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Some Remarks on the Varieties of Groups

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Abstract Let $\mathcal{V}$ be a variety of groups defined by a set V of laws. Let (N, G) be a pair of groups in which N is a normal subgroup of G. We define the lower and upper $\mathcal{V}$-marginal series of the pair (N, G) and prove some results on $\mathcal{V}$-nilpotent pairs of groups. Moreover, we extend some properties of the Baer-invariant and isologism of a pair of groups.

Keywords Pair of groups · Baer-invariant · Isologism

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1 Introduction and Preliminaries

Let F be a free group on a set $\{x_1, x_2, \ldots\}$, V be a non-empty subset of F and $\mathcal{V}$ be a variety of groups defined by a set V of laws. For a pair of groups (N, G) in which N is a normal subgroup of G, we define

$$V(N, G) = \langle v(g_1, \ldots, g_i n, \ldots, g_r)v(g_1, \ldots, g_r)^{-1} : v \in V,\ n \in N,\ g_i \in G,\ 1 \le i \le r\rangle,$$

and

$$V^*(N, G) = \{n \in N : v(g_1, \ldots, g_i n, \ldots, g_r) = v(g_1, \ldots, g_r), \forall v \in V,\ g_i \in G,\ 1 \le i \le r\}.$$

In particular, if $N = G$, then $V(N, G) = V(G)$ and $V^*(N, G) = V^*(G)$ are ordinary verbal and marginal subgroups of G (see [5,10,15]).

If $\mathcal{V}$ is the variety of nilpotent groups of class at most n, then

$$V^*(N, G) = Z_n(N, G) \quad \textit{and} \quad V(N, G) = \gamma_{n+1}(N, G),$$

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where $\gamma_1(N,G) = N$ and $\gamma_{n+1}(N,G) = [\gamma_n(N,G), G]$ for all $n \geq 1$. Moreover, $Z_0(N,G) = 1$ and for all $n \geq 1$,

$$\frac{Z_n(N,G)}{Z_{n-1}(N,G)} = Z(\frac{N}{Z_{n-1}(N,G)}, \frac{G}{Z_{n-1}(N,G)}). \tag{1}$$

Let $\mathcal{V}$ and $\mathcal{W}$ be two arbitrary varieties of groups defined by the sets of laws V and W, respectively, and G a group with a free presentation

$$1 \to R \to F \to G \to 1.$$

Then, we define the Baer-invariant of G with respect to two varieties $\mathcal{V}$ and $\mathcal{W}$ as follows

$$\mathcal{WVM}(G) = \frac{\big(R \cap W(F)\big)\big(R \cap V(F)\big)}{\big(R \cap W(F)\big)[RV^*F]},$$

where $[RV^*F]$ is the least normal subgroup T say, of F contained in R so that $R/T \subseteq V^*(F/T)$ (see [9] for more information). In this paper, we are going to generalize some results of [3, 7, 14, 15].

2 Some inequalities on the Baer-invariant of a pair of groups

In this section, we extend some results of [3, 14, 15]. To this end, we define the lower and the upper $\mathcal{V}$-marginal series of a pair (N,G). Moreover, we prove some results on nilpotent pairs of groups with respect to a variety of groups. Also, we prove some properties on the Baer-invariant of a pair of groups.

For a pair of groups (N,G) put $V_0(N,G) = N$ and define

$$V_i(N,G) = V(V_{i-1}(N,G), G).$$

Then

$$N = V_0(N,G) \supseteq V_1(N,G) \supseteq \ldots \supseteq V_n(N,G) \supseteq \ldots$$

is the called the lower $\mathcal{V}$-marginal series of N in G. Similarly, we define the upper $\mathcal{V}$-marginal series of N in G, by setting

$$V_0^*(N,G) = \langle e \rangle, \quad \frac{V_i^*(N,G)}{V_{i-1}^*(N,G)} = V^*\left(\frac{N}{V_{i-1}^*(N,G)}, \frac{G}{V_{i-1}^*(N,G)}\right).$$

The pair (N,G) of groups is said to be $\mathcal{V}$-nilpotent pair, if $V_n(N,G) = \langle e \rangle$ for some positive integer n (see [2], section 3).

