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Numerical Solution of Degenerate Fourth Order SDE Model by Milstein Scheme

Daryoud Kalvand · Esmaeil Yousefi

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Abstract In this paper,we use a Milstein scheme to develop a numerical technique for solving Stochastic differential equation which we had its deterministic form in our last article [7], we discuss the existence and uniqueness solution of deterministic and stochastic form, and then we show the advantages of the method with numerical example.

Keywords Milstein scheme · Lax-Milgram Lemma · Degenerate Differential-Equation · Stochastic Differential Equation

Mathematics Subject Classification (2010) 65L03 · 60J74 · 60J76

1 Introduction

During last decades, many work were done on forth order degenerate differential equation which applied to solve many important class of problems, such that problems of small deformation surfaces of revolution, the membrane theory of shells, the bending of plates of variable thickness with a sharp edge and the gas dynamics. First time this class of equation on the boundary of the domain, was considered by V.K. Zakharov [14], which extended the results of M.I. Vishik [13], on the fourth-order equations on the plane, also degenerate differential equations in abstract spaces have been studied by V.P. Glushko and S.G. Krein in [4], A.A. Dezin [2].

This approach was applied in A.A. Dezin book [2], by V.K. Romanko [8], and this makes it possible to study a number of phenomena, which were not

D. Kalvand (Corresponding Author)
Department of Mathematics, Linnaeus University, 351 95,Växjö, Sweden.
E-mail: draiush_kalvand@yahoo.com, daryoush.kalvand.extern@lnu.se

E. Yousefi
Department of Mathematics, Karaj Branch, Islamic Azad University, Karaj, Iran.
E-mail: esmaeil.yousefi@kiau.ac.ir

fully explored, and at the same time it is easy to trace the connection between ordinary differential equations and operator equations.

$$Lu \equiv (t^{\alpha} u(t)''tt)''tt + au(t) = f, \tag{1.1}$$

where $t \in [0, b], 0 \leq \alpha \leq 4, f \in L^2((0, b), \mathbb{H}), a \in \mathbb{R}$ [7],for this reason,I cannot describe historical background of the stochastic form of equation.
First we start by considering the necessary proposition and corollary about the spaces $\dot{W}^2\alpha(0, b)$ and $W^2\alpha(0, b)$,they are space of our study on these equations. After that,we prove the existence and uniqueness of deterministic and stochastic equation. and then we discuss about preliminary results in 3.2.1 and also in section 4 opened the numerical discussion and in subsection4.1 the Milstein method is used as the numerical method and in the continue convergence of numerical method has been raised an example is given below.

2 Spaces $\dot{W}^2\alpha(0, b)$ and $W^2\alpha(0, b)$

In this Section we briefly discuss some contents about Spaces $\dot{W}^2\alpha(0, b)$ and $W^2\alpha(0, b)$,also assume $\dot{C}^2[0, b]$ be a set of twice continuously differentiable functions $u(t)$ defined on $[0, b]$ satisfying the conditions

$$u(0) = u'(0) = u(b) = u'(b) = 0. \tag{2.1}$$

Let $\dot{W}^2_{\alpha}(0, b)$, $\alpha \geq 0$ be the completion of $\dot{C}^2[0, b]$ in the norm

$$\|u\|\dot{W}^2\alpha(0, b) = \int_0^b t^{\alpha} \left|u''(t)\right|^2 dt. \tag{2.2}$$

It is known that the elements of $\dot{W}^2\alpha(0, b)$ are continuously differentiable functions on $[\varepsilon, b]$ for every $0 < \varepsilon < b$ whose first derivatives are absolutely continuous and $u(b) = u'(b) = 0$ therefore it is sufficient to investigate properties of the elements from $\dot{W}^2\alpha(0, b)$ for small t. [9]

Proposition 1 *For every $u \in \dot{W}^2\alpha(0, b)$ close to $t = 0$ we have the following estimates*

$$|u(t)|^2 \leq C_1 t^{3-\alpha} \|u\|\dot{W}^2\alpha(0, b), \text{ for } \alpha \neq 1, 3,$$
$$|u'(t)|^2 \leq C_2 t^{1-\alpha} \|u\|\dot{W}^2_{\alpha}(0, b), \text{ for } \alpha \neq 1.$$

For $\alpha = 3$the factor $t^{3-\alpha}$ should be replaced by $|\ln t|$for $\alpha = 1$ the factor $t^{1-\alpha}$ by $|\ln t|$ and the factor $t^{3-\alpha}$ by $t^2|\ln t|$.

