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An Iterative Method for Solving Two Dimensional Nonlinear Volterra Integral Equations

Manochehr Kazemi

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Abstract In this paper, a numerical iterative algorithm based on combination of the successive approximations method and the quadrature formula for solving two-dimensional nonlinear Volterra integral equations is proposed. This algorithm uses a trapezoidal quadrature rule for Lipschitzian functions applied at each iterative step. The convergence analysis and error estimate of the method are proved. Finally, two numerical examples are presented to show the accuracy of the proposed method.

Keywords Two-dimensional nonlinear integral equations · Trapezoidal cubature formula · Iterative method · Successive approximations

Mathematics Subject Classification (2010) 45Dxx · 45D05 · 34A12

1 Introduction

This paper is focused on a numerical iterative algorithm for solving the two-dimensional nonlinear Volterra integral equation as follows

$$X(s,t) = r(s,t) + \lambda \int_a^t \int_a^s K(s,t,x,y)\psi(X(x,y))dxdy, \ (s,t) \in I, \quad (1)$$

where $a, b \in \mathbf{R}$, $r : I = [a,b] \times [a,b] \to \mathbf{R}$, $K : I^2 \to \mathbf{R}$ and r, K are continuous. Integral equations arise in many physical applications, such as electrostatics, potential theory, electrical engineering, optimal control theory, solid and Build mechanics, heat transfer, etc. These equations also arise as representation formulas in the solutions of differential equations [1–7].

M. Kazemi
Department of Mathematics, Ashtian Branch, Islamic Azad University, Ashtian, Iran.
E-mail: m.kazemi@aiau.ac.ir, univer_ka@yahoo.com

The existing numerical methods for integral equations are based on various techniques: the well-known collocations and Galerkin methods [8–10], the meshless method [11], Bernoulli operational matrix method [12], iterated collocation method [13], Chebyshev polynomials [14,15], Haar wavelet method [16], hybrid of block-pulse and parabolic functions [17], Legendre functions [18], expansion method [19], triangular function[20], collocation method and radial basis functions [21,22], block-pulse functions [23], rationalized haar functions [24], Nyström type methods [25], Bernstein polynomials [26–28] and homotopy method [29,30]. Numerical procedures for solving integral equations of the second kind, based on the successive approximation method and other iterative techniques, have been investigated in [31–35]. Some results about the existence and uniqueness of the solution of nonlinear 2D integral equations can be found in [36–42].

In the following, we construct an iterative numerical method for Eq. (1) based on the Picard's technique of successive approximations and on a trapezoidal quadrature rule applied at each iterative step. In this method, we introduce a numerical iterative procedure using successive approximations method to approximate the solution of Eq. (1). The characteristic of this method compared with the existing methods based on the operational matrices and collocation method is that it is not necessary to solve the nonlinear system of algebraic equations for obtaining the approximate solutions of integral equations.

The rest of the paper is organized as follows: In Section 2, we give basic definitions and mathematical preliminaries of the quadrature rule for 2-D integrals. Section 3 allocated to the study of the existence and uniqueness of the solution of Eq. (1). Our numerical method for approximating the solution of Eq. (1) based on combination of the successive approximations method and quadrature formula for classes of Lipschitz two-dimensional functions is presented in this section. In addition, the convergence analysis and the error estimation of the method are proved in this section. Section 4 is devoted to present some numerical experiments to confirm the theoretical results and to illustratethe accuracy of the method. Some conclusions are drawn in Section 5.

2 Preliminaries

In this section, we review some necessary and basic definitions and results which will be further needed.

Definition 1 For $L \geq 0$, a function $g : [a,b] \times [c,d] \to \mathbf{R}$, is L-Lipschitz if

$$| g(x,y) - g(s,t) | \leq L(|x-s| + |y-t|) \qquad \forall (x,y),(s,t) \in [a,b] \times [c,d].$$

The trapezoidal cubature formula derived in [32] is:

$$\int_c^d \int_a^b g(s,t)dsdt = \frac{(b-a)(d-c)}{4}[g(a,c) + g(a,d) + g(b,c) + g(b,d)] + E(g)$$

with the remainder estimate

$$E(g) \leq \frac{L}{4}(b-a)(d-c)(b-a+d-c)$$

Let Δ_x and Δ_y denote, respectively the uniform partitions of $[a, b]$ and $[c, d]$:

$$\Delta_x : a = s_0 < s_1 < s_2 < ... < s_{n-1} < s_n = b,$$
$$\Delta_y : c = t_0 < t_1 < t_2 < ... < t_{n-1} < t_n = d,$$

with $s_i = a + ih$, $t_j = c + jh'$, where $h = \frac{b-a}{n}$, $h' = \frac{d-c}{n}$.
Concerning the remainder estimation, the following result was obtained:

Theorem 1 *[32] For uniform partitions of $[a, b]$ and $[c, d]$, the following trapezoidal inequality holds:*

$$\left| \int_c^d \int_a^b g(s,t)dsdt - T(f) \right| \leq \frac{L}{4n}(b-a)(d-c)(b-a+d-c), \tag{2}$$

where

$$T(g) = \frac{(b-a)(d-c)}{4n^2} \sum_{j=1}^{n} \sum_{i=1}^{n} [g(s_{i-1}, t_{j-1}) + g(s_{i-1}, t_j) + g(s_i, t_{j-1}) + g(s_i, t_j)] \tag{3}$$

For the case $[c, d] = [a, b]$, the trapezoidal cubature rule (3) becomes

$$\int_a^b \int_a^b g(s,t)dsdt = T_n(g) + E_n(g) \tag{4}$$

with

$$T_n(g) = \frac{(b-a)^2}{4n^2} \sum_{j=1}^{n} \sum_{i=1}^{n} [g(s_{i-1}, t_{j-1}) + g(s_{i-1}, t_j) + g(s_i, t_{j-1}) + g(s_i, t_j)] \tag{5}$$

and the remainder estimate is:

$$E_n(g) \leq \frac{L}{2n}(b-a)^3 \tag{6}$$

where $L \geq 0$ is the Lipschitz constant of g.

