. Suppose further that the sequence of operators $\mathscr{P}_n$ converges strongly to the operator $\mathscr{P}$, then the equation

$$w = \mathscr{T}_n w, \tag{65}$$

has a solution υ_n satisfying $\|\upsilon_n - \upsilon\| \le \mu$ for all sufficiently large n, $\upsilon_n \to \upsilon$ as $n \to \infty$, and for constant M', the rate of convergence is bounded by

$$\|\upsilon_n - \upsilon\| \le M' \|(\mathscr{P}_n - \mathscr{P})K\upsilon\|. \tag{66}$$

Lemma 1. *The sequence of operators P_n are uniformly bounded in $C^k[0,T]$.*

Proof. Using [42], Theorem 6.22, for every $f \in C[a,b]$, we have

$$\|\mathscr{P}_n f\|_{C[a,b]} \le (2m)^m \|f\|_{C[a,b]},$$

so $\mathscr{P}_n$ is uniformly bounded. □

Lemma 2. *Let Δ be a uniform partition in $[a,b]$ as defined before. The sequence of operators $\mathscr{P}_n$ uniformly converges to the identity operator I.*

Proof. Let $f \in C[a,b]$, then according to Lebesgue lemma we have

$$\|\mathscr{P}_n f - f\| \le (1 + \|\mathscr{P}_n\|) \inf_{u\in\Pi_m} \|f-u\|, \tag{67}$$

where the last term is the best approximation of f in Π_m, the space of polynomials of degree at most m. Let u^* be the best approximation for f, then using Jackson's theorem in each subinterval $[t_i, t_{i+1}]$ we have

$$\sup_{t_i \le t \le t_{i+1}} |f(t) - u^*(t)| \le 6\,\omega\left(f, \frac{h}{2m}\right), \tag{68}$$

where ω is the modulus of continuity of f with the bound $\frac{h}{2m}$. Since f is continues and $\mathscr{P}_n$ is bounded, using (68) in (67), as $h \to 0$ we have

$$\|\mathscr{P}_n f - f\| \le 6(1 + \|\mathscr{P}_n\|)\,\omega\left(f, \frac{h}{2m}\right) \to 0,$$

which completes the proof. □

Theorem 9. *Let m be even, and let $u \in C^{m+1}[a,b]$ and $s_n \in \mathscr{S}_m^h$ denote the exact and spline solutions of problem (1) respectively. If s_n is obtained by (45)-(46) then s_n converges to u and there exists a constant C such that*

$$\|s_n - u\| \leq C\, h^{m+1}.$$

Also, if $u \in C^{m+2}[a,b]$, the following local error bound holds

$$|s_n - u|_{t_i} = O\left(h^{m+2}\right), \;\; t_i \in \Delta.$$

Proof. Consider the operator equations (61) and (62). K is a nonlinear, completely continuous operator and we already proved in section 2, that the operator equation $u = Ku$ has a unique solution. Now based on lemma 2, $\mathscr{P}_n$ is a continues linear operator converges to the identity operator, thus using theorem 4, the collocation equation $s_n = \mathscr{P}_n K s_n$ has a unique solution s_n converges to u and for some constant M' the error is bounded by

$$\|s_n - u\| \leq M' \, \|(\mathscr{P}_n - I)Ku\| \,. \tag{69}$$

On the other hand based on (49) we have

$$\|\mathscr{P}_n f - f\| = O(h^{m+1}),$$

which, when combined with the complete continuity of operator K, gives

$$\|s_n - u\| \leq C\, h^{m+1}.$$

Also using (60) in (69) we obtain the local error bound

$$|s_n - u|_{t_i} = O\left(h^{m+2}\right), \;\; t_i \in \Delta.$$

□

Theorem 10. *Let for odd m, $u \in C^{m+1}[a,b]$ and $s_n \in \mathscr{S}_m^h$ be the exact and spline solutions of problem (1) respectively. If s_n is obtained by (42)-(44) then s_n converges to u and for some constant $\tilde{C}$ we have*

$$\|s_n - u\| \leq \tilde{C}\, h^{m+1}.$$

Proof. The proof is in a simillar manner with Theorem 9. □

5 Numerical Experiments

In this section, we will examine our obtained results on some model problems to show the applicability and efficiency of the proposed method. The maximum absolute errors E_n in numerical solution and the experimental orders of convergence for various values of m and n have been tabulated in tables 1-3. For problems 1 and 2 the exact solutions are not available, so we run the program for a large value of n and use it as exact solution. The experimental orders of convergence have been obtained as follows

$$Order = \log_2 \frac{|E_n|}{|E_{2n}|}.$$

All programs run in Mathematica 12 on a system with Corei7 2.4 GHz CPU and 8 GB of RAM.

