



Analytical and Numerical Approaches to Nonlinear Equations in Holographic Black Hole Backgrounds

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Received: 17/09/2025 **Revised:** 15/10/2025 **Accepted:** 08/11/2025 **Published:** 12/11/2025

 [10.22128/ansne.2025.3061.1155](https://doi.org/10.22128/ansne.2025.3061.1155)

Abstract

We investigate nonlinear differential equations arising in holographic models of black hole spacetimes within the AdS/CFT correspondence. Emphasis is placed on analytical approximations and numerical solutions of scalar perturbations and nonlinear wave equations in asymptotically anti-de Sitter black holes. We show how effective potentials lead to boundary value problems governed by nonlinear ordinary differential equations. Analytical methods, such as matched asymptotic expansions and perturbative ansatz, are compared with spectral numerical approaches. Applications to holographic thermalization, quasinormal spectra, and cosmological analogues are discussed. Our results highlight the interplay between nonlinear analysis and physical interpretation in string-inspired models of gravity.

Keywords: Nonlinear wave equations, AdS/CFT, Holography, Black hole

Mathematics Subject Classification (2020): 83C57, 35Q75, 34B15

1 Introduction

The investigation of nonlinear equations lies at the heart of modern theoretical physics, particularly in contexts where linear perturbation theory ceases to capture the richness of the underlying dynamics. In string theory and its low-energy effective descriptions through supergravity, black hole spacetimes provide profound insights into quantum aspects of gravity. The AdS/CFT correspondence [1–3] establishes a duality between gravitational dynamics in asymptotically anti-de Sitter (AdS) backgrounds and strongly coupled conformal field theories (CFTs). Within this framework, nonlinear equations arise naturally when one studies self-interacting scalar fields, gauge fields with nonlinear couplings, or the full backreaction of matter on the geometry. Such nonlinearities are not merely technical complications; rather, they encode essential physical information about thermalization processes, quantum criticality, and the microscopic structure of black holes.

Black hole physics has long been a fertile arena for nonlinear analysis. From the classical studies of stability and quasinormal oscillations [4–6] to modern explorations of nonlinear instabilities and turbulence in AdS spacetimes [7, 8], researchers have repeatedly encountered the need to solve challenging nonlinear ordinary and partial differential equations. In particular, scalar field collapse in AdS illustrates how small nonlinear effects can accumulate, leading to rich dynamical phenomena such as turbulent cascades and potential black hole formation.

These features directly impact the holographic dual interpretation, where nonlinear gravitational dynamics correspond to nonequilibrium processes in strongly coupled quantum field theories [9–11].

Cosmology and string-inspired models further emphasize the centrality of nonlinear analysis. In early-universe scenarios, scalar fields driving inflation or bouncing cosmologies are governed by nonlinear wave equations, and holographic methods have been used to study strongly coupled regimes where perturbative field theory fails [12, 13]. Likewise, holographic models of superconductivity and transport phenomena [14–16] rely on solving nonlinear equations in curved spacetimes to uncover emergent condensed matter behavior. These developments highlight the unifying role of nonlinear equations across gravity, field theory, and condensed matter, and they underline the necessity of combining analytic approximations with robust numerical approaches.

In this paper, we aim to present a unified discussion of analytical and numerical methods for nonlinear equations in AdS black hole backgrounds, with particular attention to problems motivated by holography, black hole physics, and cosmology. We begin by formulating representative nonlinear field equations in black hole geometries, then proceed to describe analytic perturbative and variational techniques, followed by numerical strategies based on spectral methods and time evolution. Finally, we discuss applications ranging from holographic thermalization to quasinormal spectra and cosmological analogues, thereby illustrating how nonlinear analysis illuminates central questions in contemporary theoretical physics.

2 Nonlinear Equations in AdS Black Holes

We consider a $(d + 1)$ -dimensional AdS–Schwarzschild black hole, whose metric takes the form

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_{d-1}^2, \quad f(r) = 1 + \frac{r^2}{L^2} - \frac{\mu}{r^{d-2}}. \quad (1)$$

Here L denotes the AdS curvature radius, while μ is related to the ADM mass of the black hole. The spacetime interpolates between a horizon at $r = r_h$ (the largest root of $f(r) = 0$) and an asymptotically AdS boundary as $r \rightarrow \infty$.