Remark 1 The relation (5) can be written as follows:

$$\int_a^b \int_a^b g(s,t)dsdt = \frac{(b-a)^2}{4n^2} \sum_{j=1}^{n} \sum_{i=1}^{n} \sum_{l_2=1}^{2} \sum_{l_1=1}^{2} (C_2^{l_2})^2 (C_2^{l_1})^2 \left[g(s_{i+l_1-2}, t_{j+l_2-2}) \right]. \tag{7}$$

where

$$C_2^l = \frac{2!}{l!(2-l)!}$$

3 Main results

3.1 The existence results

In the sequel let $I := [a,b] \times [a,b] \subset \mathbf{R}^2$, be a closed and bounded interval. Let (M,d) be a metric space with metric d on $\mathbf{R}^2$, and Ω be Banach space of all continuous mappings f from I into $\mathbf{R}$ with uniform norm. Now, we will prove the existence and uniqueness of the solution of Eq. (1) using the method of successive approximations.
Consider the integral equation (1) under the following conditions:

(i) $r \in C(I,\mathbf{R}), K \in C(I \times I,\mathbf{R}), \psi \in C(\mathbf{R},\mathbf{R})$,

(ii) there exists $\alpha > 0$, such that
$\mid \psi(u) - \psi(u') \mid \leq \alpha \mid u - u' \mid, \forall u,u' \in \mathbf{R}$.

(iii) $\alpha\lambda M_K(b-a)^2 < 1$, where $M_K > 0$ is such that $\mid K(s,t,x,y) \mid \leq M_K$, $\forall s,x \in [a,b]$ $,t,y \in [c,d]$, according continuity of K,

(iv) there exist $\zeta, \eta > 0$, such that
$\mid (K(s,t,x,y) - K(s',t',x',y') \mid \leq \zeta(\mid s-s' \mid + \mid t-t' \mid) + \eta(\mid x-x' \mid$
$+ \mid y-y' \mid),\ \forall (s,t),(s',t'),(x,y),(x',y') \in I$.

(v) there exists $\theta > 0$, such that
$\mid r(s,t) - r(s',t') \mid \leq \theta(\mid s-s' \mid + \mid t-t' \mid)\ \forall (s,t),(s',t') \in I$.

Let $\{\Psi_k\}_{k\in N}$ be a the sequence of function $\Psi_k : I \to \mathbf{R}$, defined by $\Psi_k(x,y) = \psi(X_k(x,y))$.

Theorem 2 *(a) Let the conditions (i)-(v) are satisfied. then the Eq. (1) has a unique solution $X^* \in \Omega$, and the sequence of successive approximations $\{X_k\}_{k\in N}$ such that*

$$X_0(s,t) = r(s,t),$$
$$X_k(s,t) = r(s,t) + \lambda \int_a^t \int_a^s K(s,t,x,y)\psi(X_{k-1}(x,y))dxdy, \qquad k \geq 1, \quad (8)$$

converges to the solution $X^* \in \Omega$. Furthermore, the following a priori and a posteriori error estimates hold

$$\|X^* - X_k\|_u \leq \frac{(\alpha\lambda M_K(b-a)^2)^k}{1-\alpha\lambda M_K(b-a)^2}\|X_0 - X_1\|_u, \quad (9)$$

$$\|X^* - X_k\|_u \leq \frac{(\alpha\lambda M_K(b-a)^2)}{1-\alpha\lambda M_K(b-a)^2}\|X_{k-1} - X_k\|_u, \quad (10)$$

Moreover, the sequence of successive approximations is uniformaly bounded, that is, there exists a constant $\sigma \geq 0$ such that $|X_k(s,t)| \leq \sigma$.
(b) If all the condition (i)-(vi) are satifed, then the sequence $\{X_k\}_{k\in N}$ and $\{\Psi_k\}_{k\in N}$ are uniformly lipschitz with constants $L_0 = \lambda M_k M(b-a) + \lambda M\zeta(b-a)^2$ and $L' = \alpha\big(\lambda M_K M(b-a) + \lambda M\zeta(b-a)^2\big)$, respectively. where M is given in (16).

Proof (a) Consider the iterative scheme

$$X_{k+1}(s,t) = r(s,t) + \lambda \int_a^t \int_a^s K(s,t,x,y)\psi(X_k(x,y))dxdy, \qquad k = 1,2,... \tag{11}$$

we have

$$\begin{aligned}\big|X_{k+1}(s,t) - \ X_k(s,t)\big| \leq & \Big|\lambda \int_a^t \int_a^s K(s,t,x,y)\psi(X_k(x,y))dxdy - \\ & \lambda \int_a^t \int_a^s K(s,t,x,y)\psi(X_{k-1}(x,y))dxdy\Big| \\ & \leq \lambda\big|K(s,t,x,y)\big| \int_a^t \int_a^s \big|\psi(X_k(x,y)) - \psi(X_{k-1}(x,y))\big|dxdy \\ & \leq \alpha\lambda M_K \int_a^t \int_a^s \big|\psi(X_k(x,y)) - \psi(X_{k-1}(x,y))\big|dxdy \\ & \leq \alpha\lambda M_K(t-a)(s-a)\big\|X_k - X_{k-1}\big\|_u \\ & \leq \alpha\lambda M_K(b-a)^2\big\|X_k - X_{k-1}\big\|_u\end{aligned}$$