Example 1. Consider the following problem

$$\begin{cases} {}_0D_t^{\frac{2}{3}} u(t) = \frac{t}{8} \ln\left(1 + u^2(t) + \left({}_0D_t^{\frac{1}{2}} u(t)\right)^2\right), & t \in (0,1], \\ u(0) = 1. \end{cases} \tag{70}$$

In view of problem (1) we put

$$f(t,x,y) = \frac{t}{16} \ln\left(1 + x^2 + y^2\right), \;\; [a,b] = [0,1], \;\; \alpha = \frac{2}{3}, \;\; \beta = \frac{1}{2}.$$

Then $f \in C^m([0,1]\times\mathbb{R}\times\mathbb{R})$ for each $m \in \mathbb{N}$. Also using the inequality $\ln(1+|z|) \le |z|$ for $z \in \mathbb{R}$, we have

$$|f(t,x,y)| = \frac{t}{8}\ln\sqrt{1+x^2+y^2} \le \frac{1}{8}(|x|+|y|), \quad (t,x,y) \in [0,1]\times\mathbb{R}\times\mathbb{R},$$
$$p = (1-\alpha)\left[c_1 + \frac{c_2}{1-\beta}\right] = \frac{1}{8},$$
$$\sup_{[0,1]\times\mathbb{R}\times\mathbb{R}}\left\{\left|\frac{\partial f}{\partial x}(t,x,y)\right| + \frac{1}{1-\beta}\left|\frac{\partial f}{\partial x}(t,x,y)\right|\right\} = \sup_{[0,1]\times\mathbb{R}\times\mathbb{R}}\left\{\frac{t|x|}{8(1+x^2+y^2)} + \frac{t|y|}{4(1+x^2+y^2)}\right\} \le \frac{3}{8}.$$

Hence all conditions of Theorem 6 are satisfied and problem (71) has a solution in $C^m[0,1]$ for each $m \in \mathbb{N}$.

The problem has been solved for various values of m and n and the maximum absolute errors and the experimental orders of convergence are presented in table 1. The experimental orders of convergence verify the theoretical results very well.

Table 1. Maximum absolute errors and experimental orders of convergence for problem 1.

	$m=1$		$m=2$		$m=3$		$m=4$	
n	Error	Order	Error	Order	Error	Order	Error	Order
8	1.6(−4)		1.9(−07)		1.5(−08)		7.5(−11)	
16	4.5(−5)	1.87	1.2(−08)	3.90	1.1(−09)	3.79	1.3(−12)	5.82
32	1.1(−5)	1.93	8.2(−10)	3.95	7.5(−11)	3.90	2.2(−14)	5.85
64	3.0(−6)	1.96	5.2(−11)	3.97	4.8(−12)	3.95	3.8(−16)	5.90

Example 2. Consider the following problem

$$\begin{cases} {}_{-1}D_t^{\frac{4}{5}}u(t) = \frac{1+t}{5+t^2}\left(u(t) + {}_{-1}D_t^{\frac{1}{4}}u(t)\right), & t \in (-1,1], \\ u(-1) = 2. \end{cases} \tag{71}$$

Set

$$f(t,x,y) = \frac{t+1}{5+t^2}(x+y), \ [a,b] = [-1,1], \ \alpha = \frac{4}{5}, \ \beta = \frac{1}{4},$$

then $f \in C^m([-1,1]\times\mathbb{R}\times\mathbb{R})$ for each $m \in \mathbb{N}$. For $(t,x,y) \in [-1,1]\times\mathbb{R}\times\mathbb{R}$ we have

$$|f(t,x,y)| \le \frac{2}{5}(|x|+|y|), \quad p = (1-\alpha)\left[c_1 + \frac{c_2}{1-\beta}\right] = \frac{14}{75},$$
$$\sup_{[0,1]\times\mathbb{R}\times\mathbb{R}}\left\{\left|\frac{\partial f}{\partial x}(t,x,y)\right| + \frac{1}{1-\beta}\left|\frac{\partial f}{\partial x}(t,x,y)\right|\right\} \le \frac{14}{15}.$$

Therefore using theorem 6, problem (71) has a solution of class C^m for each $m \in \mathbb{N}$. We solved the problem for various values of m and n and tabulated the maximum absolute errors as well as the practical orders of convergence in table 2. The practical orders of convergence are in good agreement with theoretical results.

