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Research article

A Bernoulli Operational Matrix Method for Solving Nonlinear Multi-Term Fractional Variable-Order Delay Differential Equation

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Abstract

This paper investigates the generalized non-linear multi-term delay fractional differential equation of variable order. The Bernoulli operational matrix method is utilized to address a category of these equations, transforming the original problem into a system of algebraic equations amenable to numerical solutions. Sufficient and complete numerical tests are presented to showcase the accuracy, generality, and efficiency of the presented technique, as well as the adaptability of this approach. The numerical results of this method are compared with the exact solution. A comparison of this scheme's results with the exact solution illustrates the method's effectiveness and validity.

Keywords: Fractional variable-order delay differential equations, Nonlinear multi-term differential equations, Bernoulli operational matrix, Caputo differential operator

Mathematics Subject Classification (2020): 65M99

1 Introduction

The uses and study of fractional order calculus have been vibrant and among the quickest expanding fields of research over the past thirty years. It has now turned into a vital instrument owing to its extensive application across various scientific fields, including blood flow dynamics, biophysics, chemistry, physics, ongoing thermodynamic variations, the electro-dynamics of complex materials, capacitor principles, polymer rheology, dynamic systems, experimental data fitting, and more ([1–7] and references therein). The growing advancement of suitable and effective techniques for addressing fractional differential equations (FDEs) has generated heightened interest among scholars in this area. Owing to the various uses of these equations, several techniques have been developed to solve them, including Fractional Adams-Moulton methods [8], fractional linear multi-steps methods [9], trapezoidal methods [10], the iterative reproducing kernel Hilbert space method based on the Fibonacci polynomials [11] , and many others.

Introducing delay in fractional order differential equations offers new insights, particularly in bioengineering [12], where the understanding of dynamics occurring in biological tissues is enhanced by fractional derivatives [12, 13]

In mathematical sciences, delay differential equations (DDEs) are a specific type of differential equations where the derivative of an unknown function at a specific moment is expressed in terms of the function's values from previous moments. These DDEs are also referred to as time-delay systems, systems of deed-time or aftereffect, hereditary systems, and equations with deviating arguments [14].

Fractional DDEs differ from the ordinary kind, where the derivative at any given time pertains to the solution (or, in neutral equations, depends on the derivative) at earlier times. Numerous real-world phenomena can be represented by the delay FDEs [5].

Lately, Margado et al in [15] investigated numerical solutions and approximated them for $FDDE$s. The stability regions of $FDDE$ systems are assessed by Čermák et al in [16]. Stability of $FDDE$ systems through Grünwalds approach has been examined by Lazarević and Spansic [17]. Daftardar-Gejji et al in [18] introduced a New Predictor-Corrector technique ($NPCM$) and introduced new iteration technique with Jafari [19], for the numerical solution of FDEs. Bhalekar and Daftardar-Gejji suggested a Predictor-Corrector technique for addressing non-linear $FDDE$s in [20]. In [21], the author expanded the Adams-Bashforth-Moulton algorithm detailed in [3, 22, 23] for solving the $FDDE$s. Varsha et al [12], introduced a new strategy for tackling non-linear $FDDE$s. Ghasemi et al [14], utilized the Reproducing kernel Hilbert Space method for addressing nonlinear $FDDE$s. Khodabandehlo et al. in [24–27] introduced an innovative shifted Jacobi operational matrix technique for nonlinear variable-order $FDDE$s ($VFDDE$s); also, Jhing and Daftardar-Gejji developed a fresh numerical approach for solving $FDDE$s [13].

Finding exact solutions for the majority of FDEs is not straightforward, leading to the need for analytical and numerical techniques to be employed, On the other hand, it is known that acquiring analytical solutions to these equations is quite challenging. Therefore, in many cases, the precise solution remains unknown, and it is necessary to pursue a numerical estimation. Consequently, numerous investigators have devised and advanced numerical techniques to facilitate to acquire estimated solutions for this category of equations.

A recent extension of fractional calculus theory permits the fractional order of derivatives to vary with time, meaning it can be nonconstant or of variable order. In [28–30], the authors explored operators for cases where the fractional derivative's order changes with time. Moreover, there has been a growing interest among researchers in this area (the area of variable FDEs) [31, 32].

Recently, Bernoulli polynomials have demonstrated their strength as a mathematical tool for addressing involving a range of dynamic issues, such as numerically addressing high-order Fredholm integro-differential equations [33], multi-order nonlinear fractional differential equations [34], nonlinear Volterra integro-differential equations [35], fractional convection-diffusion systems in complex 2D and 3D geometries [36], pantograph equations [37], PDEs [38], linear Volterra and nonlinear Volterra-Fredholm-Hammerstein IEs [39], variable-order FDEs [40], alongside optimal control challenges [41].

As far as we know, there has not been any numerical method created utilizing the Bernoulli Operational Matrix approach for the nonlinear variable-order $FDDE$s At present, the primary objective of this article is to extend the classical polynomials in the foundation of the solution.

We propose a Novel Operational Matrix technique that relies on Bernoulli polynomials to compute the solution of variable $FDDE$s numerically. This approach employs the ($NBOM$) method to convert the primary problem into a set of algebraic equations that are solvable by well-known numerical methods. Thus, we present a ($NBOM$) for the derivatives of fractional variable-order for solving a category of $VDDE$s which are as follows:

$$\sum_{s=1}^{n} \beta_s D^{\zeta_s(t)} z(t) + \beta_{n+1} z(t-\tau) = F\left(t, z(t), D^{\zeta_1(s)} z(t), D^{\zeta_2(s)} z(t), \ldots, D^{\zeta_n(s)} z(t), z(t-\tau)\right), \quad 0 \le t \le T, \tag{1}$$

$$z(t) = g(t), \quad t \in [-\tau, 0],$$

$$z(0) = z_0, \tag{2}$$

where $\beta_s \in \mathbb{R}$ $(s = 1,2,3,4,\ldots,n+1)$, $\beta_{n+1} \neq 0$, $T > 0$, and $D^{\zeta_s} z(t)$ $(s = 1,2,3,4,\ldots,n)$ represent the Caputo's derivative of fractional-order variable.

Note 1. If $\zeta_s(t)(s = 1,2,3,4,\ldots,n)$ represent constants, then equations (1) − (2) will appear as follow:

$$\sum_{s=1}^{n} \beta_s D^{\zeta_s} z(t) + \beta_{n+1} z(t-\tau) = F\left(t, z(t), D^{\zeta_1} z(t), D^{\zeta_2} z(t), \ldots, D^{\zeta_n} z(t), z(t-\tau)\right), \quad 0 \le t \le T,$$

$$z(t) = g(t), \quad t \in [-\tau, 0],$$

$$z(0) = z_0.$$

2 Preliminaries

In this part, we evalution several key and essential aspects of fractional calculus theory. Next, we highlight some significant characteristics of Bernoulli polynomials that assist us in formulating the proposed technique.

2.1 The Fractional Order Derivative

Various definitions exist and are utilized for the fractional derivative, yet the three most common definitions of them are presented by Caputo, Grünwald-Letincov, and Riemann-Liouville. Due to the Caputo fractional derivative is the only model that behaves like the integer-order DE, in this article we use it.