Proposition 2 *For every $0 \leq \alpha \leq 4$ we have a continuous embedding*

$$\dot{W}^2_{\alpha}(0, b) \hookrightarrow L^2(0, b), \tag{2.3}$$

it is compact for $0 \leq \alpha < 4$ and for $\alpha = 4$ is not compact.

Remark 1 The embedding(2.3) for $\alpha > 4$ is fail.

When $\alpha > 4$ we use the function $u(t) = t^{\frac{-1}{2}}\varphi(t)$ that $\varphi(t) \in C^2[0,b]$ $\varphi(b) = \varphi'(b) = 0$ and $\varphi(0) \neq 0, u \in \dot{W}^2_\alpha(0,b)$ but $u \notin L^2(0,b)$.[5]

Proposition 3 *For every function* $u \in \dot{W}^2\alpha(0,b)$ *and* $\alpha \neq 1,3,$ *the norm* $\|u_h - u\|\dot{W}^2_\alpha(0,b)$ *tends to zero by* $h \to 0$.

Corollary 1 *For every* $u, v \in \dot{W}^2\alpha(0,b)$ *and* $\alpha \neq 1,3,$ *we have*

$$\lim_{h\to 0}\{u, v_h\}\alpha = \{u,v\}_\alpha.$$

Corollary 2 *If the function u has a bounded piecewise-continuous derivative of the second order in* $[\varepsilon, b]$ *For arbitrary* $0 < \varepsilon < b, \|u\|\dot{W}^2\alpha(0,b) < \infty,$ *then* (1) *is valid for* $u \in \dot{W}^2_\alpha(0,b)$.

Also,let $W^2_\alpha(0,b)$ be a set of the functions $u(t)$ which have a generalized derivative of the second order such that the following semi-norm

$$\|u\|_1^2 = \int_0^b t^\alpha |u''(t)|^2\,dt, \tag{2.4}$$

is finite. [5]

Proposition 4 *For every* $0 \leq \alpha \leq 4$ *we have the embedding*

$$W^2_\alpha(0,b) \subset L^2(0,b). \tag{2.5}$$

It follows from remark 1 that the embedding (2.5) also fails for the space $W^2\alpha(0,b)$ since it is larger than the space $\dot{W}^2\alpha(0,b)$ to work within the space $L^2(0,b)$ further we will assume that $0 \leq \alpha \leq 4$ and we can define the following norm in the space $W^2_\alpha(0,b)$

$$\|u\|W^2\alpha(0,b) = \int_0^b \left(t^\alpha\,|u''(t)|^2 + a.|u(t)|^2\right)\,dt, \tag{2.6}$$

Proposition 5 *For* $u \in W^2_\alpha(0,b)$ *we have*

$$\begin{aligned} &\textbf{\textit{(i)}}|u(t)|^2 \leq (c_1 + c_2 t^{3-\alpha})\|u\|W^2\alpha(0,b), \text{ for } \alpha \neq 1,3, \\ &\textbf{\textit{(ii)}}|u'(t)|^2 \leq (c_3 + c_4 t^{1-\alpha})\|u\|W^2\alpha(0,b), \text{ for } \alpha \neq 1. \end{aligned} \tag{2.7}$$

Following [10] in the above solutions when $\alpha = 1$ *and* $\alpha = 3$ *the factors* $t^{3-\alpha}$ *in* (i) *must be replaced by* $t^2|\ln t|$ *and* $|\ln t|$ *respectively.when* $\alpha = 1$ *the factor* $t^{1-\alpha}$ *in* (ii) *must be replaced by* $|\ln t|$.

3 Solution of Differential Equation

We analyze the solution of equation

$$Lu \equiv (t^{\alpha}u(t)''tt)''tt + au(t) = f + \sigma.\dot{W}(t),$$
$$0 \leq \alpha \leq 4, \quad t \in [0,b], \quad f \in L^2(0,b), \qquad \sigma \in \mathbb{R}, \tag{3.1}$$

where the $\dot{W}(t)$ is White noise,also the condition on equation is:

$$u(0) = 0, \qquad u't(0) = 0, \qquad u''tt(0) = 0, \qquad u'''_{ttt}(0) = 0. \tag{3.2}$$

In the following we discuss about existence and uniqueness of solution in deterministic and Stochastic states.