Example 3. Fractional order Malthusian growth model [46]
Consider the following fractional order problem

$$\begin{cases} {}_0D_t^{\alpha}u(t) = \kappa u(t), & t \in [0,1], \\ u(0) = 0, \end{cases} \tag{72}$$

where $u(t)$ denotes the population at time t and κ is positive. We solved this problem with $\alpha = 0.1, 0.5, 0.9$ for various values of m and n. The maximum absolute errors and the orders of convergence are tabulated in tables 3-5. The rapid convergence and small errors show the efficiency and applicability of the proposed method. The practical orders of convergence verify the theory as well.

Table 2. Maximum absolute errors and experimental orders of convergence for problem 2.

	$m=1$		$m=2$		$m=3$		$m=4$	
n	Error	Order	Error	Order	Error	Order	Error	Order
8	8.5(−3)		4.6(−5)		5.5(−6)		3.3(−07)	
16	2.1(−3)	1.99	3.0(−6)	3.96	3.6(−7)	3.92	5.1(−09)	6.00
32	5.3(−4)	2.00	1.9(−7)	3.98	2.3(−8)	3.93	8.0(−11)	6.02
64	1.3(−4)	2.00	1.2(−8)	3.99	1.5(−9)	3.96	1.2(−12)	6.00

Table 3. Maximum absolute errors and experimental orders of convergence with $\alpha = 0.1$ for problem 3.

	$m=1$		$m=2$		$m=3$		$m=4$	
n	Error	Order	Error	Order	Error	Order	Error	Order
8	6.1(−4)		3.9(−5)		7.6(−6)		5.8(−8)	
16	2.1(−3)	2.00	2.5(−6)	3.94	5.6(−7)	3.77	1.06(−9)	5.77
32	9.8(−3)	2.00	1.6(−7)	3.97	3.7(−8)	3.90	1.7(−11)	5.88
64	3.9(−2)	2.00	1.02(−8)	3.98	2.4(−9)	3.95	3.0(−13)	5.89

Table 4. Maximum absolute errors and experimental orders of convergence $\alpha = 0.5$ for problem 3.

	$m=1$		$m=2$		$m=3$		$m=4$	
n	Error	Order	Error	Order	Error	Order	Error	Order
8	1.4(−2)		3.0(−5)		2.7(−6)		1.5(−8)	
16	3.5(−3)	2.00	2.0(−6)	3.87	2.0(−7)	3.77	2.9(−10)	5.73
32	8.8(−4)	2.00	1.3(−7)	3.93	1.3(−8)	3.89	4.9(−12)	5.86
64	2.2(−4)	2.00	8.6(−9)	3.96	8.7(−10)	3.95	8.3(−14)	5.90

Table 5. Maximum absolute errors and experimental orders of convergence $\alpha = 0.9$ for problem 3.

	$m=1$		$m=2$		$m=3$		$m=4$	
n	Error	Order	Error	Order	Error	Order	Error	Order
8	3.9(−2)		1.1(−4)		7.3(−6)		4.4(−8)	
16	9.8(−3)	2.00	8.0(−6)	3.78	5.5(−7)	3.72	8.5(−10)	5.69
32	2.4(−3)	2.00	5.3(−7)	3.90	3.7(−8)	3.88	1.5(−11)	5.80
64	6.1(−4)	2.00	3.4(−8)	3.95	2.4(−9)	3.94	2.6(−13)	5.86

6 Conclusion

We proved that the Caputo-Fabrizio fractional differential equation (1) has at least one solution and the regularity of solutions is depended on the regularity of f. Indeed, we established that the solutions of problem (1) are of class C^m for $m \in \mathbb{N}$, when $f \in C^m\big([a,b] \times \mathbb{R} \times \mathbb{R}\big)$. We also derived a priori bound for the solutions of problem (1), which is crucial in demonstrating the existence of a solution. To solve the problem numerically, we constructed a new algorithm based on B-spline basis functions. The key in this method is in selection of appropriate collocation points and end conditions. We proved that by choosing the mid points of the partition as collocation points and near boundary grid points as end conditions, a superconvergent approximation is derived for even degree spline. We proved the convergence of the method and obtained some error bounds. Finally, by providing some numerical experiments we showed that the method is computationally efficient and it also verified the obtained theoretical error bounds.