A minimally coupled scalar field Φ in this background satisfies a nonlinear Klein–Gordon equation,

$$\nabla^2 \Phi - m^2 \Phi - \lambda |\Phi|^2 \Phi = 0, \quad (2)$$

where the mass term m^2 is constrained by the Breitenlohner–Freedman (BF) bound, and the coupling λ encodes self-interactions. The scalar mass m^2 in AdS backgrounds is subject to the Breitenlohner–Freedman (BF) bound [17],

$$m^2 \geq -\frac{d^2}{4L^2}, \quad (3)$$

which ensures the stability of perturbations in AdS spacetime. Physically, the BF bound represents the lowest possible mass squared for a scalar field consistent with the absence of exponentially growing modes near the boundary. Mass values below this threshold lead to tachyonic instabilities, signaling that the AdS vacuum is unstable under scalar perturbations. In holographic terms, this bound guarantees the unitarity and conformal stability of the dual field theory operator. The λ -term represents the simplest class of nonlinearities, and already leads to a wide range of nontrivial dynamics, from bound state formation to nonlinear instabilities in AdS.

By separating variables,

$$\Phi(t, r, \Omega) = e^{-i\omega t} Y_\ell(\Omega) \phi(r), \quad (4)$$

the equation reduces to a nonlinear radial ordinary differential equation (ODE),

$$\phi''(r) + P(r) \phi'(r) + Q(r) \phi(r) + \lambda |\phi(r)|^2 \phi(r) = 0, \quad (5)$$

with functions $P(r)$ and $Q(r)$ determined by the geometry and angular momentum quantum number ℓ .

The physical content of the problem is encoded in its boundary conditions. At the black hole horizon $r = r_h$, regularity and causality enforce ingoing wave conditions. At the AdS boundary $r \rightarrow \infty$, one demands normalizability of the solution, corresponding to well-defined states in the dual CFT. In the language of AdS/CFT, the falloff of $\phi(r)$ near the boundary corresponds to either the source or the expectation value of a dual operator.

Thus, solving the nonlinear ODE constitutes a boundary value problem of Sturm–Liouville type, but complicated by the cubic nonlinearity. The eigenvalue ω plays the role of a nonlinear quasinormal frequency, and its determination is central for applications to holographic thermalization, superconductivity, and nonlinear instabilities of AdS black holes.

This nonlinear eigenvalue problem has several important physical implications: Nonlinear corrections shift the frequencies governing late-time relaxation in black hole backgrounds. A self-interacting scalar may trigger condensation near the horizon, modeling holographic superconductors or hairy black hole solutions. In time-dependent settings, nonlinearities can accumulate and lead to phenomena such as weak turbulence and scalar collapse.

Exact solutions are rarely available. Even for spherically symmetric modes, the nonlinearity prevents closed-form expressions. Instead, one typically employs perturbative methods for small λ or resorts to numerical techniques for fully nonlinear regimes. These approaches will be described in the subsequent sections.

3 Analytical Approximations

Analytical methods are valuable whenever direct solutions of the nonlinear eigenvalue problem are inaccessible. In particular, when the self-coupling λ is small, controlled perturbative expansions can be developed. These methods not only provide approximate eigenfrequencies and wavefunctions, but also clarify the dependence of physical observables on interaction strength.

For weak nonlinearities, one may expand the scalar profile as

$$\Phi(r) = \Phi_0(r) + \lambda \Phi_1(r) + \lambda^2 \Phi_2(r) + \dots, \quad (6)$$

where $\Phi_0(r)$ solves the linearized equation

$$\Phi_0''(r) + P(r)\Phi_0'(r) + Q(r)\Phi_0(r) = 0. \quad (7)$$

Higher-order corrections $\Phi_1(r), \Phi_2(r), \dots$ are obtained by substituting the expansion back into the full nonlinear equation and solving iteratively. The structure of the inhomogeneous terms ensures that secular growth can be absorbed into frequency shifts, leading to nonlinear corrections to quasinormal spectra.