Theorem 1 *Let $\mathcal{V}$ be a variety of groups. If (N,G) is a $\mathcal{V}$-nilpotent pair of groups and M is a non-trivial normal subgroup of G such that $M \cap N \neq \langle e \rangle$, then $M \cap V^*(N,G) \neq \langle e \rangle$.*

Proof Since (N, G) is $\mathcal{V}$-nilpotent, then there exists a positive integer c such that $V_c^*(N, G) = N$. Let i be the least integer such that $M \cap V_i^*(N, G) \neq \langle e \rangle$. We have

$$V\left(M \cap V_i^*(N, G), G\right) \leq M \cap V_{i-1}^*(N, G) = \langle e \rangle$$

and

$$M \cap V_i^*(N, G) \leq M \cap V^*(N, G)$$

Hence,

$$M \cap V^*(N, G) = M \cap V_i^*(N, G) \neq \langle e \rangle.$$

The following corollary is an immediate result of Theorem 1.

Corollary 1 *If (N, G) is a $\mathcal{V}$-nilpotent pair of groups with $N \neq \langle e \rangle$, then $V^*(N, G) \neq \langle e \rangle$.*

Theorem 2 *Let $\mathcal{V}$ be a variety of groups and (N, G) be a $\mathcal{V}$-nilpotent pair of groups, then every maximal subgroup of G which does not contain N, is normal.*

Proof By the assumption there exists a positive integer c such that

$$V_c^*(N, G) = N,$$

thus for every maximal subgroup M of G, the following series is a subnormal series for M

$$M \trianglelefteq MV^*(N, G) \trianglelefteq MV_2^*(N, G) \trianglelefteq ... \trianglelefteq MV_c^*(N, G) = MN = G.$$

If $c = 1$, then $M \trianglelefteq G$. Since M is a maximal subgroup of G, then there exists a least positive integer j where $MV_j^*(N, G) = G$. Then, $MV_{j-1}^*(N, G) = M$, and so $M \trianglelefteq G$.

Theorem 3 *Let $\mathcal{V}$ be a variety of groups. If (N, G) is a pair of groups such that $K \leq V^*(N, G)$ and $(N/K, G/K)$ is a $\mathcal{V}$-nilpotent pair of groups, then (N, G) is a $\mathcal{V}$-nilpotent pair.*

Proof Since $(N/K, G/K)$ is a $\mathcal{V}$-nilpotent pair of groups. So, there exist a normal series as

$$1 = \frac{N_1}{K} \leq \frac{N_2}{K} \leq ... \leq \frac{N_n}{K} = \frac{N}{K}$$

such that

$$\frac{N_{i+1}/N}{N_i/N} \leq V^*\left(\frac{N/K}{N_i/K}, \frac{G/K}{N_i/K}\right).$$

Now, we have $N_{i+1}/N_i \leq V^*\left(N/N_i, G/N_i\right)$. Hence, we obtain the following normal series.

$$1 = N_0 \leq N_1 \leq ... \leq N_n = N.$$

Thus, the pair (N, G) is a $\mathcal{V}$-nilpotent pair.

Theorem 4 *Let $\mathcal{V}$ be a variety of groups. If (N,G) is a $\mathcal{V}$-nilpotent pair and $K \trianglelefteq N$ such that $|K| = p^n$. Then $K \leq V_n^*(N,G)$.*

Proof We prove the result by using induction. Let $n = 1$, then

$$\langle e\rangle \neq K \cap V^*(N,G) \leq K,$$

so $|K \cap V^*(N,G)| = p = |K|$, hence $K \cap V^*(N,G) = K$ and so, $K \leq V^*(N,G)$. Let the result holds for every number less than n and $M = K \cap V^*(N,G) \neq \langle e\rangle$, then $|K/M| = p^m$ where $m < n$. Now we have

$$\frac{K}{M} = \frac{K}{K \cap V^*(N,G)} \cong \frac{KV^*(N,G)}{V^*(N,G)}.$$

By induction hypothesis, we have

$$\begin{aligned}\frac{KV^*(N,G)}{V^*(N,G)} &\leq V_m^*\left(\frac{N}{V^*(N,G)}, \frac{G}{V^*(N,G)}\right)\\ &= \frac{V_{m+1}^*(N,G)}{V^*(N,G)}.\end{aligned}$$

Thus, $K \leq V_{m+1}^*(N,G) \leq V_n^*(N,G)$.