Therefore, we obtained

$$\|X_{k+1} - X_k\|_u \leq \alpha\lambda M_K(b-a)^2\|X_k - X_{k-1}\|_u.$$

Hence

$$\|X_{k+1} - X_k\|_u \leq (\alpha\lambda M_K(b-a)^2)^k\|X_2 - X_1\|_u.$$

Since Ω is a complete metric space, and $\alpha\lambda M_K(b-a)^2 < 1$, then we conclude by using the Weierstrass M-test that the series

$$\sum_{k=1}^{\infty}(X_{k+1}(s,t) - X_k(s,t)), \tag{12}$$

is absolutely and uniformly convergent on $[a,b] \times [c,d]$. On the other hand, $X_k(s,t)$ can be written as

$$X_k(s,t) = X_1(s,t) + \sum_{m=1}^{k-1}(X_{m+1}(s,t) - X_m(s,t)),$$

therefore from uniform convergence of the series (12), we conclude that $\lim_{k\to\infty} X_k(s,t)$ exists for all $(s,t) \in [a,b] \times [c,d]$, that is, there exists a unique solution $X^* \in \mathbf{X}$ such that

$$\lim_{k\to\infty} \| X_k - X^* \| = 0.$$

Taking limit of both sides of Eq. (11), we obtain

$$\begin{aligned}\lim_{k\to\infty} X_{k+1}(x,y) &= \lim_{k\to\infty} r(s,t) + \lambda \int_a^t \int_a^s K(s,t,x,y)\psi(\lim_{k\to\infty} X_k(x,y))dxdy \\ &= r(s,t) + \lambda \int_a^t \int_a^s K(s,t,x,y)\psi(X(x,y))dxdy = X(s,t)\end{aligned}$$

that is, $X^*(s,t)$ is the unique solution of (1).
Moreover, by the Banach's fixed point principle we obtain the estimates (9) and (10).
Let $\Psi_0 : [a,b]^2 \to \mathbf{R}, \Psi_0(x,y) = \psi(r(x,y))$. Since ψ, r are continuous, we infer that Ψ_0 is continuous on the compact set $[a,b]^2$ and therefore $M_0 \geq 0$ exist, such that

$$|\psi_0(x,y)| \leq M_0 \qquad \forall (x,y) \in [a,b]\times[c,d]. \tag{13}$$

For $(s,t) \in [a,b]^2$, it follows that

$$\begin{aligned}&\big|X_k(s,t) - X_{k-1}(s,t)\big| \\ &\leq \lambda\big|K(s,t,x,y)\big| \int_a^t \int_a^s \big|\psi(X_{k-1}(x,y)) - \psi(X_{k-2}(x,y))\big|dxdy \\ &\leq \lambda M_K \int_a^t \int_a^s \big|\psi(X_{k-1}(x,y)) - \psi(X_{k-2}(x,y))\big|dxdy \\ &= (\alpha\lambda M_K(b-a)^2)\big\|X_{k-1} - X_{k-2}\big\|_u.\end{aligned}$$

and by induction,

$$\big|X_k(s,t) - X_{k-1}(s,t)\big| \leq (\alpha\lambda M_K(b-a)^2)^{k-1}\big\|X_1 - X_0\big\|_u.$$

Choosing $X_0 \in \Omega, X_0 = r$ we have

$$\begin{aligned}&\big|X_0(s_p,t_q) - X_1(s_p,t_q)| \\ &= \Big|r(s_p,t_q) - r(s_p,t_q) + \lambda \int_a^t \int_a^s K(s_p,t_q,x,y)\psi(X_0(x,y))dxdy\Big| \qquad (14) \\ &\leq \lambda \int_a^t \int_a^s \big|K(s_p,t_q,x,y)\psi(X_0(x,y))\big|dxdy \\ &= \lambda M_k M_0 (t-a)(s-a) \leq \lambda M_k M_0 (b-a)^2 \qquad (15)\end{aligned}$$

So,

$$\begin{aligned}\big|X_k(s,t) - X_0(s,t)\big| &\leq \big|X_k(s,t) - X_{k-1}(s,t)\big| + \big|X_{k-1}(s,t) - X_{k-2}(s,t)\big| \\ &+ ... + \big|X_1(s,t) - X_0(s,t)\big| \\ &\leq ((\alpha\lambda M_K(b-a)^2)^{k-1} + (\alpha\lambda M_K(b-a)^2)^{k-2} \\ &+ ... + \alpha\lambda M_K(b-a)^2 + 1)\big\|X_1 - X_0\big\|_u \\ &= \frac{1-(\alpha\lambda M_K(b-a)^2)^k}{1-(\alpha\lambda M_K(b-a)^2)}.\big\|X_1 - X_0\big\|_u \\ &\leq \frac{\lambda M_K(b-a)^2 M_0}{1-\alpha\lambda M_K(b-a)^2} \quad \forall (s,t) \in [a,b]\times[c,d].\end{aligned}$$