3.1 Deterministic Differential Equation

In equation (3.1) if $\sigma = 0$ we get the deterministic form which we investigated the existence and uniqueness its of solution. [11]

Theorem 1 *(Existences and Uniqueness of solution)*
For every $f \in L^2(0,b)$ the solution of the equation (3.1) *under boundary condition3.2 when $\sigma = 0$ exists and is unique.*

Proof It is obvious that $W^2_{\alpha}(0,b)$ is Hilbert space.
Let

$$\mathbf{B} : W^2\alpha(0,b) \times W^2\alpha(0,b) \to \mathbb{R},$$
$$\mathbf{B}[u,v] = \int_0^b t^{\alpha}u''v''dt + \int_0^b auvdt, \;\; \forall u,v \in W^2_{\alpha}(0,b). \tag{3.3}$$

We define the linear map:

$$l_f : W^2_{\alpha}(0,b) \ni v \mapsto l_f(v) = \int_0^b fvdt \in \mathbb{R},$$

which is a functional on $W^2_{\alpha}(0,b)$, since

$$|l_f(v)|^2 = |\int_0^b fvdt|^2 \leq \|f\|^2_{L^2(0,b)}\|v\|^2_{L^2(0,b)} \leq \|f\|^2_{L^2(0,b)}\|v\|^2_{W^2_{\alpha}(0,b)}. \tag{3.4}$$

We should show that there exists $u \in W^2\alpha(0,b)$ such that $\mathbf{B}[u,v] = l_f(v)$ for all $v \in W^2\alpha(0,b)$, when $l_f(v) = \int_0^b fvds$ that can be obtained by the

Lax-Milgram Lemma.

$$
\begin{aligned}
&\|u\|W^2\alpha(0,b)\|v\|W^2\alpha(0,b)\\
&\qquad = (\int_0^b (t^\alpha|u''|^2 + a|u|^2)dt)^{\frac{1}{2}}(\int_0^b (t^\alpha|v''|^2 + a|v|^2)dt)^{\frac{1}{2}},\\
&\qquad = (\int_0^b (\sqrt{t^\alpha u''^2 + au^2})^2 dt)^{\frac{1}{2}}(\int_0^b (\sqrt{t^\alpha v''^2 + av^2})^2 dt)^{\frac{1}{2}},\\
&\qquad \geq (\int_0^b (\sqrt{(t^\alpha u''^2 + au^2).(t^\alpha v''^2 + av^2)})^2 dt)^{\frac{1}{2}},\\
&\qquad \geq \int_0^b |\sqrt{(t^{2\alpha} u''^2 v''^2 + at^\alpha u''^2 v^2 + t^\alpha a.v''^2 u^2 + au^2)}|, \qquad (3.5)\\
&\qquad \geq \int_0^b \sqrt{(t^{2\alpha} u'' v'' + auv)^2} dt,\\
&\qquad \geq \int_0^b |(t^\alpha u'' v'' + auv)| dt,\\
&\qquad = \int_0^b |t^\alpha u'' v'' + auv| dt\\
&\qquad \geq |\mathbf{B}[u,v]|,
\end{aligned}
$$

for all $u \in W^2\alpha(0,b)$, $\beta\|u\|^2 W^2_\alpha(0,b) \leq \mathbf{B}[u,u]$, we also have the relation for $\beta = 1$ in fact with equality

$$\mathbf{B}[u,u] = \int_0^b (t^\alpha u''^2 + au^2)dt = \|u\|^2_{W^2_\alpha(0,b)}. \qquad (3.6)$$

By the Lax-Milgram Lemma there exists $u \in W^2\alpha(0,b)$ such that $\mathbf{B}[u,v] = \int_0^b fvdt$ for all $v \in W^2\alpha(0,b)$.

3.2 Stochastic Differential Equation

We start with a deterministic or random outlook and introduce preliminary background material on differential equations.

3.2.1 Preliminary Results

$(\Omega, \mathbb{F}, \Im_t, \mathbb{P})$ is complete probability space that $\mathbb{F}$ is the σ-algebra $\Im_t$ is filtration and $\mathbb{P}$ is Probability measure.

Definition 1 (filtration)
Let $(\Omega, \mathbb{F}, \mathbb{P})$ be a probability space.
A filtration $\{\Im_t : t \geq 0\}$ is a family of sub σ-algebras of $\mathbb{F}$ that are increasing that is $\Im_s$ is a sub σ-algebra of $\Im_t$ for $s \leq t$ each $(\Omega, \mathbb{F}, \mathbb{P})$ is a measure space and we assume it is complete.