Definition 1. *The $\zeta-$order ($m-1<\zeta\leq m$) fractional derivatives of Caputo (right and left-sided) are presented as [42]:*

$$D_{-}^{\zeta}z(t)=\frac{(-1)^{m}}{\Gamma(m-\zeta)}\int_{t}^{T}\frac{z'(s)}{(s-t)^{\zeta-m+1}}ds,$$
$$D_{+}^{\zeta}z(t)=\frac{1}{\Gamma(m-\zeta)}\int_{0}^{t}\frac{z'(s)}{(t-s)^{\zeta-m+1}}ds, \tag{3}$$

that

$$D_{+}^{\zeta}t^{w}=\begin{cases}0, & for\, w\in M_0\, and\, w<\lceil\zeta\rceil,\\ \dfrac{\Gamma(w+1)}{\Gamma(w-\zeta+1)}t^{w-\zeta}, & for\, w\in M_0\, and\, w>\lceil\zeta\rceil,\end{cases} \tag{4}$$

and

$$D_{-}^{\zeta}(T-t)^{w}=\begin{cases}0, & for\, w\in M_0\, and\, w<\lceil\zeta\rceil,\\ \dfrac{(-1)^{w}\Gamma(w+1)}{\Gamma(w-\zeta+1)}(T-t)^{w-\zeta}, & for\, w\in M_0\, and\, w>\lceil\zeta\rceil,\end{cases} \tag{5}$$

where $M_0=\{0,1,2,\dots\}$ and $\lceil . \rceil$ is the ceiling function. Also

$$D_{\pm}^{\zeta}(\gamma\varphi(t)\,+\,\delta\phi(t))\,=\,\gamma D_{\pm}^{\zeta}(\varphi(t))\,+\delta D_{\pm}^{\zeta}(\phi(t)),$$

where $\gamma$and δ are constants.

Definition 2. *The Caputos derivative of fractional-order variable $\zeta(t)$ for $z(t)\in C^{m}[0,T]$ is expressed as [32, 43]:*

$$D^{\zeta(t)}z(t)=\frac{1}{\Gamma(1-\zeta(t))}\int_{0^{+}}^{t}\frac{z'(g)}{(t-g)^{\zeta(t)}}dg+\frac{z(0^{+})-z(0^{-})}{\Gamma(1-\zeta(t))}t^{-\zeta(t)}. \tag{6}$$

In the initial moment and for $0<\zeta(t)<1$, possess:

$$D^{\zeta(t)}z(t)=\frac{1}{\Gamma(1-\zeta(t))}\int_{0^{+}}^{t}\frac{z'(g)}{(t-g)^{\zeta(t)}}dg, \tag{7}$$

and, for the constants kand r, we possess

$$D^{\zeta(t)}(kz_1(t)+brz_2(t))=kD^{\zeta(t)}z_1(t)+rD^{\zeta(t)}z_2(t)\,. \tag{8}$$

Based on Eq. (6), for a fixed C we will have:

$$D^{\zeta(t)}C\,=\,0. \tag{9}$$

On the other hand

$$D^{\zeta(t)}t^{k}=\begin{cases}0, & for\, k\,=\,0,\\ \dfrac{\Gamma(k+1)}{\Gamma(k+1-\zeta(t))}t^{k-\zeta(t)}, & for\, k=1,2,\dots.\end{cases} \tag{10}$$

2.2 Bernoulli Polynomials and Their Properties

The polynomials of Bernoulli form a set of independent polynomials that constitute a complete basis for all square-integrable function spaces over the interval $[0,1]$ (known as the space $L^2[0,1]$).

Let $B_n(s)$ represents the $n-$th degree Bernoulli polynomial in s, is described as follows [40,44]:

$$B_n(s) = \sum_{j=0}^{n} \binom{n}{j} b_{n-i}\, s^j, \tag{11}$$

where $b_j, j=0,1,\ldots,n$, denote the numbers of Bernoulli that found in the series expansion of trigonometric functions [40,45] and can be characterized using the subsequent identification:

$$\frac{s}{e^s-1} = \sum_{j=0}^{\infty} b_j \frac{s^j}{j!}, \tag{12}$$

thus

$$\begin{aligned}
&B_0(s) = 1,\\
&B_1(s) = s - \frac{1}{2},\\
&B_2(s) = s^2 - s + \frac{1}{6},\\
&B_3(s) = s^3 - \frac{3}{2}s^2 + \frac{1}{2}s,\\
&B_4(s) = s^4 - 2s^3 + s^2 - \frac{1}{30},
\end{aligned}$$

the initial five Bernoulli polynomials.

The subsequent property holds for Bernoulli polynomials [45]:

$$\int_0^1 B_k(s)\, B_m(s) = (-1)^{-1+k} \frac{m!\, k!}{(m+k)!} b_{m+k},\ m,k \geq 1.$$

Conversely, the Bernoulli polynomials can be conveniently produced using the subsequent recursive relation

$$\sum_{j=0}^{n-1} \binom{n}{j} B_j(s) = ns^{n-1}\ ,\ j = 2,3,\ldots. \tag{13}$$

We remember that the benefits of Bernoulli polynomials in estimating any unknown arbitrary function, compared to certain traditional orthogonal polynomials, are:

I. The operational matrix of derivatives for Bernoulli contains fewer nonzero elements compared to that of some shifted classical orthogonal polynomials. The nonzero entries of the Bernoulli operational matrix exist solely in the first subdiagonal. In contrast, the shifted Jacobi and Chebyshev polynomials form a strictly lower triangular matrix, as noted in [46,47].

II. The Bernoulli polynomials contain fewer terms than the several traditional orthogonal polynomials. For instance, the sixth Bernoulli polynomial contains five terms, whereas the sixth shifted Chebyshev polynomial consists of seven terms, and this disparity will grow as the degree increases. Thus, when approximating any arbitrary function, we utilize more *CPU* time by employing classical orthogonal polynomials in contrast to Bernoulli polynomials; for further reading refer to [48].

III. The coefficients of separate terms in Bernoulli polynomials are less than such coefficients in the traditional orthogonal polynomials. As the calculation errors in the product are tied to the coefficients of separate terms, employing Bernoulli polynomials reduces these errors.

Therefore, considering the above-mentioned issues and due to its high accuracy and easy implementation, using Bernoulli operational matrix method is economical.

3 Approximating Function using Bernoulli Polynomials

Assume the function $z(t)$ is a random square integrable function ($z(t) \in L^2[0,1]$), consequently, it can be stated in the following form [40]:

$$z(t) = \sum_{j=0}^{\infty} f_j B_j(t), \tag{14}$$

where f_j (the coefficients of the series) are obtained using the formula presented in [39,41] as follows:

$$f_j = \frac{1}{j!} \int_0^1 \frac{d^j z(t)}{dt^j} dt. \tag{15}$$

Thus, we can approximate the solution utilizing $(M+1)-$terms of the given series in Eq. (14) and we shall possess

$$z(t) \simeq z_M(t) = \sum_{j=0}^{M} f_j B_j(t) = F^T \Phi_M(t), \tag{16}$$

where $F = [f_0, f_1, f_2, f_3, \ldots, f_M]^T$ and $\Phi_M(t) = [B_0(t), B_1(t), B_2(t), B_3(t), \ldots, B_M(t)]^T$.