Let $M_r \geq 0$ such that $|r(s,t)| \leq M_r$ for all $(s,t) \in I$. Then

$$\left|X_k(s,t)\right| \leq \left|X_k(s,t) - X_0(s,t)\right| + \left|X_0(s,t)\right| \leq \frac{\lambda M_K (b-a)^2 M_0}{1 - \alpha\lambda M_K (b-a)^2} + M_r = \sigma.$$

for all $(s,t) \in [a,b]$.
(b) cosidering

$$M = max\big(M_0, max\{|\psi(u)| : u \in [-\sigma, \sigma]\}\big) \tag{16}$$

we get

$$|\Psi_k(s,t)| = |\psi(X_k(s,t))| \leq M$$

for all $(s,t) \in [a,b] \times [a,b]$ and $m \in \mathbf{N}$. Let $(s,t),(s',t') \in [a,b] \times [a,b]$, we obtain

$$\left|X_0(s,t) - X_0(s',t')\right| \leq \theta(\left|s-s'\right| + \left|t-t'\right|)$$

For arbitrary $\psi \in C(\mathbf{R},\mathbf{R})$ and $s,t,s',t' \in [a,b]$, since

$$\begin{aligned}
&\int_a^{t'}\int_a^{s'} K(s,t,x,y)\psi\big(X(x,y)\big)dxdy \\
&= \int_a^{t}\int_a^{s} K(s,t,x,y)\psi\big(X(x,y)\big)dxdy + \int_t^{t'}\int_a^{s} K(s,t,x,y)\psi\big(X(x,y)\big)dxdy \\
&+ \int_a^{t}\int_s^{s'} K(s,t,x,y)\psi\big(X(x,y)\big)dxdy + \int_t^{t'}\int_s^{s'} K(s,t,x,y)\psi\big(X(x,y)\big)dxdy
\end{aligned}$$

we obtain

$$\begin{aligned}
\left|X_k(s,t) - X_k(s',t')\right| &\leq \left|r(s,t) - r(s',t')\right| \\
&+ \mid \lambda \int_a^{t}\int_a^{s} K(s,t,x,y)\psi\big(X_{k-1}(x,y)\big)dxdy \\
&- \lambda \int_a^{t'}\int_a^{s'} K(s^{'},t^{'},x,y)\psi\big(X_{k-1}(x,y)\big))dxdy \mid \\
&\leq \mid \lambda \int_a^{t}\int_a^{s} K(s,t,x,y)\psi\big(X_{k-1}(x,y)\big)dxdy \\
&- \lambda \int_a^{t'}\int_a^{s'} K(s,t,x,y)\psi\big(X_{k-1}(x,y)\big))dxdy \mid \\
&+ \mid \lambda \int_a^{t'}\int_a^{s'} K(s,t,x,y)\psi\big(X_{k-1}(x,y)\big)dxdy \\
&- \lambda \int_a^{t'}\int_a^{s'} K(s^{'},t^{'},x,y)\psi\big(X_{k-1}(x,y)\big))dxdy \mid \\
&\leq \mid \int_t^{t'}\int_a^{s} K(s,t,x,y)\psi\big(X_{k-1}(x,y)\big)dxdy \mid
\end{aligned}$$

$$\begin{aligned}
&+ \mid \int_a^t \int_s^{s'} K(s,t,x,y)\psi\big(X(x,y)\big)dxdy \mid \\
&+ \mid \int_t^{t'} \int_s^{s'} K(s,t,x,y)\psi\big(X_{k-1}(x,y)\big)dxdy \mid \\
&+ \mid \lambda M \int_a^{t'} \int_a^{s'} (K(s,t,x,y) - K(s^{'},t^{'},x,y))dxdy \mid \\
&\leq \lambda M_k M|t'-t||s-a| + \lambda M_k M|t-a||s'-s| \\
&+ \lambda M_k M|t'-t||s'-s| \\
&+ \lambda M(\zeta(\mid s-s' \mid + \mid t-t' \mid))|t'-a||s'-a| \\
&\leq (\lambda M_k M(b-a) + \lambda M\zeta(b-a)^2)|t'-t| \\
&+ (\lambda M_k M(b-a) + \lambda M\zeta(b-a)^2)|s'-s| \\
&\leq (\lambda M_k M(b-a) + \lambda M\zeta(b-a)^2)(|t'-t| + |s'-s|)
\end{aligned}$$

with $L_0 = \lambda M_k M(b-a) + \lambda M\zeta(b-a)^2$ and

$$\begin{aligned}
\big|\Psi_0(s,t) - \Psi_0(s',t')\big| &\leq \alpha\big|X_0(s,t) - X_0(s',t')\big| \\
&\leq \alpha\theta(\big|s-s'\big| + \big|t-t'\big|)
\end{aligned}$$

$$\begin{aligned}
\big|\Psi_m(s,t) - \Psi_k(s',t')\big| &\leq \alpha\big|X_k(s,t) - X_k(s',t')\big| \\
&\leq \alpha L_0(\big|s-s'\big| + \big|t-t'\big|) \\
&= \alpha L_0(\big|s-s'\big| + \big|t-t'\big|) \\
&= L'(\big|s-s'\big| + \big|t-t'\big|).
\end{aligned}$$

where

$$L' = \alpha L_0 = \alpha\big(\lambda M_k M(b-a) + \lambda M\zeta(b-a)^2\big)$$

Corollary 1 *The functions* $K(s_p,t_q,x,y)\psi(x,y,X_k(x,y)), p = \overline{0,n}\ q = \overline{0,n}, k \in N$ *are uniformly lipschitz with constant*

$$L = \eta M + M_K\alpha\big(\lambda M_k M(b-a) + \lambda M\zeta(b-a)^2\big)$$

Proof Let arbitrary $(s,t),(s',t') \in [a,b]\times[c,d]$. We define the function $\Psi_{k,p,q} : I \to \mathbf{R}, \Psi_{k,p,q}(x,y) = K(s_p,t_q,x,y)\psi(X_k(x,y)), (x,y) \in I, p,q = \overline{0,n}$. Then