Definition 2 (adapted)
Let $(\Omega, \mathbb{F}, \Im_t, \mathbb{P})$ be a filtered probability space a stochastic process $\{U(t) : t \in [0, b]\}$ is $\Im_t$-adapted if the variable $U(t)$ is $\Im_t$-measurable for all $t \in [0, b]$.

Definition 3 ($\Im_t$ -Brownian motion)
A real-valued process $\{W(t) : t \in [0, b]\}$ is an Brownian motion on a filtered probability space $(\Omega, \mathbb{F}, \Im_t, \mathbb{P})$
if:
(i) $W(0) = 0$ a.s.
(ii) $W(t)$ is continuous as a function of t.
(iii) $W(t)$ is $\Im_t$-adapted and $W(t) - W(s)$ is independent of $\Im_s, s < t$.
(iv) $W(t) - W(s) \sim \mathbb{N}(0, t - s)$ for $0 \leq s \leq t$.

Definition 4 (Hilbert space of Stochastic Processes)
Let $(\Omega, \mathbb{F}, \Im_t, \mathbb{P})$ be a filtered probability space. We define $\mathbb{H}$ as

$$\begin{aligned}\mathbb{H} = \{u | u : (0, b) \times \Omega \mapsto \mathbb{R}^m, \|u\|_{\mathbb{H}} := \mathbb{E}[\int_0^b t^\alpha |u(t)''|^2 + a|u(t)|^2 dt] < \infty, \\ u(0) = 0, u't(0) = 0, u''tt(0) = 0, u'''_{ttt}(0) = 0\},\end{aligned} \tag{3.7}$$

3.2.2 Solution of Stochastic Differential Equation

The Matrix form of the equation (3.1) is

$$dU(t) = A(t)U(t)dt + B(t)dt + \Sigma(t)dW(t), \tag{3.8}$$

when:

$$A(t) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{-a}{t^\alpha} & 0 & \frac{-\alpha(\alpha-1)}{t^2} & \frac{-2\alpha}{t} \end{pmatrix},$$

$$B(t) = \begin{pmatrix} 0, & 0, & 0, & \frac{f(t)}{t^\alpha} \end{pmatrix}^T,$$

$$\Sigma(t) = \begin{pmatrix} 0, & 0, & 0, & \frac{\sigma}{t^\alpha} \end{pmatrix}^T,$$

when $0 < t < b$ with the boundary $U_0 = (u(0), u't(0), u''tt(0), u'''_{ttt}(0))$, the analytical solution of equation (3.8) has the following form

$$U(t) = U(0) + \int_0^t A(t).U(t)dt + \int_0^t B(t)dt + \int_0^t \Sigma(t).dW(t). \tag{3.9}$$

Theorem 2 *(Existence and Uniqueness of Solution)*
For every $f \in L^2(0, b)$ the generalized solution of the equation (3.1) *exists and is unique.*

Proof It is obvious that $\mathbb{H}$ is a Hilbert space. Let

$$\begin{aligned} &\mathbb{B} : \mathbb{H} \times \mathbb{H} \to \mathbb{R}, \\ &\mathbb{B}[u, v] = \mathbb{E}[\int_0^b t^\alpha u''v'' + auvdt],\ \forall u, v \in \mathbb{H}, \end{aligned} \tag{3.10}$$

we should show that $\mathbb{B}[u, v] = \langle f, v\rangle, \quad \forall v \in \mathbb{H}$ that it can be obtained by the Lax-Milgram Lemma.