Let $\mathcal{V}$ and $\mathcal{W}$ be two varieties of groups defined by the sets of laws V and W, respectively and G be a finite group in $\mathcal{W}$ with a free presentation $1 \to R \to F \to G \to 1$. If N is a normal subgroup of G and S is a normal subgroup of F such that $N \cong S/R$, then the Baer-invariant of the pair (N,G) with respect to $\mathcal{V}$ and $\mathcal{W}$ is defined as

$$\mathcal{WVM}(N,G) = \frac{W(F)\big(R \cap [SV^*F]\big)}{W(F)[RV^*F]},$$

where $[XV^*Y] = V(X,Y)$. One can check that $\mathcal{WVM}(N,G)$ is abelian and independent of the choice of the free presentation of G. In the special case where $\mathcal{W}$ is the variety of all groups, then

$$\mathcal{WVM}(N,G) = \mathcal{VM}(N,G)$$

is the Baer-invariant of the pair (N,G) with respect to the variety $\mathcal{V}$ (see [2, 8,11–14] for more information).
In the following theorems we generalize some results of M.R. Rismanchian and M. Araskhan in [14] .

Theorem 5 *Let (N,G) be a pair of finite groups and $1 \to R \to F \to G \to 1$ be a free presentation of G such that $N \cong S/R$ for a normal subgroup S of F. If N is a subgroup $\mathcal{V}$-nilpotent of G of class $c \geq 2$, then*

(i) $|V_{c-1}(N,G)||\mathcal{WVM}(N,G)| = \left|\mathcal{WVM}\big(\frac{N}{V_{c-1}(N,G)}, \frac{G}{V_{c-1}(N,G)}\big)\right|\left|\frac{[V_{c-1}(S,F)RV^*F]}{[RV^*F]}\right|$;

(ii) $d\big(\mathcal{WVM}(N,G)\big) \leq d\Big(\mathcal{WVM}\big(\frac{N}{V_{c-1}(N,G)}, \frac{G}{V_{c-1}(N,G)}\big)\Big) + d\big(\frac{[V_{c-1}(S,F)RV^*F]}{[RV^*F]}\big)$;

(iii) $e\big(\mathcal{WVM}(N,G)\big)$ *divides*

$$e\Big(\mathcal{WVM}\big(\tfrac{N}{V_{c-1}(N,G)},\tfrac{G}{V_{c-1}(N,G)}\big)\Big)e\big(\tfrac{[V_{c-1}(S,F)RV^*F]}{[RV^*F]}\big).$$

where $e(X)$ and $d(X)$ are the exponent and the minimal number of generators of a group X, respectively.

Proof Let $\mathcal{V}$ and $\mathcal{W}$ be two varieties of groups and G be a group in the variety $\mathcal{W}$ with two normal subgroups K and N such that $K \subseteq N$. Then, by Theorem 2.2 of [12], the following sequence is exact:

$$1 \to \mathcal{WVM}(G,K) \overset{\alpha}{\to} \mathcal{WVM}(N,G)$$
$$\to \mathcal{WVM}(N/K,G/K) \overset{\beta}{\to} \frac{K \cap [NV^*G]}{[KV^*G]} \to 1.$$

Thus,

$$|\mathcal{WVM}(N,G)| = |Im(\alpha)|\Big|\frac{R\cap[V_{c-1}(S,F)RV^*F]}{[RV^*F]}\Big|$$

and

$$\frac{\mathcal{WVM}(N/K,G/K)}{Im(\alpha)} \cong K,$$

where $K = V_{c-1}(N,G)$. Hence,

$$|K||\mathcal{WVM}(N,G)| = |\mathcal{WVM}(\frac{N}{K},\frac{G}{K})|\Big|\frac{R\cap[V_{c-1}(S,F)RV^*F]}{[RV^*F]}\Big|.$$

But $[KV^*G] = [V_{c-1}(N,G)V^*G] = V_c(N,G) = \langle e\rangle$, so

$$[V_{c-1}(S,F)RV^*F] \subseteq R.$$

This implies part (i). Similarly, we can prove (ii) and (iii).

Theorem 6 *Let (N,G) be a pair of finite groups such that $V^*(G) \subseteq N$. Let $H = G/V^*(G)$ and $L = N/V^*(G)$. Then*

(i) $|[NV^*G]| \le |\mathcal{WVM}(L,H)|\ |[LV^*H]| \le |\mathcal{WVM}(N,G)||[NV^*G]|$;

(ii) $|[NV^*G]| = |\mathcal{WVM}(L,H)||[LV^*H]|$
$\iff \mathcal{WVM}(L,H) \cong V^*(G)\cap[NV^*G]$;

(iii) $|\mathcal{WVM}(L,H)||[LV^*H]| = |\mathcal{WVM}(N,G)||[NV^*G]|$
$\iff \frac{\mathcal{WVM}(L,H)}{\mathcal{WVM}(N,G)} \cong V^*(G)\cap[NV^*G]$.