In this context, we presume that

$$S(t) = \begin{bmatrix} 1 \\ t \\ t^2 \\ t^3 \\ t^4 \\ t^5 \\ \vdots \\ t^M \end{bmatrix}. \tag{17}$$

According to Eq. (16)

$$\Phi_M(t) = \begin{bmatrix} B_0(t) \\ B_1(t) \\ \vdots \\ B_i(t) \\ B_{i+1}(t) \\ \vdots \\ B_M(t) \end{bmatrix} = \begin{bmatrix} b_0 \\ \sum_{k=0}^{1} \begin{pmatrix} 1 \\ k \end{pmatrix} b_{1-k} t^k \\ \vdots \\ \sum_{k=0}^{i} \begin{pmatrix} i \\ k \end{pmatrix} b_{i-k} t^k \\ \sum_{k=0}^{i+1} \begin{pmatrix} i+1 \\ k \end{pmatrix} b_{i-k+1} t^k \\ \vdots \\ \sum_{k=0}^{M} \begin{pmatrix} M \\ k \end{pmatrix} b_{M-k} t^k \end{bmatrix} = \begin{bmatrix} \theta_{1,1} & \theta_{1,2} & \cdots & \theta_{1,M+1} \\ \theta_{2,1} & \theta_{2,2} & \cdots & \theta_{2,M+1} \\ \vdots & \vdots & \vdots & \vdots \\ \theta_{i,1} & \theta_{i,2} & \cdots & \theta_{i,M+1} \\ \theta_{i+1,1} & \theta_{i+1,2} & \cdots & \theta_{i+1,M+1} \\ \vdots & \vdots & \vdots & \vdots \\ \theta_{M+1,1} & \theta_{M+1,2} & \cdots & \theta_{M+1,M+1} \end{bmatrix} \begin{bmatrix} 1 \\ t \\ \vdots \\ t^i \\ t^{i+1} \\ \vdots \\ t^M \end{bmatrix} = \Theta S(t), \tag{18}$$

where Θ is a square matrix described as follows:

$$\theta_{l+1,k+1} = \begin{cases} \begin{pmatrix} l \\ k \end{pmatrix} b_{l-k}, \ l \geq k, \\ \\ 0, \ otherwise, \end{cases} \tag{19}$$

for $0 \leq l, k \leq M$.

Then Θ as follows

$$\Theta = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ -\frac{1}{2} & 1 & 0 & 0 & 0 & \cdots & 0 \\ \frac{1}{6} & -1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{2} & -\frac{3}{2} & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & 0 \\ b_M & \binom{M}{1} b_{M-1} & \binom{M}{2} b_{M-2} & \binom{M}{3} b_{M-3} & \binom{M}{4} b_{M-4} & \cdots & 1 \end{bmatrix}_{(M+1)\times(M+1)}. \tag{20}$$

Hence, using Eq. (18), we get

$$S(t) = \Theta^{-1}\,\Phi_M(t). \tag{21}$$

4 Novel Bernoulli Polynomials Operational Matrix $(NBOM)$

Operational matrices, utilized across various fields of numerical analysis, address diverse issues of various kinds and subjects are particularly significant, including IEs, DEs, and integro-differential equations, partial and ordinaryFDEs [42, 49–57]. Now, we explore the $(NBOM)$ of fractional order to assist in the computational solution of Eqs.$(1)-(2)$. Thus, we transform the original problem into a set of algebraic equations that can be solved using numerical techniques in collocation points.

Initially, we infer $D^{\zeta_i(t)}\Phi_M(t), (i = 1, 2, \ldots n)$ as follows:
based on the previous content, have: $\Phi_M(t) = \Theta S(t)$, so

$$D^{\zeta_i(t)}\Phi_M(t) = D^{\zeta_i(t)}(\Theta S(t)) = \Theta D^{\zeta_i(t)}\,[1, t, \ldots, t^M]^T, \quad i = 1, 2, 3, 4, \ldots, n. \tag{22}$$

Combining Eqs. (10) and (22), in general, it gives:

$$\begin{aligned} D^{\zeta_i(t)}\Phi_M(t) &= \Theta D^{\zeta_i(t)}(S(t)) = \Theta\left[0, \frac{\Gamma(2)t^{(1-\zeta_i(t))}}{\Gamma(2-\zeta_i(t))}, \frac{\Gamma(3)t^{(2-\zeta_i(t))}}{\Gamma(3-\zeta_i(t))}, \ldots, \frac{\Gamma(1+M)t^{(M-\zeta_i(t))}}{\Gamma(1+M-\zeta_i(t))}\right]^T \\ &= \Theta \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & \frac{\Gamma(2)t^{-\zeta_i(t)}}{\Gamma(2-\zeta_i(t))} & 0 & \cdots & 0 \\ 0 & 0 & \frac{\Gamma(3)t^{-\zeta_i(t)}}{\Gamma(3-\zeta_i(t))} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{\Gamma(1+M)t^{-\zeta_i(t)}}{\Gamma(1+M-\zeta_i(t))} \end{bmatrix} \begin{bmatrix} 1 \\ t \\ t^2 \\ \vdots \\ t^M \end{bmatrix} \\ &= \Theta Q_i(t) S(t), \end{aligned} \tag{23}$$

where

$$Q_i(t) = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & \frac{\Gamma(2)t^{-\zeta_i(t)}}{\Gamma(2-\zeta_i(t))} & 0 & \cdots & 0 \\ 0 & 0 & \frac{\Gamma(3)t^{-\zeta_i(t)}}{\Gamma(3-\zeta_i(t))} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{\Gamma(1+M)t^{-\zeta_i(t)}}{\Gamma(1+M-\zeta_i(t))} \end{bmatrix}. \tag{24}$$

Using Eq. (21), then

$$D^{\zeta_i(t)}\Phi_M(t) = \Theta Q_i(t)\,\Theta^{-1}\Phi_M(t)\,, i = 1, 2, 3, 4, \ldots, n. \tag{25}$$

The operational matrix of $D^{\zeta_i(t)}\Phi_M(t)$, is $\Theta Q_i(t)\,\Theta^{-1}$.