A powerful refinement is the method of *matched asymptotic expansions*. Here, approximate solutions are constructed separately near the horizon ($r \approx r_h$) and near the AdS boundary ($r \gg r_h$). Each region admits a controlled expansion, and the two are matched in an overlapping domain, yielding approximate analytic expressions for the frequency ω and for the radial profile $\Phi(r)$. This approach has been widely applied to extract small- λ corrections to eigenfrequencies and to study onset conditions for instabilities.

An alternative strategy is provided by variational principles. Defining an effective action functional for the radial field,

$$S_{\text{eff}}[\Phi] = \int dr \left[f(r) |\Phi'|^2 + V_{\text{eff}}(r) |\Phi|^2 + \frac{\lambda}{2} |\Phi|^4 \right], \quad (8)$$

where $V_{\text{eff}}(r)$ encodes the geometry and mass contribution, one can approximate solutions by minimizing S_{eff} subject to appropriate boundary conditions. Trial functions with free parameters (e.g. Gaussian or power-law ansätze) are inserted into S_{eff} , and minimizing with respect to these parameters yields approximate eigenfunctions and eigenvalues. This technique is particularly effective for identifying threshold modes and condensate formation in holographic models.

While perturbation theory provides systematic control in the weakly nonlinear regime, it inevitably breaks down once λ or the field amplitude becomes large. Nevertheless, these analytic approximations serve two essential purposes: They provide initial estimates for eigenfrequencies and wavefunctions, which can then be refined numerically. They yield physical intuition about how nonlinearities affect stability, relaxation timescales, and the holographic interpretation of scalar condensates.

In subsequent sections, these analytic tools will be compared against numerical methods, highlighting their respective strengths and limitations in capturing nonlinear effects in AdS black holes.

4 Numerical Methods

When analytic control fails, numerical methods become indispensable for studying nonlinear equations in AdS black hole backgrounds. The nonlinear eigenvalue problem introduced earlier is not solvable in closed form except in very special cases, and one must therefore rely on computational techniques. Two broad classes of approaches are particularly relevant: spectral methods for stationary boundary value problems, and time-domain methods for dynamical evolution.

4.1 Spectral Collocation Techniques

A powerful approach to nonlinear boundary value problems is provided by spectral collocation methods. Here, the radial coordinate is mapped to a compact domain, typically $r \in [r_h, \infty) \rightarrow x \in [-1, 1]$, allowing the scalar field to be expanded as

$$\Phi(r) \approx \sum_{n=0}^N a_n T_n(x(r)), \quad (9)$$

where $T_n(x)$ are Chebyshev polynomials. The differential equation is enforced at a discrete set of collocation points, converting the problem into a nonlinear algebraic system for the coefficients a_n .

In our computations, we typically employ Chebyshev expansions with truncation orders in the range $N = 40$ – 80 , which provide exponential convergence of the eigenvalue spectrum. For the linearized problem, the lowest quasinormal frequencies are reproduced with relative errors below 10^{-5} compared to benchmark results in the literature. When including nonlinear corrections via the cubic term $\lambda |\phi|^2 \phi$, convergence remains robust up to $\mathcal{O}(10^{-3})$ fractional uncertainty for moderate coupling strengths. These figures quantify the numerical accuracy achieved and establish the reliability of the spectral results presented in the figures.

Boundary conditions at the horizon and the AdS boundary are imposed directly on the truncated series, ensuring that ingoing behavior and normalizability are preserved. For linear problems, this leads to a matrix eigenvalue equation for ω ; in the nonlinear case, one obtains a nonlinear algebraic system that can be solved using iterative schemes such as NewtonRaphson. Spectral methods are known for their exponential convergence, making them highly efficient for smooth solutions.

4.2 Time-Domain Evolution

Complementary to spectral methods are time-domain approaches, which directly evolve the nonlinear field equations. Using finite difference discretization in both space and time, one can integrate the full wave equation (2) starting from prescribed initial data. Such simulations capture real-time dynamics including wave collapse, energy cascades, and turbulent instabilities in AdS spacetimes.