Proof (i) By Theorem 2.2 of [12], we have

$$|\mathcal{WVM}(L,H)| = |V^*(G)\cap[NV^*G]||\ker\beta|.$$

On the other hand,

$$[LV^*H] = \frac{[NV^*G]V^*(G)}{V^*(G)} \cong \frac{[NV^*G]}{V^*(G)\cap[NV^*G]}.$$

So,

$$|V^*(G) \cap [NV^*G]| = \frac{|[NV^*G]|}{|[LV^*H]|}.$$

Hence,

$$|[NV^*G]||\ker\beta| = |\mathcal{WVM}(L,H)||[LV^*H]|.$$

This implies that

$$|[NV^*G]| \leq |\mathcal{WVM}(L,H)||[LV^*H]|.$$

Moreover, $|\ker\beta| = |Im\alpha| \leq |\mathcal{WVM}(N,G)|$. Therefore,

$$|\mathcal{WVM}(L,H)||[LV^*H]| \leq |\mathcal{WVM}(N,G)||[NV^*G]|.$$

(ii) By considering the first part, we have

$$|\ker\beta| = 1 \Longleftrightarrow \mathcal{WVM}(L,H) \cong V^*(G) \cap [NV^*G] \text{ and}$$
$$|\ker\beta| = 1 \Longleftrightarrow |[NV^*G]| = |\mathcal{WVM}(L,H)||[LV^*H]|.$$

Thus, the result holds.

(iii) By Theorem 2.2 of [12],

$$|\ker\alpha||\mathcal{WVM}(L,H)||[LV^*H]| = |\mathcal{WVM}(N,G)||[NV^*G]|.$$

Also $|\ker\alpha| = 1$ if and only if $\frac{\mathcal{WVM}(L,H)}{\mathcal{WVM}(N,G)} \cong V^*(G) \cap [NV^*G]$, which completes the proof.

By Theorem 6, we obtain the following corollary.

Corollary 2 *Let G be a finite group and $H = G/V^*(G)$. Then*

(i) $|V(G)| \leq |\mathcal{WVM}(H)||V(H)| \leq |\mathcal{WVM}(G)||V(G)|$;
(ii) $|V(G)| = |\mathcal{WVM}(H)||V(H)| \Longleftrightarrow \mathcal{WVM}(H) \cong V^*(G) \cap V(G)$;
(iii) $|\mathcal{WVM}(H)||V(H)| = |\mathcal{WVM}(G)||V(G)| \Longleftrightarrow \frac{\mathcal{WVM}(H)}{\mathcal{WVM}(G)} \cong V^*(G) \cap V(G)$.

3 Isologism of pairs of groups

In this section, we survey some results on isologism of pairs of groups. The notion of isologism of a pair of groups was discussed in [4]. Indeed, we extend some results of [6,7].

Let (N,G) and (M,H) be pairs of groups. An homomorphism from (N,G) to (M,H) is a homomorphism $f : G \to H$ such that $f(N) \subseteq M$. We say that (N,G) and (M,H) are isomorphic and write $(N,G) \simeq (M,H)$, if f is an isomorphism and $f(N) = M$.

Definition 1 Let (N,G) and (M,H) be two pairs of groups and $\mathcal{V}$ be a variety of groups defined by the set of laws V. An $\mathcal{V}$-isologism between (N,G) and (M,H) is a pair of isomorphism (α,β) with $\alpha : G/V^*(N,G) \to H/V^*(M,H)$ and $\beta : V(N,G) \to V(M,H)$, such that $\alpha\big(N/V^*(N,G)\big) = M/V^*(M,H)$ and for every $v \in V$, $n \in N$ and $g_1,\dots,g_r \in G$

$$\beta\big(v(g_1,\cdots,g_in,\cdots,g_r)v(g_1,\cdots,g_r)^{-1}\big) = \\ v(h_1,\cdots,h_im,\cdots,h_r)v(h_1,\cdots,h_r)^{-1},$$

whenever, $h_i \in \alpha\big(g_iV^*(N,G)\big)$ and $m \in \alpha\big(nV^*(N,G)\big)$. We say that (N,G) and (M,H) are v-isologic, if there exists an $\mathcal{V}$-isologism between them. In this case we write $(N,G) \sim_{\mathcal{V}} (M,H)$.