Here, we estimate the variable-order fractional of the calculated function that obtained in Eq. (16) as follows

$$D^{\zeta_i(t)}z(t) \simeq D^{\zeta_i(t)}(A^T\Phi_M(t)) = A^T D^{\zeta_i(t)}\Phi_M(t) = A^T\Theta Q_i(t)\,\Theta^{-1}\Phi_M(t), \quad i=1,2,3,4,\dots,n. \tag{26}$$

By using Eq. (26), hence the Eq. (1) turned into

$$\sum_{i=1}^{n}\beta_i(A^T\Theta Q_i(t)\,\Theta^{-1}\Phi_M(t)) + \beta_{n+1}A^T\Phi_M(t-\tau) = F\,(t, A^T\Phi_M(t), (A^T\Theta Q_1(t)\,\Theta^{-1}\Phi_M(t)), (A^T\Theta Q_2(t)\,\Theta^{-1}\Phi_M(t)),\dots, (A^T\Theta Q_n(t)\,\Theta^{-1}\Phi_M(t)), A^T\Phi_M(t-\tau)), \quad 0\le t\le T, \tag{27}$$

with condition

$$A^T\Phi_M(0) = g(0).$$

Ultimately, we utilize t_k $(k = 0,1,2,3,\dots,M)$ where selected as $t_k = \dfrac{T(2k+1)}{2M+1}$ (collocation point). Consequently Eq. (27) converted into the following form

$$\sum_{i=1}^{n}\beta_i(A^T\Theta Q_i(t_k)\,\Theta^{-1}\Phi_M(t_k)) + \beta_{n+1}A^T\Phi_M(t_k-\tau) = F\,(t_k, A^T\Phi_M(t_k), (A^T\Theta Q_1(t_k)\,\Theta^{-1}\Phi_M(t_k)), (A^T\Theta Q_2(t_k)\,\Theta^{-1}\Phi_M(t_k)),\dots, (A^T\Theta Q_n(t_k)\,\Theta^{-1}\Phi_M(t_k)), A^T\Phi_M(t_k-\tau)), \quad k=0,1,2,3,4,\dots,M. \tag{28}$$

We can numerically solve the algebraic system in Eq. (28) to identify the matrix A using Newtons iteration method for nonlinear problems and Solve function of Mathematica 13 software for linear problems. Thus, the numerical solution provided in Eq. (16) can be achieved.

5 Numerical Experiences

Through demonstrating various examples in this section, we illustrate the effectiveness of the method utilizing the Mathematica 13 software.

To evaluate our approach, we implement our technique on several variable-order $FDDE$s. These examples have been examined in terms of Absolute ($E_A = \mid Z_{Exact}(t_i) - Z_M(t_i)\mid$) and Relative Errors ($E_R = \dfrac{\mid Z_{Exact}(t_i) - Z_M(t_i)\mid}{Z_{Exact}(t_i)}$) and the time needed to compute their solution.

Collation of the outcomes produced by this method with the precise solution and with different other techniques (found in the literature) for each example indicates that this innovative technique aligns most closely with the exact solution. The consistency, stability, and straightforward application of this technique make it more usable and dependable.

Example 1. Examine the subsequent $FDDE$ for $\tau > 0, 0 < \zeta \le 1$,

$$D^{\zeta}z(t) = \frac{2z(t)^{1-\dfrac{\zeta}{2}}}{\Gamma(3-\zeta)} + z(t-\tau) - z(t) + 2\tau\sqrt{z(t)} - \tau^2, \quad z(t)=0, \quad t\le 0. \tag{29}$$

Note that $z(t) = t^2$ is the exact solution and $0\le t\le 1$, $\tau = 0.3$, $\zeta = 0.6$.

Utilizing the ideas outlined in Section 4, we regard the approximate solution featuring $(M+1)$ finite terms shown in Eq. (16) for this problem and replace in primary problem. Then by applying the Eq. (26), this problem is transformed into form of the Eq. (27), finally, utilizing t_i, we derive a system of equations that can be solved using established numerical techniques to obtain the unknown matrix A.

The Relative and Absolute Errors (at specific nodal points) of this approach, along with the required CPU time, are presented in Tables 1 and 2. From these Tables, it can be seen that the numerical outcomes are nearly identical to the exact solution, and Figure 1 confirms the reliability of $NBOM$ method compared to other techniques. Furthermore, in Figure 2 we display the Absolute Error for this case. It is important to observe that Figures 1 and 2 illustrate a good alignment between the approximate solutions and the true solution.

In this instance for $M=3$ and $M=6$, we have

$$A = [0.33333, 1, 1, -1.40032\times 10^{-9}]^T,$$

$$A = [0.33333, 1, 1, 0.66667, -7.0096\times 10^{-15}, 5.4443\times 10^{-15}, -1.32121\times 10^{-14}, -4.5030\times 10^{-15}]^T,$$

respectively.

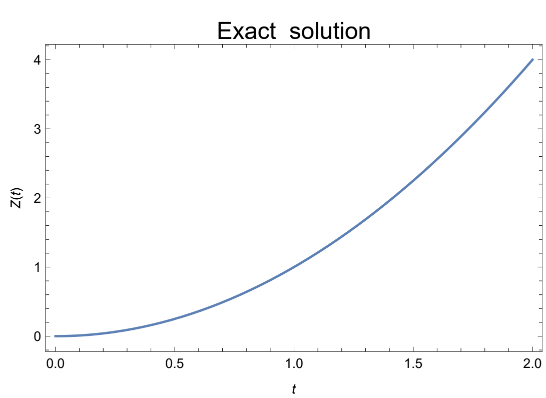
Therefore, it follows that the solution of the Eq. (29), estimated using new method, is in the best agreement with the accurate solution.

Table 1. Comparison the Absolute Errors at some nodal points with different M for Example 1.

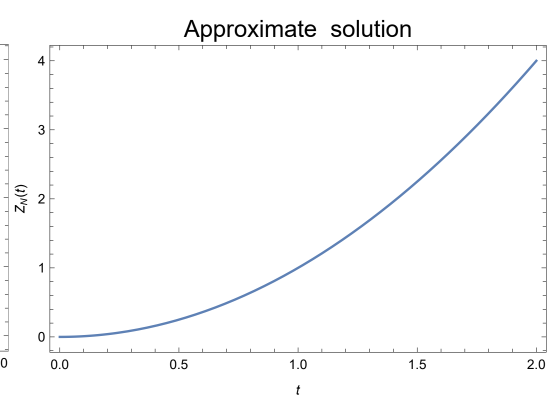
t_i	Our technique, $M=3$	Our technique, $M=6$
0.1	5.06903×10^{-10}	2.09902×10^{-16}
0.2	1.05563×10^{-9}	9.71445×10^{-17}
0.3	1.46820×10^{-9}	4.16334×10^{-17}
0.4	1.81513×10^{-9}	1.66533×10^{-16}
0.5	2.10484×10^{-9}	1.66533×10^{-16}
0.6	2.34571×10^{-9}	5.55113×10^{-17}
0.7	2.54617×10^{-9}	5.55113×10^{-17}
0.8	2.71460×10^{-9}	1.11022×10^{-16}
0.9	2.85941×10^{-9}	2.22045×10^{-16}
1.0	2.89800×10^{-9}	4.44089×10^{-16}
CPU time	$0.09375\,s$	$0.23437\,s$

Table 2. Comparison the Relative Errors at some nodal points with different M for Example 1.

t_i	Our technique, $M=3$	Our technique, $M=6$
0.1	5.69033×10^{-8}	2.09902×10^{-14}
0.2	2.63908×10^{-8}	2.42861×10^{-15}
0.3	1.63133×10^{-8}	4.62533×10^{-16}
0.4	1.13446×10^{-8}	1.40830×10^{-15}
0.5	8.41934×10^{-9}	6.66134×10^{-16}
0.6	6.51587×10^{-9}	1.66427×10^{-16}
0.7	5.54617×10^{-9}	1.25322×10^{-16}
0.8	4.24156×10^{-9}	1.84208×10^{-16}
0.9	3.53013×10^{-9}	2.43110×10^{-16}
1.0	3.09650×10^{-9}	4.02019×10^{-16}
CPU time	$0.09375\,s$	$0.23437\,s$



(a) Exact solution



(b) Approximate solution

Figure 1. Comparison of between approximate solution(z_6) of *NBOM* method and exact solution for Example 1.