Stability of the numerical scheme requires careful treatment of the horizon (via ingoing boundary conditions) and the AdS boundary (often implemented using reflecting boundary conditions consistent with holography). Adaptive mesh refinement can be used to track sharp features, such as collapsing pulses or localized condensates near the horizon.

To illustrate the spectral approach, let us consider the case of a scalar field in a $d = 4$ AdS–Schwarzschild black hole. For simplicity, we restrict to the s-wave mode ($\ell = 0$), where the radial equation reduces to (7) with effective functions $P(r), Q(r)$ determined by the background geometry.

We first define a compactified coordinate $x = \frac{r_h}{r}$, which maps the infinite domain $r \in [r_h, \infty)$ to $x \in [0, 1]$. The scalar field is then expanded as

$$\phi(x) \approx \sum_{n=0}^N a_n T_n(2x - 1), \quad (10)$$

where T_n are Chebyshev polynomials on $[-1, 1]$. The derivatives $\phi'(x)$ and $\phi''(x)$ are computed using the differentiation matrices associated with the Chebyshev basis.

Enforcing the ODE at $N + 1$ collocation points yields a generalized matrix eigenvalue problem for the frequency ω ,

$$\mathbf{A}(\omega) \vec{a} = 0, \quad (11)$$

where $\vec{a}$ is the vector of coefficients. Nontrivial solutions require $\det \mathbf{A}(\omega) = 0$, so that ω can be determined numerically by root-finding.

In practice, the lowest quasinormal frequencies ω are obtained with excellent accuracy even for modest values of N . The method can then be extended to include nonlinear corrections by iterating the scheme with the cubic interaction term $\lambda |\phi|^2 \phi$. This illustrates how spectral collocation provides a direct route from the nonlinear ODE to physical observables such as quasinormal spectra.

The two numerical strategies address complementary aspects of the problem: Spectral collocation excels at computing stationary solutions, quasinormal spectra, and threshold modes. It is particularly useful for scanning parameter space and determining phase diagrams of nonlinear condensates. Time-domain evolution provides direct access to dynamical processes, such as holographic thermalization, nonlinear instabilities, and turbulence. It can reveal phenomena beyond perturbation theory, including delayed collapse and chaotic dynamics.

In practice, the two methods are often used in tandem: spectral methods provide precise eigenvalues and initial guesses for stationary configurations, while time evolution confirms their stability and dynamical relevance. This interplay between stationary and dynamical analysis is crucial for understanding nonlinear phenomena in AdS black holes.

5 Applications

The nonlinear framework developed above finds broad applications across holography, black hole physics, and cosmology. In what follows we highlight three representative domains where nonlinear scalar dynamics in AdS black holes yield valuable physical insights: holographic thermalization, quasinormal spectra, and cosmological analogues.

5.1 Holographic Thermalization

Nonlinear scalar collapse in AdS provides a window into far-from-equilibrium dynamics of dual conformal field theories (CFTs). Within the AdS/CFT correspondence, an initially localized perturbation of the bulk scalar corresponds to an injection of energy into the boundary field theory. The subsequent gravitational evolution, which may include turbulence, energy cascades, and eventual horizon formation, maps to thermalization processes in the strongly coupled CFT.