$$\begin{aligned}
&\big|\Psi_{k,s,t}(x,y) - \Psi_{k,s,t}(x',y')\big| \\
&= \big|K(s_p,t_q,x,y)\psi(X_k(x,y)) - K(s_p,t_q,x',y')\psi(X_k(x',y'))\big| \\
&\leq \big|K(s_p,t_q,x,y)\psi(X_k(x,y)) - K(s_p,t_q,x',y')\psi(X_k(x,y))\big| \\
&+ \big|K(s_p,t_q,x',y')\psi(X_k(x,y)) - K(s_p,t_q,x',y')\psi(X_k(x',y'))\big| \\
&\leq M\big|K(s_p,t_q,x,y) - K(s_p,t_q,x',y')\big| \\
&+ M_K\big|\psi(X_k(x,y)) - \psi(X_k(x',y'))\big| \\
&\leq M\eta(\big|x-x'\big| + \big|y-y'\big|) + M_K L'(\big|x-x'\big| + \big|y-y'\big|) \\
&\leq L(\big|x-x'\big| + \big|y-y'\big|). \quad k \in \mathbf{N}
\end{aligned} \tag{17}$$

where

$$L = M\eta + M_K L' = \eta M + M_K \alpha\big(\lambda M_k M(b-a) + \lambda M\zeta(b-a)^2\big)$$

Here, we present a numerical method to solve E.q (1). We consider the uniform partitions

$$D_1 : a = s_0 < s_1 < s_2 < ... < s_{n-1} < s_n = b\ ,$$
$$D_2 : a = t_0 < t_1 < t_2 < ... < t_{n-1} < t_n = b,$$

with $s_p = a + p\frac{b-a}{n}, p = \overline{0,n}$, $t_q = c + q\frac{b-a}{n}, q = \overline{0,n}$.

Notation : The symbol $p = \overline{0,n}$ means the integer values of the p vary from 0 to n arbitrarily.

We see that on the knots of the partition the sequence of successive approximations (8) is

$$X_0(s_p,t_q) = r(s_p,t_q),$$
$$X_k(s_p,t_q) = r(s_p,t_q) + \lambda \int_a^{t_q}\int_a^{s_p} K(s_p,t_q,x,y)\psi(X_{k-1}(x,y))dxdy, \quad (18)$$

and applying the quadrature (7) to relation (18), we obtain the following iterative relation:

$$\begin{aligned}\overline{X}_k(s_p,t_q) &= r(s_p,t_q)\\ &+ \frac{(b-a)^2}{4n^2}\sum_{i=1}^{p}\sum_{j=1}^{q}\Big(K(s_p,t_q,x_{i-1},y_{j-1})\psi(X_{k-1}(t_{i-1},t_{j-1})).\\ &+ K(s_p,t_q,x_i,y_{j-1})\psi(X_{k-1}(x_i,y_{j-1}))\\ &+ K(s_p,t_q,x_{i-1},y_j)\psi(X_{k-1}(x_{i-1},y_j))\\ &+ K(s_p,t_q,x_i,y_j)\psi(X_{k-1}(x_i,y_j))\Big),\end{aligned}$$

The above recursive relation can be written as follows:

$$\begin{aligned}\overline{X}_k(s_p,t_q) &= r(s_p,t_q)\\ &+ \frac{(b-a)^2}{4n^2}\sum_{i=1}^{p}\sum_{j=1}^{q}\sum_{l_2=1}^{2}\sum_{l_1=1}^{2}(C_2^{l_2})^2(C_2^{l_1})^2\left[K(s_p,t_q,x_{i+l_1-2},y_{j+l_2-2})\right.\\ &\qquad \left.\psi(X_{k-1}(x_{i+l_1-2},y_{j+l_2-2}))\right].\end{aligned} \quad (19)$$

Also, we have

$$X_k(s_p,t_q) = \overline{X}_k(s_p,t_q) + E_{k,p,q} \quad (20)$$

with

$$\mid E_{k,p,q} \mid \leq L\frac{(b-a)^3}{2n} \qquad \forall k \in N, p = \overline{0,n}, q = \overline{0,n} \quad (21)$$

where $L > 0$ is the Lipschitz constant of $\Psi_{k,p,q}(x,y)$.

3.2 The convergence analysis

Here, we present the rate of convergence and the error estimate of the proposed approach for solving two- dimensional nonlinear Volterra integral equations.

Theorem 3 *Suppose that the conditions (i)-(v) are satisfied. Moreover, assume that $\overline{X_k}(s_p,t_q)$ is the approximation solution of (1) by using numerical successive approximations method. If $\lambda\alpha M_k(b-a)^2<1$, then, $\overline{X_k}(s_p,t_q)$ is convergent to the unique solution of Eq. (1), and the error estimate is:*

$$\|X^*-\overline{X}_k\|\leq\frac{(\lambda\alpha M_k(b-a)^2)^{k+1}}{\alpha(1-\lambda\alpha M_k(b-a)^2)}M_0+\frac{L(b-a)^2}{2n(1-\lambda\alpha M_K(b-a)^3)},\qquad(22)$$

Proof By (9) and (14) we have

$$\left|X^*(s_p,t_q)-X_k(s_p,t_q)\right|\leq\frac{(\lambda\alpha M_k(b-a)^2)^{k+1}}{\alpha(1-(\lambda\alpha M_k(b-a)^2))}M_0,\qquad(23)$$