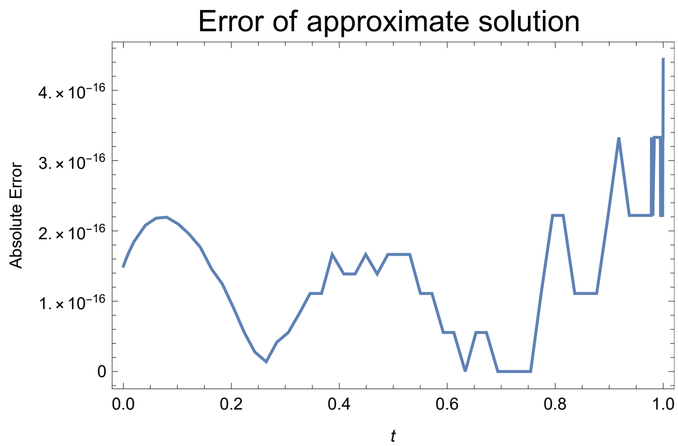


Figure 2. The absolute errors comparison between numerical solution(z_6) of *NBOM* scheme and exact solution for Example 1.

Example 2. Take into account the variable-order *FDDE* under periodic conditions

$$D^{\zeta_2(t)}z(t)+D^{\zeta_1(t)}z(t)-z(t-\tau)-z(t)-2\tau z(t)^3=$$

$$\frac{\Gamma(3)z(t)^{2-\zeta_2(t)}}{\Gamma(3-\zeta_2(t))}-\frac{\Gamma(2)z(t)^{1-\zeta_2(t)}}{\Gamma(2-\zeta_2(t))}+\frac{\Gamma(3)z(t)^{2-\zeta_1(t)}}{\Gamma(3-\zeta_1(t))}-\frac{\Gamma(2)z(t)^{1-\zeta_1(t)}}{\Gamma(2-\zeta_1(t))}-2t^2+2t\tau-\tau^2+\tau-2\tau(t^2-t)^3, \tag{30}$$

$$z(t)=t^2-t,\ t\in[-\tau,0],$$

$$z(0)=z(T).$$

In above problem $z(t)=t^2-t$ is the true solution and $0\leq t\leq T$, $T=1$, $\zeta_1(t)=0.5t$, $\zeta_2(t)=0.25t$ and $\tau=1$.

Like the previous example, in this example, the E_A and E_R (at some nodal points) of this method, as well as the *CPU* time required to compute them are given and compared in Tables 3 and 4. Figures 3–5 show the efficiency and the reliability of *NBOM* technique. By looking closely at these tables and figures, we realize that the numerical results are in the closest alignment with the precise solution.

In this instance, we have for $M=3$ and $M=5$, we have

$$A=[-0.166667,1.63433\times10^{-17},1,-6.99553\times10^{-18}]^T,$$

$$A=[-0.166667,-8.15977\times10^{-19},1,5.63725\times10^{-16},-7.67433\times10^{-16},-3.09957\times10^{-15}]^T,$$

respectively.

Example 3. Consider the below *FDDE* for $0<\zeta\leq1$,

$$D^{\zeta}z(t)-z(t-\tau)+z(t)-5\tau z(t)^2=g(t),$$

$$g(t)=\frac{2\exp(t)(-1+t)}{1+\exp(2)}-\frac{2\exp(t-\tau)(-1+t-\tau)}{1+\exp(2)}-5(1-\frac{2\exp(2)}{1+\exp(2)}+\frac{2\exp(t)(-1+t)}{1+\exp(2)})^2\tau$$

$$-\frac{2t^{\zeta}(-2+\zeta)(t^2+\exp(t)t^{\tau}(-1+t+\zeta)\Gamma(2-\eta)-\exp(t)t^{\zeta}(-1+t+\zeta)\Gamma(2-\zeta,t))}{\Gamma(3-\zeta)(1+\exp(2))}, \tag{31}$$

$$z(t)=\frac{2\exp(t)(-1+t)}{1+\exp(2)}-\frac{2\exp(2)}{1+\exp(2)}+1,\ t\in[-\tau,0],$$

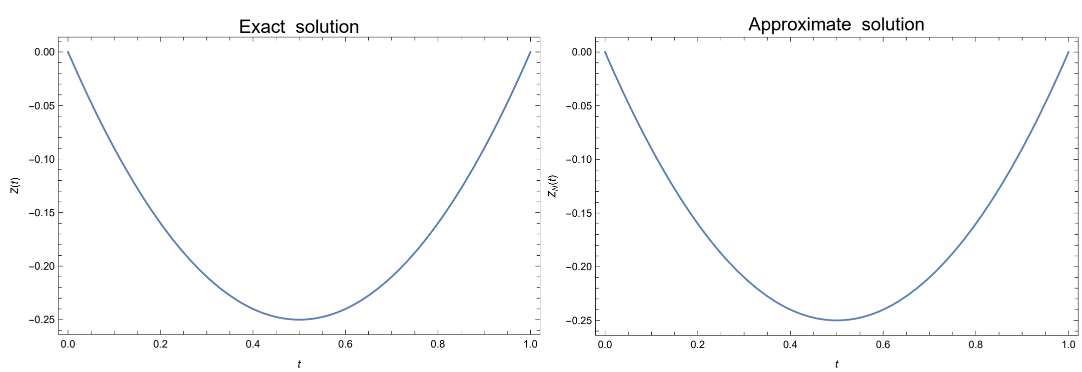
$$z(0)=-z(T).$$

Table 3. Comparison the absolute errors at some nodal points with different M for Example 2.

t_i	Our technique, $M = 3$	Our technique, $M = 5$
0.1	8.3266×10^{-17}	1.38778×10^{-16}
0.2	9.7144×10^{-17}	1.38778×10^{-16}
0.3	8.3266×10^{-17}	1.11022×10^{-16}
0.4	5.5511×10^{-17}	2.77556×10^{-17}
0.5	1.3877×10^{-16}	0
0.6	1.1102×10^{-16}	5.55113×10^{-17}
0.7	1.1102×10^{-16}	2.77556×10^{-17}
0.8	8.3266×10^{-17}	8.32667×10^{-17}
0.9	1.3877×10^{-16}	1.38778×10^{-17}
1.0	8.3266×10^{-17}	3.74079×10^{-17}
CPU time	$0.1285\,s$	$0.125\,s$

Table 4. Comparison the relative errors at some nodal points with different M for Example 2.

t_i	Our technique, $M = 3$	Our technique, $M = 5$
0.1	1.07936×10^{-15}	1.54198×10^{-15}
0.2	5.2041×10^{-16}	8.67362×10^{-16}
0.3	2.6433×10^{-16}	5.28678×10^{-16}
0.4	5.7824×10^{-16}	1.15648×10^{-16}
0.5	4.4408×10^{-16}	0
0.6	4.6259×10^{-16}	2.31296×10^{-16}
0.7	3.9650×10^{-16}	1.32169×10^{-16}
0.8	8.6726×10^{-16}	5.20417×10^{-16}
0.9	9.25186×10^{-16}	1.54198×10^{-16}
CPU time	$0.0625\,s$	$0.125\,s$



(a) Exact solution **(b)** Approximate solution

Figure 3. Comparison of between approximate solution(z_3) of *NBOM* method and exact solution for Example 2.