$$\begin{aligned} &\|u\|_{\mathbb{H}}\|v\|_{\mathbb{H}} \\ &= (\mathbb{E}[\int_0^b (t^\alpha|u''|^2 + a|u|^2)dt])^{\frac{1}{2}}.(\mathbb{E}[\int_0^b (t^\alpha|v''|^2 + a|v|^2)dt])^{\frac{1}{2}}, \\ &= (\mathbb{E}[\int_0^b (\sqrt{t^\alpha u''^2 + au^2})^2 dt])^{\frac{1}{2}}.(\mathbb{E}[\int_0^b (\sqrt{t^\alpha v''^2 + av^2})^2 dt])^{\frac{1}{2}}, \\ &\geq (\mathbb{E}[\int_0^b (\sqrt{(t^\alpha u''^2 + au^2).(t^\alpha v''^2 + av^2)})^2 dt])^{\frac{1}{2}}, \\ &\geq \mathbb{E}[\int_0^b |\sqrt{(t^{2\alpha} u''^2 v''^2 + at^\alpha u''^2 v^2 + t^\alpha a v''^2 u^2 + au^2)} dt|], \\ &\geq \mathbb{E}[\int_0^b \sqrt{(t^{2\alpha} u''v'' + auv)^2} dt], \\ &\geq \mathbb{E}[\int_0^b |(t^\alpha u''v'' + auv)| dt], \\ &= \mathbb{E}[\int_0^b |(t^\alpha u''v'' + auv)| dt] \\ &= |\mathbb{B}[u, v]| \\ &\geq \mathbb{B}[u, v], \end{aligned} \tag{3.11}$$

for $\forall u \in \mathbb{H},\ \beta\|u\|^2_{\mathbb{H}} \leq \mathbb{B}[u, u]$, we have:

$$\begin{aligned} \mathbb{B}[u, v] &= \mathbb{E}[\sqrt{\int_0^b (t^\alpha u''u'' + auu)dt}] \\ &\geq \mathbb{E}[\sqrt{\int_0^b (t^\alpha u''^2 + au^2)dt}] \\ &= \|u\|^2_{\mathbb{H}}, \end{aligned} \tag{3.12}$$

for $f : \mathbb{H} \longrightarrow \mathbb{R}$, we define the functional

$$l_f(v) = \mathbb{E}[\int_0^b fvdt + \int_0^b \sigma.\dot{W}(t)vdw],$$

for every $v \in \mathbb{H}, \sigma \in \mathbb{R}$.

$$\begin{aligned}
&|l_f(v)|^2 \\
&= |\mathbb{E}[\int_0^b fvdt + \int_0^b \sigma \dot{W}(t) vdw]|^2, \\
&\leq 2\mathbb{E}\int_0^b |fv|^2 dt + 2\mathbb{E}\int_0^b |\sigma \dot{W}(t) v|^2 dt, \\
&\leq 2(\|f\|L^2(0,b) \times \Omega + \sigma^2 b^2) v \|v\| L^2(0,b) \times \Omega, \\
&\leq 2(\|f\|L^2(0,b) \times \Omega + \sigma^2 b^2)\|v\|\mathbb{H}
\end{aligned} \tag{3.13}$$

Hence, $l_f(v)$ is a linear continuous functional over the space $\mathbb{H}$, by Lax-Milgram Lemma :

$$\mathbb{B}[u, v] = l_f(v).$$

Remark 2 (Existence and Uniqueness of Solution of SDE form when $\epsilon < t < b$) When $\epsilon < t < b$, the equation (3.8) with the boundary

$$U_0 = (u(\epsilon), u't(\epsilon), u''tt(\epsilon), u'''_{ttt}(\epsilon))$$

has the following form

$$\begin{aligned}
U(t) = U(\epsilon) + \int_\epsilon^t A(t).U(t)dt + \int_\epsilon^t B(t)dt, \\
+ \int_\epsilon^t \Sigma(t).dW(t).
\end{aligned} \tag{3.14}$$

For every $f \in L^2(0, b)$ the generalized solution of the equation (3.1) exists and is unique.

4 Numerical Scheme of SDE

Generally for some drift or diffusion functions of stochastic phenomena models the available analytical solutions can not easily found the numerical schemes are useful methods.

We discus the numerical approximation u_i where $t_i = t_0 + i.dt,\ i = 1, ..., N$ and $dt = \frac{b-a}{N}$ on $[a, b]$.

To solve a SDE and using numerical method at first needed to have a model that investigated the subject to find a reasonable mathematical interpretation that here the Milstein method is considered.