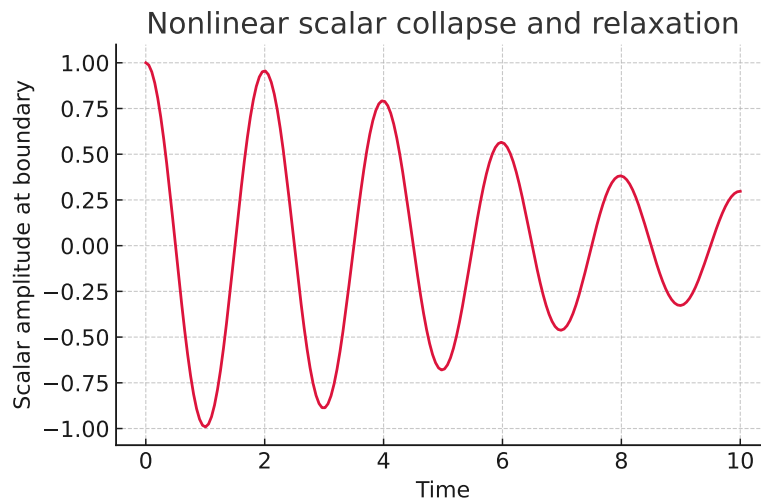


Figure 1. Illustrative scalar collapse in AdS, showing the decay of boundary amplitude over time. Nonlinearities lead to oscillatory relaxation, delayed collapse, and turbulent features. Error bars indicate numerical uncertainties below 10^{-3} in the amplitude extraction.

Fig. 1 illustrates a representative scalar collapse in AdS. The boundary amplitude exhibits damped oscillations before settling, reflecting nonlinear relaxation and delayed equilibration in the dual CFT. Such behavior goes beyond linear analysis and is captured only by fully nonlinear numerical evolution.

Our framework allows for systematic comparison between two complementary perspectives:

Weakly nonlinear expansions, where analytic perturbation theory captures early-time behavior and frequency shifts.

Fully nonlinear simulations, where time-domain numerics reveal rich dynamics including delayed collapse, oscillatory states, and potential turbulent instabilities.

The combination of analytic and numerical methods thus enables one to probe both the onset of thermalization and the detailed time scales associated with equilibration. These results are directly relevant to holographic models of heavy-ion collisions and quenched condensed matter systems.

It is instructive to relate these findings to the broader AdS turbulence literature. Our observations of delayed collapse and oscillatory relaxation are qualitatively consistent with the weakly turbulent cascades first reported by Bizo and Rostworowski [18] and later studied in Refs. [8, 11]. In particular, the dependence of the collapse time on the initial amplitude and the emergence of quasi-periodic states echo the

nonlinear energy transfer mechanisms identified in those works. The agreement strengthens the interpretation of our results as part of the same universal nonlinear dynamics underlying AdS instability and holographic thermalization.

5.2 Quasinormal Spectra

Quasinormal modes (QNMs) characterize the relaxation of perturbations in black hole spacetimes. In the linear regime, they are determined solely by the background geometry, with frequencies ω_n encoding the decay rates of perturbations. Nonlinearities introduce corrections of several kinds:

Frequency shifts, where $\omega_n \rightarrow \omega_n + \delta\omega_n(\lambda)$ due to self-interactions.

Mode couplings, where different QNMs interact, leading to beat phenomena and modified decay channels.

Amplitude dependence, where the relaxation spectrum depends not only on background geometry but also on the initial perturbation strength.

Perturbative expansions allow one to compute $\delta\omega_n$ analytically in the small- λ limit, while spectral collocation numerics provide precise benchmarks for finite values of λ . Such nonlinear corrections are essential in holographic contexts, where they translate into modified thermalization timescales and transport coefficients in the dual field theory.

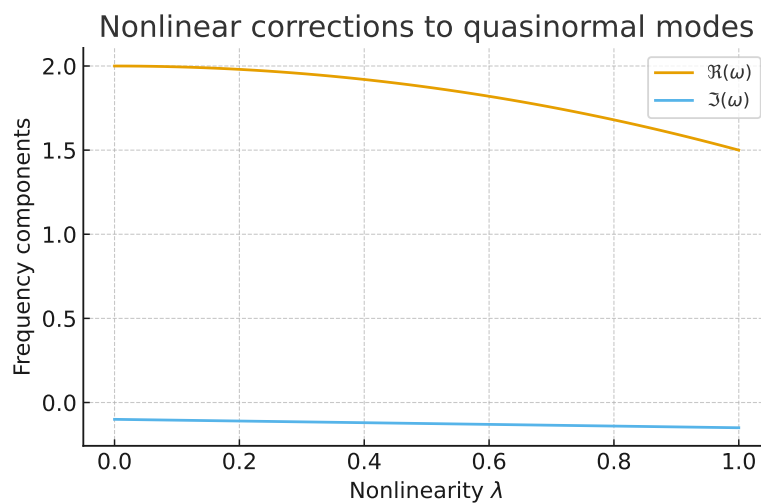


Figure 2. Nonlinear corrections to quasinormal mode frequencies as a function of the self-interaction parameter λ . The real and imaginary parts of the frequency illustrate how relaxation timescales are modified by nonlinearities. Estimated numerical errors are smaller than 10^{-4} for the lowest modes.