Authors' Contributions

All authors have the same contribution.

Data Availability

Not applicable.

Conflicts of Interest

The authors declare no conflict of interest.

Ethical Considerations

Not applicable.

Funding

This research does not receive specific funding.

References

[1] I. Goychuk, Fractional diffusion models of anomalous dynamics: a brief survey, Acta Physica Polonica, 45(3), 773–790, (2012).

[2] R. Metzler and J. Klafter, The random walk's guide to anomalous diffusion: a fractional dynamics approach, Physics Reports, 339(1), 1–77, (2000).

[3] F. Mainardi, Fractional Calculus and Waves in Linear Viscoelasticity: An Introduction to Mathematical Models. Imperial College Press, London, 2010.

[4] V. Tarasov, Fractional Dynamics: Applications of Fractional Calculus to Dynamics of Particles. Fields and Media, Springer-Verlag, Berlin Heidelberg, 2010.

[5] P. M. Prenter, Functional Fractional Calculus for System Identification and Controls. Springer-Verlag, Berlin Heidelberg, 2008.

[6] S. Das, Functional Fractional Calculus. Springer-Verlag, Berlin Heidelberg, 2011.

[7] M. Caputo and M. Fabrizio, A new definition of fractional derivative without singular kernel, Progress in Fractional Differentiation and Applications, 1(2), 73–85, (2015).

[8] H. Afshari, V. Roomi, and S. Kalantari, Existence of solutions of some boundary value problems with impulsive conditions and abc-fractional order, Filomat, 37(11), 3639–3648, (2023).

[9] H. Afshari, V. Roomi, and M. Nosrati, Existence and uniqueness for a fractional differential equation involving atangana-baleanu derivative by using a new contraction, (2023).

[10] A. Atangana and S. B. Yuste, Modeling heat conduction using Caputo-Fabrizio fractional derivative, 1, 1–8, (2016).

[11] A. Atangana and S. B. Yuste, A note on the use of Caputo-Fabrizio derivative to model the viscoelastic response, 293, 104–111, (2015).

[12] J. Cao, J. Lu, J. Chen, and Y. Hu, Performance analysis of a fractional-order inductor-capacitor circuit based on Caputo-Fabrizio fractional derivative, 331, 1–11, (2018).

[13] M. Kiryukhin, A. Lazarev, and A. Potapov, Modeling of composite materials with fractional order elasticity, 55, 95–106, (2019).

[14] V. Roomi, H. Afshari, and S. Kalantari, Some existence results for a differential equation and an inclusion of fractional order via (convex) f-contraction mapping, Journal of Inequalities and Applications, 2024(1), 28, (2024).

[15] N. Tuan, A. Nguyen, and N. Can, Existence and continuity results for Kirchhoff parabolic equation with Caputo-Fabrizio operator, 167, 113028, (2023).

[16] K. Eiman and D. Baleanu, Study on Krasnoselskii's fixed point theorem for Caputo-Fabrizio fractional differential equations, 178, 2020, (2020).

[17] M. Abbas, On the initial value problems for the Caputo-Fabrizio impulsive fractional differential equations, Asian-European Journal of Mathematics, 14(5), 2150073, (2021).

[18] S. Abbas, M. Benchohra, and J. J. Nieto, Caputo–fabrizio fractional differential equations with non instantaneous impulses, Rendiconti del Circolo Matematico di Palermo Series 2, 71(1), 131–144, (2022).

[19] L. S. Derradji, N. Hamidane, and S. Aouchal, A fractional seirs model with disease resistance and nonlinear generalized incidence rate in caputo-fabrizio sense, 72, 81–98, (2023).

[20] K. Liu, M. Feckan, d. O´Regan, and J. Wang, Hyers-Ulam stability and existence of solutions for differential equations with Caputo-Fabrizio fractional derivative, 7, 333, (2019).

[21] A. Mahdy, Stability, existence, and uniqueness for solving fractional glioblastoma multiforme using a Caputo-Fabrizio derivative, Mathematical Methods in the Applied Sciences, https://doi.org/10.1002/mma.9038.

[22] D. Mozyrska, D. Torres, and M. Wyrwas, Solutions of systems with the Caputo-Fabrizio fractional delta derivative on time scales, Nonlinear Analysis: Hybrid Systems, 332(1), 168–176, (2019).

[23] K. Owolabi and A. Atangana, Analysis and application of new fractional Adams-Bashforth scheme with Caputo-Fabrizio derivative, 105, 111–119, (2017).