With anti-periodic condition type. The precise solution is

$$z(t) = \frac{2\exp(t)(-1+t)}{1+\exp(2)} - \frac{2\exp(2)}{1+\exp(2)} + 1,$$

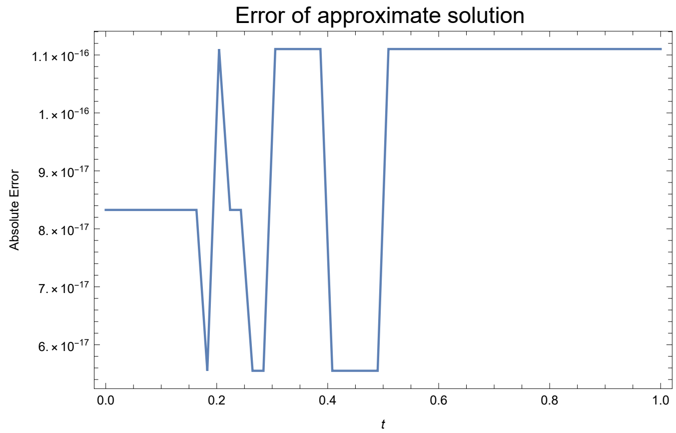


Figure 4. The absolute errors comparison between numerical solution(z_3) of $NBOM$ scheme and exact solution for Example 2.

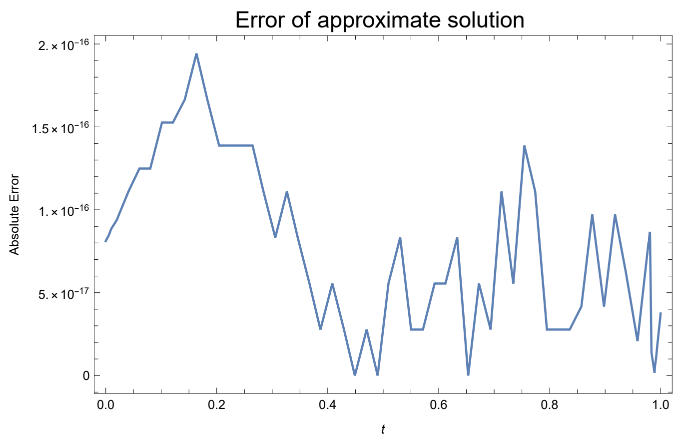


Figure 5. The absolute errors comparison between numerical solution(z_5) of $NBOM$ scheme and exact solution for Example 2.

and $0 \leq t \leq T\,, T = 2, \tau = 0.01\exp(-t)\,, \zeta = 0.2$.

We solve Example 3 in a similar manner to the previous examples and obtain the results. Then, we set up tables of absolute and relative errors and CPU time needed for our method, as shown in Tables 5 and 6. We also draw Figures 5 and 6 to demonstrate the accuracy and efficiency of the proposed method.

In this example for $M = 15$, we have

$$A = [0.33333, 1, 1, -1.40032 \times 10^{-9}]^T,$$

$$A = [-0.622464, 0.396745, 0.192682, 0.049892, 0.0087147, 0.00114289, 0.000119683, 0.0000104165,$$

$$7.74935 \times 10^{-7}, 5.03127 \times 10^{-8}, 2.89662 \times 10^{-9}, 1.49774 \times 10^{-10}, 7.01614 \times 10^{-12}, 3.10203 \times 10^{-13},$$

$$8.84712 \times 10^{-15}, 3.77588 \times 10^{-15}]^T.$$

6 Conclusions

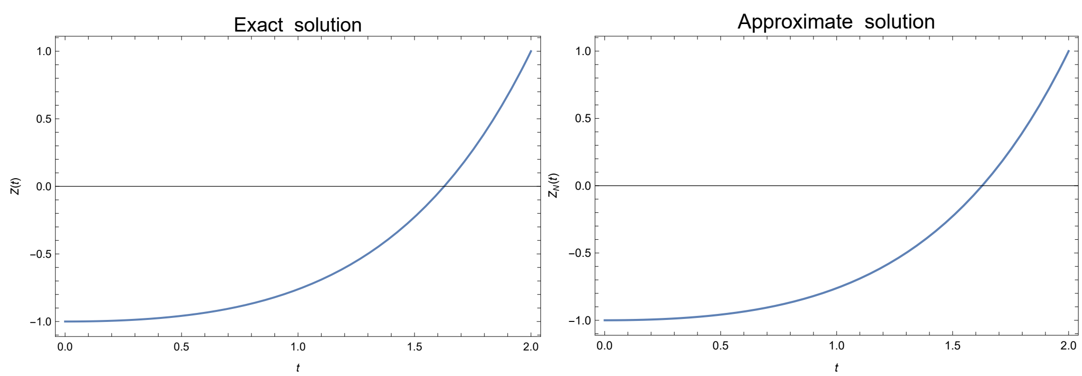
In this paper, the fundamental objective of the paper is to introduce a novel computational methods based on Bernoulli operational matrix $(NBOM)$ for the generalized non-linear variable-order $FDDE$s Eqs.(1)–(2). We investigated the generalized $VFDDE$s by the $(NBOM)$ scheme. In this method, an algebraic equations system is obtained and solved by using a appropriate numerical method. The precision and effectiveness of this method is shown by solving some numerical examples.

Table 5. Comparison the absolute errors at some nodal points with different M for Example 3.

t_i	Our technique, $M=15$	Our technique, $M=10$
0.2	9.3258×10^{-15}	1.0248×10^{-10}
0.4	8.7929×10^{-14}	3.8000×10^{-9}
0.6	4.24105×10^{-14}	9.0013×10^{-10}
0.8	2.5313×10^{-14}	7.8627×10^{-10}
1.0	2.2204×10^{-14}	6.3847×10^{-10}
1.2	2.0206×10^{-14}	6.4103×10^{-10}
1.4	1.8651×10^{-14}	6.8871×10^{-10}
1.6	1.8207×10^{-14}	1.0665×10^{-9}
1.8	1.9984×10^{-14}	6.2548×10^{-17}
2.0	6.7723×10^{-14}	1.0248×10^{-17}
CPU time	$2.0156\,s$	$0.6224\,s$

Table 6. Comparison the relative errors at some nodal points with different M for Example 3.

t_i	Our technique, $M=15$	Our technique, $M=10$
0	1.025×10^{-15}	1.024×10^{-10}
0.2	3.779×10^{-14}	3.779×10^{-9}
0.4	9.779×10^{-15}	8.776×10^{-10}
0.6	6.661×10^{-15}	7.354×10^{-10}
0.8	3.614×10^{-15}	5.453×10^{-10}
1.0	6.661×10^{-15}	4.882×10^{-10}
1.2	5.453×10^{-15}	4.154×10^{-10}
1.4	4.185×10^{-15}	3.998×10^{-10}
1.6	3.271×10^{-15}	3.321×10^{-10}
1.8	6.981×10^{-15}	4.185×10^{-10}
2.0	2.003×10^{-15}	1.024×10^{-10}
CPU time	$0.2625\,s$	$0.125\,s$



(a) Exact solution **(b)** Approximate solution

Figure 6. Comparison of between approximate solution(z_{15}) of *NBOM* method and exact solution for Example 3.