4.1 Milstein Method

We have

$$du''' = (-\frac{a}{t^{\alpha}}u - \frac{2\alpha}{t}u''' - \frac{\alpha(\alpha-1)}{t^2}u'')dt + (\frac{f}{t^{\alpha}})dt + (\frac{\sigma}{t^{\alpha}})dW_{(t)}, \tag{4.1}$$

from equation (3.1) also extend with the Taylor series,truncated them and by considering

$$\begin{aligned} f(t) &= -\frac{a}{t^{\alpha}}u - \frac{2\alpha}{t}u''' - \frac{\alpha(\alpha-1)}{t^2}u'' + \frac{f}{t^{\alpha}}, \\ G(t) &= \frac{\sigma}{t^{\alpha}}, \end{aligned} \tag{4.2}$$

have the following equation:

$$\begin{aligned} u'''(t) =& u'''(s) + f(s)(t-s) + G(s)\int_s^t dW(p), \\ &+ \int_s^t D_x G(s)(G(s)\int_s^t dW(p)W(r) + R_m(s), \end{aligned} \tag{4.3}$$

which remainder term $R_m(s)$ has the form:

$$R_m(s) := \int_s^t R_f(r)dr + \int_s^t R_G(r)dW(r) + \int_s^t DG(r)R_1(r)dw(r), \tag{4.4}$$

when:

$$\begin{aligned} R_f &:= Df(r-s) + \int_0^1 (1-\theta)h^2 D^2 f(\theta)d\theta, \\ R_G &:= \int_0^1 (1-\theta)h^2 D^2 G(\theta)d\theta, \\ R_1 &:= f(t-s) + R_E(s), \\ R_E &:= \int_s^t R_f(r)dr + \int_s^t R_G(r)dW(r) + \int_s^t DG(r)dW(r). \end{aligned} \tag{4.5}$$

Definition 5 (mesh-point form of Milstein method)
When $dt_{i+1} = (t_{i+1} - t_i)$ and $dW_{i+1} = (W_{i+1} - W_i)$ and boundary $u_0 = (u(t_0), u'(t_0), u''(t_0), u'''(t_0)) \in W^2_{\alpha}(0, b)$, the equation (4.3) can has the following form :

$$\begin{aligned} U(t_{i+1}) = U(t_i) + A(t_i)U(t_i)dt_{i+1} + B(t_i)dW_{i+1}, \\ + \frac{1}{2}B(t_i)B'x(ti)\{dW^2_{i+1} - dt_{i+1}\} \end{aligned} \tag{4.6}$$

that is mesh-point form of Milstein method.

4.1.1 Convergence of Milstein Method

For step dt the Milstein method generates Stochastic variables u_i that approximate the variables $u(t_i)$ given by the exact solution of the SDE at t_i and we investigated the convergence of u_i to $u(t_i)$ as dt tends to zero.

Assumption 3 *Assume the drift and diffusion functions of Milstein model hold in linear growth condition and the global Lipschitz condition. [3]*

Assumption 4 *Let Assumption 3 hold and suppose that the drift and diffusion functions are twice continuously differentiable and the second derivatives are uniformly bounded. [3]*

Theorem 5 *(convergence of Milstein method)*
Let Assumption 4 hold and for all $u_1, u_2 \in W^2\alpha(0,b)$, $\varepsilon \in \mathbb{R}$,$L \geq 0$ *that* $\|DG(u_1)(G(u_1)\varepsilon) - DG(u_2)(G(u_2)\varepsilon)\|L^2(0,b) \leq L\|u_1 - u_2\|2\|\varepsilon\|$ *and* $T \geq 0$ *exists* $K > 0$ *such that* $\sup 0 \leqslant t_i \leqslant T\|u(t_i) - u_i\|_{L^2(0,b)} \leqslant Kdt$*.[3]*

4.1.2 Analysis of Milstein Convergence

Consider convergence numerically by

$$\|u(t_i) - u_i\|L^2(0,b) \approx (\frac{1}{M}\Sigma j = 1^M\|u^j(t_i) - u_i^j\|_2^2)^{\frac{1}{2}}. \qquad (4.7)$$

It is unusual to have explicit solutions and in practice u_i are approximated with two different time steps by using the same sample path of $W(t)$ steps dt and dt_{ref} such that $dt = \kappa dt_{ref}$ for some $\kappa \in \mathbb{N}$that $T = N_{ref}dt_{ref} = Ndt$ and $N_{ref} = \kappa N$ that are using this trick in following example.

5 Numerical Illustration

Consider the boundary value problem on interval $[1e-4, 1]$

$$\begin{aligned}
&(t.u'')'' + u = (t^5x^4 + 240t^2x^4).\dot{W}(t),\\
&u(1.0e-4, x) = 1.0e-20x^4,\\
&u'_t(1.0e-4, x) = 5.0e-16x^4,\\
&u''_{tt}(1.0e-4, x) = 2.0e-11x^4,\\
&u'''_{ttt}(1.0e-4, x) = 6.0e-7x^4.
\end{aligned}$$

This problem has been solved with different values of kappa and has discussed about approximate of solution with dt_{ref} and dt.