Using (23) we have

$$\begin{aligned}|X^*(s_p,t_q)-\overline{X}_k(s_p,t_q)|&\leq|X^*(s_p,t_q)-X_k(s_p,t_q)|+|X_k(s_p,t_q)-\overline{X}_k(s_p,t_q)|\\&\leq\frac{(\lambda\alpha M_k(b-a)^2)^{k+1}}{\alpha(1-(\lambda\alpha M_k(b-a)^2))}M_0+\|X_k-\overline{X}_k\|_u\end{aligned}$$

therefore, we shall obtain the estimates for $\|X_k-\overline{X}_k\|_u$.
Form (8), (19) and (21) for $k=1$, we obtain

$$\left|X_1(s_p,t_q)-\overline{X}_1(s_p,t_q)\right|\leq|E_{1,p,q}|\leq\frac{L(b-a)^3}{2n}$$

Using (20) and (21) we obtain

$$\begin{aligned}\left|X_k(s_p,t_q)-\overline{X}_k(s_p,t_q)\right|&\leq|E_{k,p,q}|\\&+\lambda\frac{(b-a)^2}{4n^2}\sum_{i=1}^{p}\sum_{j=1}^{q}\sum_{l_2=1}^{2}\sum_{l_1=1}^{2}(C_2^{l_2})^2(C_2^{l_1})^2\\&\times|K(s_p,t_q,x_{i+l_1-2},y_{j+l_2-2})|\big(\psi(X_{k-1}(x_{i+l_1-2},y_{j+l_2-2})\\&-\psi(\overline{X}_{k-1}(x_{i+l_1-2},y_{j+l_2-2}))\big)\end{aligned}$$

Also, for $k=2$ it follow that

$$\begin{aligned}\left|X_2(s_p,t_q)-\overline{X}_2(s_p,t_q)\right|\leq&\frac{L(b-a)^3}{2n}\\&+\lambda\alpha M_k\frac{(b-a)^2}{4}\sum_{l_2=0}^{2}\sum_{l_1=0}^{2}(C_2^{l_2})^2(C_2^{l_1})^2||X_1-\overline{X}_1||_u\\&\leq\frac{L(b-a)^3}{2n}+\lambda\alpha M_kL(b-a)^2||X_1-\overline{X}_1||_u\end{aligned}$$

$$\leq \frac{L(b-a)^3}{2n} + \lambda\alpha M_k L(b-a)^2 \frac{L(b-a)^3}{2n}$$
$$= (1 + \lambda\alpha M_k L(b-a)^2)\frac{L(b-a)^3}{2n}$$

By induction, for $k \in N$, $k \geq 3$, we obtain

$$| X_m(s,t) - \overline{X}_m(s,t) |$$
$$\leq [1 + \lambda\alpha M_K(b-a)^2 + ... + (\lambda\alpha M_K(b-a)^2)^{m-1}]\frac{L(b-a)^3}{2n}$$
$$= \frac{1-(\lambda\alpha M_K(b-a)^2)^m}{1-\lambda\alpha M_K(b-a)^2}\frac{L(b-a)^3}{2n}$$
$$\leq \frac{1}{1-\lambda\alpha M_K(b-a)^2}\frac{L(b-a)^3}{2n}$$

therefore

$$\|X_k - \overline{X}_k\|_u \leq \frac{L(b-a)^2}{2n(1-\lambda\alpha M_K(b-a)^3)}. \tag{24}$$

Hence, from (23) and (24) we conclude that

$$\|X^* - \overline{X}_k\|_u \leq \frac{(\lambda\alpha M_k(b-a)^2)^{k+1}}{\alpha(1-\lambda\alpha M_k(b-a)^2)}M_0 + \frac{L(b-a)^2}{2n(1-\lambda\alpha M_K(b-a)^3)}.$$

Since $\lambda\alpha M_k(b-a)^2 < 1$, it is easy to see that

$$\lim_{k,n\to\infty} \|X^* - \overline{X}_k\| = 0, \tag{25}$$

which is the convergence of the proposed method.

3.3 Algorithm of the approach

Applying the above presented trapezoidal cubature rule we obtain the following iterative algorithm:

Step 1: Input the values $a, b, \lambda, \varepsilon', n$ and the functions r, K, ψ.

Step 2: Choose $\varepsilon' > 0$ and for $p = \overline{0,n}$, $q = \overline{0,n}$, set $\overline{X}_0(s_p, t_q) = r(s_p, t_q)$.

Step 3: For all $p = \overline{0,n}$, $q = \overline{0,n}$, Compute $\overline{X}_k(s_p, t_q)$ by (19).

Step 4: We use the values computed at the previous step and obtain for $p = \overline{0,n}$, $q = \overline{0,n}$, the values:

$$| \overline{X}_k(s_p, t_q) - \overline{X}_{k-1}(s_p, t_q) |$$

Step 5: If $| \overline{X}_k(s_p, t_q) - \overline{X}_{k-1}(s_p, t_q) | < \varepsilon'$, print k and print $\overline{X}_k(s_p, t_q)$, for all $p = \overline{0,n}$, $q = \overline{0,n}$, stop.; otherwise, set $k = k + 1$ and go to Step 3.

This algorithm has a practical criterion presented below in Remark (2).