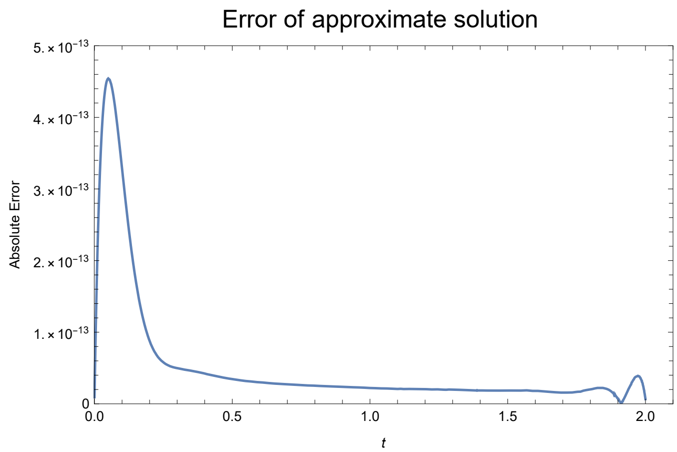


Figure 7. The absolute errors comparison between numerical solution(z_{15}) of *NBOM* scheme and exact solution for Example 3.

Authors' Contributions

All authors have the same contribution.

Data Availability

The manuscript has no associated data or the data will not be deposited.

Conflicts of Interest

The authors declare that there is no conflict of interest.

Ethical Considerations

The authors have diligently addressed ethical concerns, such as informed consent, plagiarism, data fabrication, misconduct, falsification, double publication, redundancy, submission, and other related matters.

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References

[1] D. Bojović and J. Boško, Fractional order convergence rate estimates of finite difference method on nonuniform meshes, J. Comput. Meth. Appl. Math., 1(3), 213–221, (2001).

[2] D. Baleanu, R. L. Magin, S. Bhalekar, and V. Daftardar-Gejji, Chaos in the fractional order nonlinear Bloch equation with delay, Commun. Nonlinear. Sci. Numer. Simul., 25(1-3), 41–49, (2015).

[3] K. Diethelm, N. J. Ford, and A. D. Freed, Detailed error analysis for a fractional Adams method, Numer. Algorithms, 36(1), 31–52, (2004).

[4] Y. Kuang, Delay differential equations: with applications in population dynamics, vol. 191, Academic Press, 1993.

[5] M. Saedshoar Heris and M. Javidi, On fractional backward differential formulas for fractional delay differential equations with periodic and anti-periodic conditions, Appl. Numer. Math., 118, 203–220, (2017).

[6] E. Shivanian, A. Jafarabadi, T. G. Chegini, and A. Dinmohammadi, Analysis of a time-dependent source function for the heat equation with nonlocal boundary conditions through a local meshless procedure, Computational and Applied Mathematics, 44(6), 282, (2025).

[7] N. Hosseinzadeh, E. Shivanian, M. Z. Fairooz, and T. G. Chegini, A robust RBF-FD technique combined with polynomial enhancements for valuing European options in jump-diffusion frameworks, International Journal of Dynamics and Control, 13(6), 212, (2025).

[8] C. Lubich, Discretized fractional calculus, SIAM J. Math. Anal., 17(3), 704–719, (1984).

[9] L. Galeone and R. Garrappa, On multistep methods for differential equations of fractional order, Mediterr. J. Math., 3(3-4), 565–580, (2006).

[10] R. Garrappa, Trapezoidal methods for fractional differential equations: theoretical and computational aspects, Math. Comput. Simul., 110, 96–112, (2015).

[11] B. Azarnavid, M. Nabati, and M. Emamjomeh, The iterative reproducing kernel Hilbert space method based on the Fibonacci polynomials for the nonlinear fractional differential equations with fractional integral conditions, Journal of Advanced Mathematical Modeling, 13(3), 344–354, (2023).

[12] V. Daftardar-Gejji, Y. Sukale, and S. Bhalekar, Solving fractional delay differential equations: a new approach, Frac. Calculus Appl. Anal., 18(2), 400–418, (2015).

[13] A. Jhinga and V. Daftardar-Gejji, A new numerical method for solving fractional delay differential equations, J. Comput. Appl. Math., 38, 166, (2019).

[14] M. Ghasemi, M. Fardi, and R. Khoshsiar Ghaziani, Numerical solution of nonlinear delay differential equations of fractional order in reproducing kernel Hilbert space, Appl. Math. Comput., 268, 815–831, (2015).

[15] M. L. Morgado, N. J. Ford, and P. Lima, Analysis and numerical methods for fractional differential equations with delay, J. Comput. Appl. Math., 252, 159–168, (2013).

[16] J. Čermák, J. Horníček, and T. Kisela, Stability regions for fractional differential systems with a time delay, Commun. Nonlinear Sci. Numer. Simul., 31(1), 108–123, (2016).

[17] M. P. Lazarević and A. M. Spasić, Finite-time stability analysis of fractional order time-delay systems: Gronwall's approach, Math. Comput. Model., 49(3), 475–481, (2009).

[18] V. Daftardar-Gejji and H. Jafari, An iterative method for solving non linear functional equations, J. Math. Anal. Appl., 316, 753–763, (2006).

[19] V. Daftardar-Gejji, Y. Sukale, and S. Bhalekar, A new predictor-corrector method for fractional differential equations, Appl. Math. Comput., 244, 158–182, (2014).

[20] S. Bhalekar and V. Daftardar-Gejji, A predictor-corrector scheme for solving non-linear delay differential equations of fractional order, J. Frac. Calculus Appl., 1(5), 1–8, (2011).

[21] Z. Wang, A numerical method for delayed fractional-order differential equations, J. Appl. Math., 2013, 256071, (2013).

[22] K. Diethelm, N. J. Ford, and A. D. Freed, A predictor-corrector approach for the numerical solution of fractional differential equations, Nonlinear Dynamics, 29, 3–22, (2002).

[23] K. Diethelm and N. J. Ford, Analysis of fractional differential equations, J. Math. Anal., 265(2), 229–248, (2002).

[24] H. R. Khodabandehlo, E. Shivanian, and S. Abbasbandy, Numerical solution of nonlinear delay differential equations of fractional variable-order using a novel shifted Jacobi operational matrix, Eng. Comp., 38(Suppl 3), S2593–S2607, (2022).