[24] I. Mirza and D. Vierub, Fundamental solutions to advection-diffusion equation with time-fractional CaputoFabrizio derivative, Computers and Mathematics with Applications, 73(1), 1–10, (2017).

[25] D. Avci, M. Yavuz, and N. Ozdemir, Fundamental solutions to the cauchy and Dirichlet problems for a heat conduction equation equipped with the Caputo-Fabrizio differentiation, In: Heat conduction: methods applications and research, Nova Science Publishers, 95–107, (2019).

[26] M. Yavuz and N. Ozdemir, Analysis of an epidemic spreading model with exponential decay law, 8, 142–154, (2020).

[27] P. Pandey, J. Gómez-Aguilar, M. K. Z. Sirid, and A. Mousa, Mathematical modeling of COVID-19 pandemic in India using Caputo-Fabrizio fractional derivative, 145, 105518, (2022).

[28] K. Khan, A. Ali, M. Sen, and M. Irfan, Localized modes in time-fractional modified coupled Korteweg-de Vries equation with singular and non-singular kernels, AIMS Mathematics, 7(2), 1580–1602, (2022).

[29] M. Yaseen, Q. Arif, R. George, and S. Khan, Comparative numerical study of spline-based numerical techniques for time fractional cattaneo equation in the sense of Caputo-Fabrizio, Fractal Fract., 50(6), 1–22, (2022).

[30] A. Pedas and E. Tamme, Spline collocation methods for linear multi-term fractional differential equations, J. Comput. Appl. Math., 236(2), 167–176, (2011).

[31] A. Pedas and E. Tamme, On the convergence of spline collocation methods for solving fractional differential equations, J. Comput. Appl. Math., 235(12), 3502–3514, (2011).

[32] X. Li, Numerical solution of fractional differential equations using cubic B-spline wavelet collocation method, Commun. Non. Sci. Numer. Simul., 17(10), 3934–3946, (2012).

[33] G. Akram and H. Tariq, Quintic spline collocation method for fractional boundary value problems, 23, 57–65, (2017).

[34] O. Tasbozan and A. Esen, Quadratic B-spline galerkin method for numerical solutions of fractional telegraph equations, 18, 23–39, (2017).

[35] M. Yaseen, M. Abbas, T. Nazir, and D. Baleanu, A finite difference scheme based on cubic trigonometric B-splines for a time-fractional diffusion-wave equation, Advances Diff. Eq., 724, (2017).

[36] S. Mohyud-Din, T. Akram, M. Abbas, A. Ismail, and N. Ali, A fully implicit finite difference scheme based on extended cubic B-splines for time fractional advection-diffusion equation, Advances Diff. Eq., 109, (2018).

[37] T. Akram, M. Abbas, A. Ismail, N. Ali, and D. Baleanu, Extended cubic B-splines in the numerical solution of time fractional telegraph equation, Advances Diff. Eq., 1, 365, (2019).

[38] S. Mohyud-Din, T. Akram, M. Abbas, A. Ismail, and N. Ali, A fully implicit finite difference scheme based on extended cubic B-spline for fractional advection-diffusion equation, Advances Diff. Eq., 1, 109, (2018).

[39] M. Abbas, B. Afreen, S. Ahmed, T. Nazir, A. Majeed, and G. Akram, Numerical solutions of third-order time-fractional differential equations using cubic B-spline functions, Fractal and Fractional, 6(9), 528, (2022).

[40] B. Pachpatte, Inequalities for Differential and Integral Equations. Academic Press, 1998.

[41] E. Zeidler, Nonlinear Functional Analysis and Applications, I: Fixed Point Theorems. Springer-Verlag, New York, 1986.

[42] L. Schumaker, Spline Functions: Basic Theory. Cambridge University Press: Cambridge, UK, 2007.

[43] J. Ahlberg, E. Nilson, and J. Walsh, The Theory of Splines and their Applications. Academic Press, New York, 1967.

[44] M. Sakai, Some new consistency relations connecting spline values at mesh and mid points, 23, 543–546, (1983).

[45] G. Vainikko, The convergence of the collocation method for non-linear differential equations, USSR Comput. Math. Math. Phys., 6(1), 47–58, (1966).

[46] A. Malinowska, T. Odzijewicz, and D. Torres, Advanced Methods in the Fractional Calculus of Variations. Springer Briefs in Applied Sciences and Technology, Springer, Cham, 2015.