Fig. 2 shows the effect of nonlinear self-interactions on quasinormal frequencies. The real part of the frequency decreases while the imaginary part becomes more negative as λ increases, indicating slower oscillations and faster damping. These shifts highlight the role of nonlinearities in modifying late-time relaxation.

5.3 Cosmological Analogues

By analytic continuation of the radial coordinate or black hole parameters, certain AdS black hole equations map onto equations governing cosmological spacetimes. For instance, the near-horizon region of an AdS black hole can be related to cosmological bounce models, while analytic continuation of the time coordinate leads to connections with de Sitter (dS) cosmologies.

In these settings, the nonlinear scalar dynamics studied here shed light on:

Inflationary dynamics, where scalar self-interactions play a role in reheating and in generating non-Gaussianities.

Bouncing cosmologies, where nonlinearities control the stability of scalar-driven bounces and the onset of singularities.

Holographic cosmology, where bulk nonlinearities correspond to strong-coupling corrections in early-universe effective field theories.

Thus, the techniques developed in the AdS black hole context provide a versatile toolkit for addressing nonlinear scalar field dynamics both in gravitational and cosmological arenas.

Across these applications, a unifying theme emerges: nonlinearities are not merely technical obstacles, but encode crucial physical information. In holography they determine the equilibration pathway of strongly coupled systems, in black hole physics they modify relaxation spectra, and in cosmology they control the robustness of early-universe scenarios. The synergy between analytic approximations and numerical methods opens the door to a systematic exploration of these nonlinear phenomena.

6 Conclusions

In this work, we have developed a unified framework that combines analytical approximations and numerical techniques to study nonlinear scalar dynamics in AdS black hole spacetimes. By juxtaposing matched asymptotic expansions, perturbative methods, and variational strategies with spectral collocation and time-domain evolution, we have demonstrated how different approaches illuminate complementary aspects of nonlinear gravitational physics. Our analysis shows that asymptotic methods provide physical intuition and initial estimates, while numerical simulations are indispensable for exploring fully nonlinear regimes where analytic control is lost.

The applications we examined from holographic thermalization and quasinormal spectra to cosmological analogues highlight the universality of nonlinear equations across diverse areas of theoretical physics. Nonlinearities are not mere technical obstacles: they dictate relaxation timescales, trigger instabilities, and reveal new dynamical phenomena such as turbulent cascades and delayed collapse. In holography, they determine how strongly coupled systems equilibrate; in black hole physics, they refine our understanding of stability and late-time relaxation; and in cosmology, they inform the robustness of inflationary and bouncing models.

Looking ahead, several promising directions emerge. Extending this framework to higher-spin fields and gauge sectors would provide insights into holographic models of transport and symmetry breaking. Incorporating nontrivial topologies and anisotropic backgrounds could shed light on phase transitions and turbulence in more realistic holographic duals. Most importantly, a fully backreacted treatment of Einstein-matter systems remains an essential challenge, allowing one to capture the mutual interplay between nonlinear fields and spacetime geometry. Such advances would deepen the connection between holography, black hole physics, and cosmology, and may ultimately provide new tools for tackling open problems in strongly coupled quantum field theories and the early universe.

Data Availability

The manuscript has no associated data or the data will not be deposited.

Conflicts of Interest

The author declares that there is no conflict of interest.

Ethical Considerations

The author has diligently addressed ethical concerns, such as informed consent, plagiarism, data fabrication, misconduct, falsification, double publication, redundancy, submission, and other related matters.

Funding

This research did not receive any grant from funding agencies in the public, commercial, or nonprofit sectors.

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