[25] H. R. Khodabandelo, E. Shivanian, and S. Abbasbandy, A novel shifted Jacobi operational matrix method for nonlinear multi-terms delay differential equations of fractional variable-order with periodic and anti-periodic conditions, Math. Meth. Appl. Sci., 45(16), 10116–10135, (2022).

[26] H. R. Khodabandehlo, E. Shivanian, and S. Abbasbandy, A Novel Shifted Jacobi Operational Matrix for Solution of Nonlinear Fractional Variable-Order Differential Equation with Proportional Delays, Int. J. Indust. Math., 14(4), 415–432, (2022).

[27] H. R. Khodabandehlo, E. Shivanian, and S. Abbasbandy, A novel shifted Jacobi operational matrix method for linear multi-terms delay differential equations of fractional variable-order with periodic and anti-periodic conditions, Kragujevac Journal of Mathematics, 50(1), 39–69, (2026).

[28] S. G. Samko and B. Ross, Integration and differentiation to a variable fractional order, Integr. Transf. Spec. Funct., 1, 277–300, (1993).

[29] T. Odzijewicz, A. B. Malinowska, and D. F. M. Torres, Noethers theorem for fractional variational problems of variable order, Cent. Eur. J. Phys., 11, 691–701, (2013).

[30] S. Chen, F. Liu, and K. Burrage, Numerical simulation of a new two-dimensional variable-order fractional percolation equation in non-homogeneous porous media, Comput. Math. Appl., 67, 1673–1681, (2014).

[31] D. Tavares, R. Almeida, and D. F. M. Torres, Caputo derivatives of fractional variable order: numerical approximations, Commun. Nonlinear Sci. Numer. Simul., 35, 69–87, (2016).

[32] J. Liu, X. Li, and L. Wu, An operational matrix of fractional differentiation of the second kind of Chebyshev polynomial for solving multi-term variable order fractional differential equation, Math. Probl. Eng., 2016, 7126080, (2016).

[33] A. H. Bhrawy, E. Tohidi, and F. Soleymani, A new Bernoulli matrix method for solving high-order linear and nonlinear Fredholm integro-differential equations with piecewise intervals, Appl. Math. Comput., 219, 482–497, (2012).

[34] B. Azarnavid, M. Emamjomeh, M. Nabati, and A. Dinmohammadi, An efficient iterative method for multi-order nonlinear fractional differential equations based on the integrated Bernoulli polynomials, Computational and Applied Mathematics, 43(1), 43–68, (2024).

[35] B. Azarnavid, The Bernoulli polynomials reproducing kernel method for nonlinear Volterra integro-differential equations of fractional order with convergence analysis, Computational and Applied Mathematics, 42(1), (2022).

[36] M. Fardi, M. A. Zaky, and B. Azarnavid, Fractional Convection-Diffusion Systems in Complex 2D and 3D Geometries: A Bernoulli Polynomial-Based Kernel Method, Applied Numerical Mathematics, 218, 275–297, (2025).

[37] E. Tohidi, A. H. Bhrawy, and K. A. Erfani, Collocation method based on Bernoulli operational matrix for numerical solution of generalized pantograph equation, Appl. Math. Model., 37, 4283–4294, (2013).

[38] F. Toutounian and E. Tohidi, A new Bernoulli matrix method for solving second order linear partial differential equations with the convergence analysis, Appl. Math. Comput., 223, 298–310, (2013).

[39] S. Bazm, Bernoulli polynomials for the numerical solution of some classes of linear and nonlinear integral equations, J. Comput. Appl. Math., 275, 44–60, (2015).

[40] S. Nemati, P. M. Lima, and D. F. M. Torres, Numerical Solution of Variable-Order Fractional Differential Equations Using Bernoulli Polynomials, Fractal Fract., 5, 219, (2019).

[41] E. Keshavarz, Y. Ordokhani, and M. A. Razzaghi, Numerical solution for fractional optimal control problems via Bernoulli polynomials, J. Vib. Control, 22, 3889–3903, (2016).

[42] A. H. Bhrawy and M. A. Zaky, A method based on the Jacobi tau approximation for solving multi-term time-space fractional partial differential equations, J. Comput. Phys., 281, 876–895, (2015).

[43] Y. M. Chen, L. Q. Liu, B. F. Li, and Y. N. Sun, Numerical solution for the variable-order linear cable equation with Bernstein polynomials, Appl. Math. Comput., 238, 329–341, (2014).

[44] F. Costabile, F. Dellaccio, and M. I. Gualtieri, A new approach to Bernoulli polynomials, Rend. Mat. Ser. VII, 26, 1–12, (2006).

[45] G. Arfken, Mathematical Methods for Physicists, 3rd ed., Academic Press, 1985.

[46] E. H. Doha, A. H. Bhrawy, and S. S. Ezz-Eldien, A new Jacobi operational matrix: an application for solving fractional differential equations, Appl. Math. Model., 36, 4931–4943, (2012).

[47] E. H. Doha, A. H. Bhrawy, and S. S. Ezz-Eldien, A Chebyshev spectral method based on operational matrix for initial and boundary value problems of fractional order, Comput. Math. Appl., 62, 2364–2373, (2011).

[48] S. Mashayekhi, Y. Ordokhani, and M. Razzaghi, Hybrid functions approach for nonlinear constrained optimal control problems, Commun. Nonlinear. Sci. Numer. Simulat., 17, 1831–1843, (2012).

[49] S. A. Yousefi and M. Behroozifar, Operational matrices of Bernstein polynomials and their applications, Inter. Systems Sci., 32, 709–716, (2010).

[50] W. Labecca, O. Guimaraes, and J. R. C. Piqueira, Dirac's formalism combined with complex Fourier operational matrices to solve initial and boundary value problems, Commun. Nonlinear Sci. Numer. Simul., 19(8), 2614–2623, (2014).

[51] M. Razzaghi and S. Yousefi, Legendre wavelets method for the nonlinear Volterra-Fredholm integral equations, Math. Comput. Simul., 70, 1–8, (2005).

[52] H. Danfu and S. Xufeng, Numerical solution of integro-differential equations by using CAS wavelet operational matrix of integration, Appl. Math. Comput., 194, 460–466, (2007).

[53] S. H. Behiry, Solution of nonlinear Fredholm integro-differential equations using a hybrid of block pulse functions and normalized Bernstein polynomials, J. Comput. Appl. Math., 260, 258–265, (2014).

[54] A. Saadatmandi and M. Dehghan, A new operational matrix for solving fractional-order differential equations, Comput. Math. Appl., 59, 1326–1336, (2010).

[55] A. Saadatmandi, Bernstein operational matrix of fractional derivatives and its applications, Appl. Math. Model., 38, 1365–1372, (2014).

[56] M. H. Atabakzadeh, M. H. Akrami, and G. H. Erjaee, Chebyshev operational matrix method for solving multi-order fractional ordinary differential equations, Appl. Math. Model., 37, 8903–8911, (2013).

[57] A. H. Bhrawy and A. S. Alofi, The operational matrix of fractional integration for shifted Chebyshev polynomials, Appl. Math. Lett., 26, 25–31, (2013).