Remark 2 The "a-posteriori error" estimate is useful to get the stopping criterion. Such estimate can be obtained as follows:
For given $\varepsilon^{'} > 0$ (previously chosen), there is determined the first natural number m for which

$$\mid \overline{X}_k(s_p, t_q) - \overline{X}_{k-1}(s_p, t_q) \mid < \varepsilon^{'} \tag{26}$$

and we stop to this m retaining the approximations $u_k(s,t)$ of solution. We observe

$$\begin{aligned} \|X^* - \overline{X}_k\|_u &\leq \|X^* - X_k\|_u + \|X_k - \overline{X}_k\|_u \\ &\leq \frac{(\alpha\lambda M_K(b-a)^2)}{1-(\alpha\lambda M_K(b-a)^2)}\|X_{k-1} - X_k\|_u + \frac{L(b-a)^3}{2n(1-\lambda\alpha M_K(b-a)^2)} \end{aligned}$$

and

$$\begin{aligned} \|X_k - X_{k-1}\|_u &\leq \|X_k - \overline{X}_k\|_u + \|\overline{X}_k - \overline{X}_{k-1}\|_u + \|\overline{X}_{k-1} - X_{k-1}\|_u \\ &\leq \frac{L(b-a)^3}{n(1-\lambda\alpha M_K(b-a)^2)} + \|\overline{X}_k - \overline{X}_{k-1}\|_u \end{aligned}$$

So,

$$\begin{aligned} \|X^* - \overline{X}_k\|_u &\leq \frac{(\alpha\lambda M_K(b-a)^2)}{1-(\alpha\lambda M_K(b-a)^2)}\|\overline{X}_k - \overline{X}_{k-1}\|_u, \\ &+ \frac{(\alpha\lambda M_K(b-a)^2)+1}{(1-(\alpha\lambda M_K(b-a)^2))^2}\frac{L(b-a)^3}{2n} \end{aligned}$$

and therefore, in order to obtain $\mid X^*(s,t) - \overline{X}_k(s,t) \mid < \varepsilon$, we require

$$\frac{(\alpha\lambda M_K(b-a)^2)+1}{(1-(\alpha\lambda M_K(b-a)^2))^2}\frac{L(b-a)^3}{2n} < \frac{\varepsilon}{2}, \tag{27}$$

and

$$\frac{(\alpha\lambda M_K(b-a)^2)}{1-(\alpha\lambda M_K(b-a)^2)}\|\overline{X}_k - \overline{X}_{k-1}\| < \frac{\varepsilon}{2}.$$

We can choose the least natural number n, for which inequality (29) holds.

$$\frac{(\alpha\lambda M_K(b-a)^2)+1}{(1-(\alpha\lambda M_K(b-a)^2))^2}\frac{L(b-a)^3}{2n} < \frac{\varepsilon}{2}, \tag{28}$$

this is

$$\frac{(\alpha\lambda M_K(b-a)^2)+1}{(1-(\alpha\lambda M_K(b-a)^2))^2}\frac{L(b-a)^3}{\varepsilon} < n, \tag{29}$$

Finally, we find the smallest natural number $k \in \mathbf{N}$ (this is the last iterative step to be made) for which,

$$\|\overline{X}_k - \overline{X}_{k-1}\|_u < \frac{\varepsilon}{2}.\frac{1-\alpha\lambda M_K(b-a)^2}{\alpha\lambda M_K(b-a)^2} = \varepsilon^{'}.$$

With these, the inequality $|\ \overline{X}_k(s_p,t_q) - \overline{X}_{k-1}(s_p,t_q)\ | < \varepsilon^{'}$ leads to $|\ X^*(s_p,t_q) - \overline{X}_k(s_p,t_q)\ | < \varepsilon$, and the desired accuracy ε is obtained.

4 Numerical experiments

In this section, we report the numerical results of the implementation of the proposed scheme to show the applicability and efficiency of the methods. We introduce the notations

$$E_{p,q} := |X^*(s_p,t_q) - \overline{X}_k(s_p,t_q)|, \tag{30}$$

and

$$\|E_n\|_\infty := \max\{E_{p,q}|p,q = 0,1,...,n\} \tag{31}$$

where X^* and $\overline{X}_k$ denote the exact solution of integral equation (1) and its approximation of order n obtained by the method presented in Sect. 3, respectively. All results computed by programming in Maple 17.

Example 1 [12, 24] Consider two-dimensional nonlinear Fredholm integral equation

$$X(s,t) = r(s,t) + \int_0^t \int_0^s K(s,t,x,y)(X(x,y))^2 dxdy, \qquad (s,t) \in [0,1] \times [0,1], \tag{32}$$

where

$$r(s,t) = s + t - \frac{1}{12} st(s^3 + 4s^2t + 4st^2 + t^3),$$
$$K(s,t,x,y) = s + t - y - x.$$

The exact solution of this equation is

$$X(s,t) = s + t.$$

Applying the algorithm for $n = 10, \varepsilon^{'} = 10^{-20}$, we obtain the number of iterations $k = 17$ iterations. The numerical results of this example at the various values of s and t in the interval $[0,1]$ by proposed method are shown in *Table 1* for $n = 10$.

In order to more detailed testing of convergence, we consider $n = 20$ and for $\varepsilon^{'} = 10^{-25}$ the number of iterations is $k = 22$. It is seen that $E_{p,q}, p,q = \overline{0,n}$, tend to zero as h decrease. The numerical results are shown in *Table 2*. For $n = 40, \varepsilon^{'} = 10^{-25}$, we have $k = 23$ iterations and the results are in *Table 3*. The results $\|E_n\|_\infty$ for $\varepsilon^{'} = 10^{-15}$ and $n \in \{10,20,40\}$, respectively, are $1.015 \times 10^{-5}, 1.761 \times 10^{-6}$ and 2.152×10^{-7}. The results in *Table 1-3* confirm the convergence of the numerical method, that is $E_{p,q} \to 0$ as $n \to 0$.