Table 1 approximate of solution with dt_{ref} and dt

N_{ref}	kappa	dt	dt_{ref}	Error on u and u_{ref}
1000	5	4.9e-3	9.8e-4	3.7e-5
	100	9.9e-2	9.9e-4	6.5e-5
	500	4.9e-1	9.8e-4	4.9e-6
2000	5	2.4e-3	4.8e-4	4.3e-4
	100	4.9e-2	4.9e-4	5.2e-4
	500	2.4e-1	4.8e-4	1.6e-5
3000	5	1.6e-3	3.2e-4	2.6e-5
	100	3.3e-2	3.3e-4	3.7e-4
	500	1.6e-2	3.2e-4	1.8e-5

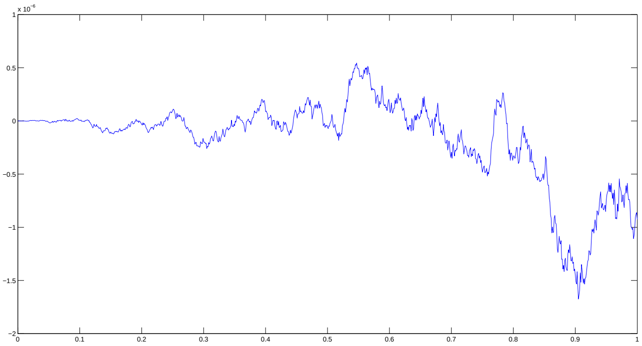

Fig. 1 Diagram of the u_{ref} as a consequence of time

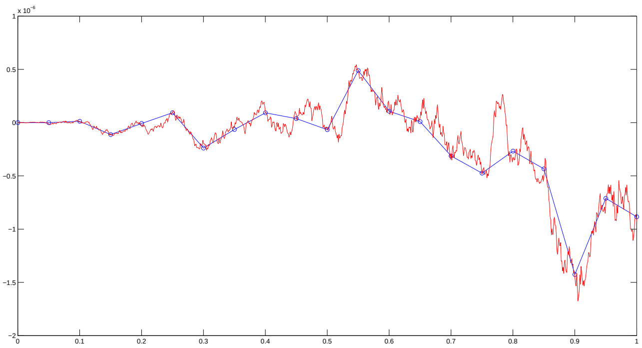

Fig. 2 Approximation by the Milestin scheme with $dt = 4.9e-3$ and $dt_{ref} = 9.8e-4$.

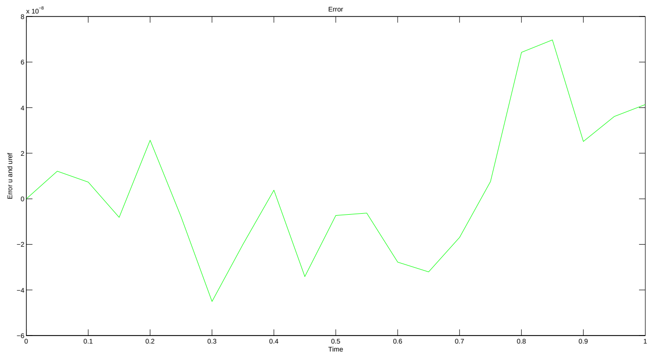

Fig. 3 Diagram of the error $\|u - u_{ref}\|$ as a consequence of t_i

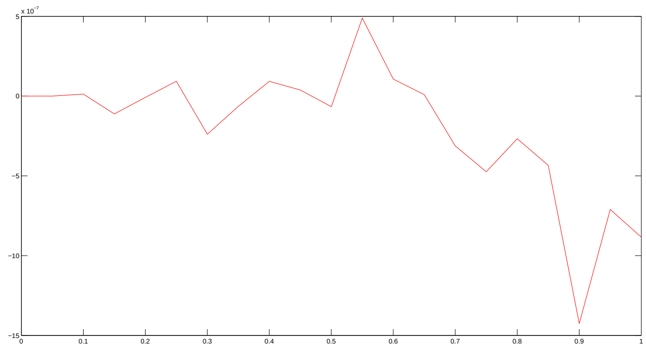

Fig. 4 Diagram of the u as a consequence for this reason

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