If $\mathcal{V}$ is the variety of abelian groups or nilpotent groups of class at most n, then $\mathcal{V}$-isologism coincides with isoclinism and n-isoclinism between pairs of groups. In addition, if $N = G$ and $M = H$, then $\mathcal{V}$-isologism between two pairs of groups is an $\mathcal{V}$-isologism between G and H.
The following Lemma is proved by the authors in ([4], Lemma 5).

Lemma 1 *Let (N,G) be a pair of groups. If M is a normal subgroup of G with $M \le N$ and H is a subgroup of G, then*

(a) $(H \cap N, H) \sim_{\mathcal{V}} \big((H \cap N)V^*(N,G), HV^*(N,G)\big)$. *In particular if* $G = HV^*(N,G)$, *then* $(H\cap N,H) \sim_{\mathcal{V}} (N,G)$. *Conversely, if* $\frac{H}{V^*(H\cap N,H)}$ *satisfies the ascending chain condition on normal subgroups and* $(H \cap N,H) \sim_{\mathcal{V}} (N,G)$, *then* $G = HV^*(N,G)$;
(b) $(N/M, G/M) \sim_{\mathcal{V}} \big(N/M \cap V(N,G), G/M \cap V(N,G)\big)$. *In particular if* $M \cap V(N,G) = \langle e\rangle$, *then* $(N,G) \sim_{\mathcal{V}} (\frac{N}{M},\frac{G}{M})$. *Conversely, if* $V(N,G)$ *satisfies the ascending chain condition on normal subgroups and* $(N,G) \sim_{\mathcal{V}} (\frac{N}{M},\frac{G}{M})$, *then* $M \cap V(N,G) = \langle e\rangle$.

A pair of groups (N,G) is said to be $\mathcal{V}$-perfect, if $N = V(N,G)$.
The following results give the connections between $\mathcal{V}$-perfect and $\mathcal{V}$-isologism of pairs of groups.

Corollary 3 *Let (N,G) be a $\mathcal{V}$-perfect pair of groups such that $V^*(N,G) = \langle e\rangle$. Then any $\mathcal{V}$-isologic (K,H) to (N,G) is isomorphic to the direct product of N by the marginal subgroup of (K,H).*

Proof By the assumption, we have

$$K/V^*(K,H) \cong N/V^*(N,G) \cong V \sim_{\mathcal{V}} K$$

Now, by Lemma 1, we obtain

$$K = V(K,H)V^*(K,H) \quad \textit{and} \quad V^*(K,H) \cap V(K,H) = \langle e\rangle$$

Thus, $K \cong N \times V^*(K,H)$.

Corollary 4 *Let (N, G) be a pair of finite groups. If H is a normal subgroup of G such that $H \subseteq N$ and the pair (H, N) is $\mathcal{V}$-perfect. Also, let $(H, N) \sim_{\mathcal{V}} (N, G)$, then $N = V(N, G)V^*(N, G)$.*

Proof By Lemma 1, we have $N = HV^*(N, G)$. On the other hand, $H = V(H, N)$. Therefore, $N = V(N, G)V^*(N, G)$.

Corollary 5 *Let (N, G) be a $\mathcal{V}$-perfect pair of groups. Then (N, G) can not be $\mathcal{V}$-isologic to any pair of groups (H, N), in which H is a proper subgroup of G such that $H \subseteq N$ or factor pair of groups of itself.*

Proof If H is a proper subgroup of G such that $H \subseteq N$ and $(N, G) \sim_{\mathcal{V}} (H, N)$, then by using Lemma 1, we have $H \cap V(N, G) = \langle e \rangle$. Therefore, $H = \langle e \rangle$.

Corollary 6 *Let (N, G) and (K, H) be two pairs of groups such that $|N| = |K|$ and $(K, H) \sim_{\mathcal{V}} (N, G)$. If (N, G) is $\mathcal{V}$-perfect or $V^*(N, G) = \langle e \rangle$, then $(N, G) \cong (K, H)$.*

Proof By the definition of isologism, there are isomorphisms

$$\alpha : \frac{N}{V^*(N, G)} \to \frac{K}{V^*(K, H)} \quad \textit{and} \quad \beta : V(N, G) \to V(K, H).$$

Now, if $N = V(N, G)$, then $|N| = |V(K, H)|$ since $|N| = |K|$, it implies that $K = V(K, H)$ and hence, $(N, G) \cong (K, H)$. If $V^*(N, G) = \langle e \rangle$, then the result holds.