Table 1 Numerical results for $n = 10$, in Example 1.

s t	0.1	0.3	0.5	0.7	0.9
0.1	5.7821×10^{-5}	6.0157×10^{-5}	6.7442×10^{-5}	8.0510×10^{-5}	1.0075×10^{-4}
0.3	5.8285×10^{-5}	5.5779×10^{-5}	5.7072×10^{-5}	6.1603×10^{-5}	6.8924×10^{-5}
0.5	6.1581×10^{-5}	5.3752×10^{-5}	4.8619×10^{-5}	4.4167×10^{-5}	3.8048×10^{-5}
0.7	7.0985×10^{-5}	5.5914×10^{-5}	4.2435×10^{-5}	2.6858×10^{-5}	4.7163×10^{-6}
0.9	9.1215×10^{-5}	6.4017×10^{-5}	3.7157×10^{-5}	4.7692×10^{-6}	4.0212×10^{-5}

Table 2 Numerical results for $n = 20$, in Example 1.

s t	0.1	0.3	0.5	0.7	0.9
0.1	9.2769×10^{-6}	9.6517×10^{-6}	1.0820×10^{-5}	1.2917×10^{-5}	1.6164×10^{-5}
0.3	9.3513×10^{-6}	8.9494×10^{-6}	9.1568×10^{-6}	9.8837×10^{-6}	1.1058×10^{-6}
0.5	9.8801×10^{-6}	8.6241×10^{-6}	7.8005×10^{-6}	7.0862×10^{-6}	6.1046×10^{-6}
0.7	1.1389×10^{-5}	8.9709×10^{-6}	6.8083×10^{-6}	4.3092×10^{-6}	7.5669×10^{-7}
0.9	1.4635×10^{-5}	1.0271×10^{-5}	5.9609×10^{-6}	7.6516×10^{-7}	6.4517×10^{-6}

Table 3 Numerical results for $n = 40$, in Example 1.

s t	0.1	0.3	0.5	0.7	0.9
0.1	1.1872×10^{-6}	1.2352×10^{-6}	1.3848×10^{-6}	1.6531×10^{-6}	2.0686×10^{-6}
0.3	1.1968×10^{-6}	1.1453×10^{-6}	1.1719×10^{-6}	1.2649×10^{-6}	1.4152×10^{-6}
0.5	1.2644×10^{-6}	1.1037×10^{-6}	9.9839×10^{-7}	9.0695×10^{-7}	7.8126×10^{-7}
0.7	1.4575×10^{-6}	1.1481×10^{-6}	8.7134×10^{-7}	5.5154×10^{-7}	9.6840×10^{-8}
0.9	1.8728×10^{-6}	1.3144×10^{-6}	7.6285×10^{-7}	9.7854×10^{-8}	8.2567×10^{-7}

Example 2 Consider two-dimensional nonlinear Fredholm integral equation

$$X(s,t) = r(s,t) + \int_0^t \int_0^s K(s,t,x,y)\sin(X(x,y))dxdy, \qquad (s,t) \in [0,1]\times[0,1], \tag{33}$$

where

$$r(s,t) = 1 + ts + \sin(st+1)s^2t^2 + \sin(1)s^2t^2 - 2\cos(1)st + 2ts\cos(st+1),$$
$$K(s,t,x,y) = syx^2t^2,$$

and exact solution

$$X(s,t) = 1 + st.$$

By applying the same procedure as described in the proposed technique, we have the following numerical results at different values of s and t in the interval $[0,1]$, which are shown in *Table 4* for $\varepsilon' = 10^{-20}$ and $n \in \{10, 20, 40\}$. Also, we obtain the accuracy $O(10^{-4} - 10^{-7})$, $O(10^{-5} - 10^{-8})$ and $O(10^{-6} - 10^{-8})$ respectively.

Table 4 Numerical results for $n = 10, n = 20, n = 40$, in Example 2.

(s,t)	$E_{p,q}, n = 10$	$E_{p,q}, n = 20$	$E_{p,q}, n = 40$
(0.0,0.0)	0	0	0
(0.1,0.1)	1.854351×10^{-7}	4.637924×10^{-8}	1.159607×10^{-8}
(0.2,0.2)	1.483481×10^{-6}	3.710339×10^{-7}	9.276854×10^{-8}
(0.3,0.3)	5.006748×10^{-6}	1.252239×10^{-6}	3.130938×10^{-7}
(0.4,0.4)	1.186784×10^{-5}	2.968271×10^{-6}	7.421482×10^{-7}
(0.5,0.5)	2.317939×10^{-5}	5.797404×10^{-6}	1.449508×10^{-6}
(0.6,0.6)	4.005398×10^{-5}	1.001791×10^{-5}	2.504750×10^{-6}
(0.7,0.7)	6.360425×10^{-5}	1.590807×10^{-5}	3.977450×10^{-6}
(0.8,0.8)	9.494278×10^{-5}	2.374617×10^{-5}	5.937186×10^{-6}
(0.9,0.9)	1.351822×10^{-4}	3.381046×10^{-5}	8.453532×10^{-6}
(0.1,0.1)	1.854351×10^{-4}	4.637921×10^{-5}	9.989606×10^{-6}
k	23	23	24
$\|E_n\|_\infty$	1.854×10^{-4}	4.638×10^{-5}	9.990×10^{-6}

5 Conclusions

In this investigation, a computational method has been provided to find the approximate solution of two-dimensional nonlinear Volterra integral equations based on the successive approximation method and the trapezoidal quadrature rule. In Theorem 2, we obtain the existence and uniqueness of the solution and prove some the uniformly boundedness and uniformly Lipschitz properties for the terms of the sequence of successive approximations. The convergence and the error estimation of this presented iterative method are proved in Theorem 3. The numerical results reported in tables show that only a small number of iteration is required to obtain a good approximate solution.

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