Definition 2 Let (N, G) be a pair of groups. If G contains no proper subgroup H satisfying $G = HV^*(N, G)$, then (N, G) is called *subgroup irreducible with respect to* $\mathcal{V}$-isologism. If the group G contains no normal subgroup M with $N \cap M \neq \langle e \rangle$ and $M \cap V(N, G) = \langle e \rangle$, then (N, G) is called *quotient irreducible with respect to* $\mathcal{V}$-isologism.

Lemma 2 *If (N, G) is a $\mathcal{V}$-perfect pair of groups. Then (N, G) is subgroup and quotient irreducible pair of groups.*

Proof Assume that (N, G) be a $\mathcal{V}$-perfect pair of groups and H be a subgroup of G such that $N = HV^*(N, G)$. Thus, we have $V(N, G) = V(H, N)$. So, $H = N$. Now, we can see that (N, G) is quotient irreducible pair of groups.

Theorem 7 *Let (N_1, G_1) and (N_2, G_2) be two $\mathcal{V}$-isologic pairs of groups. If (N_1, G_1) is subgroup and quotient irreducible pair of groups, then so is (N_2, G_2).*

Proof Let H be a normal subgroup of G_1 such that $H \subseteq N_1$ and $H \cap V(N_1, G_1) = \langle e \rangle$. Then, we have

$$H \subseteq V^*(N_1, G_1) \quad \textit{and} \quad V^*\big(N_1/H, G_1/H\big) = \frac{V^*(N_1, G_1)}{H}.$$

Now, assume that

$$N_2 = N_1/H = K/HV^*\big(N_1/H, G_1/H\big) = \frac{K}{H} \cdot \frac{V^*(N_1, G_1)}{H}.$$

So, $N_1 = KV^*(N_1, G_1)$, which implies that $N_1 = K$ and hence $N_2 \cong N_1/H = K/H$. Thus, the result holds when N_1 is assumed to be quotient irreducible pair of groups.

References

1. G. Ellis, The Schur multiplier of a pair of groups, Appl. Categ. Structures, 6, 355-371 (1998).
2. A. Gholami, Z. Mohammad Abadi, S. Heidarian, Some generalized varietal properties of a pair of groups, Southeast Asian Bull. Math., 36, 301–308 (2012).
3. M. Hassanzadeh, A. Pourmirzaei, S. Kayvanfar, On the nilpotency of a pair of groups, Southeast Asian Bull. Math., 37, 67–77 (2013).
4. S. Heidarian, A. Gholami, Z. Mohammad Abadi, The structure of ν-isologic pairs of groups, Filomat, 26, 67–79 (2011).
5. C. R. Leedham-Green, S. Mackay, Baer- Invariants, Isologism, Varietal Lows and Homology, Acta Math., 137, 99–150 (1976).
6. S. Lotfi, M. Taheri, Some relations between isologic and varietal perfect groups, Hindawi Journal of Mathematics, 4 pages (2016).
7. M. R. R. Moghaddam, A. R. Salemkar, Varietal isologism and covering groups, Arch. Math., 75, 8–15 (2000).
8. M. R. R. Moghaddam, A. R. Salemkar, H. M. Sanny, Some inequalities for the Baer-invariant of a finite groups, Indag. Math., 18, 73–82 (2007).
9. M. R. R. Moghaddam, A. R. Salemkar, P. Niroomand, Covering groups respect to two varieties of groups, Southeast Asian Bull. Math., 31, 313–320 (2007).
10. H. Neuman, Varieties of groups, Speringer- Verlag Berlin Heidelberg, New York, 1967.
11. A. Onsory, M. Araskhan, The Generalization of the Stalling's Theorem, Journal of Generalized Lie Theory and Applications, 3 pages, 2015.
12. M. R. Rismanchian, M. Araskhan, Generalized Baer-Invariant of a pair of groups and the direct limit, Journal of Sciences, Islamic Republic of Iran, 22, 159–164 (2011).
13. M. R. Rismanchian, M. Araskhan, Generalized Baer-Invariant of a Pair of Groups and Marginal Extension, Journal of Sciences, Islamic Republic of Iran, 23, 251–256 (2012).
14. M. R. Rismanchian, M. Araskhan, Some properties on the Baer-invariant of a pair of groups and ν_G-marginal series, Turk. J. Math., 37, 259–266 (2013).
15. S. M. Taheri, F. Shahini, Varietal nilpotent groups, World Applied Sciences Journal. 12, 2314–2316 (2011).