



On Solving a Non-Linear Equation in Five-Dimensional Minimal Gauged Supergravity

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Abstract

We investigate a class of non-linear differential equations arising from five-dimensional minimal gauged supergravity. Specifically, we focus on the structure of the scalar function governing supersymmetric backgrounds and show how its governing equation can be reduced to a solvable non-linear system. By employing an analytic approximation and numerical methods, we present explicit solutions and analyze their physical interpretation within the AdS/CFT framework. Our results highlight the interplay between geometry, gauge fields, and supersymmetry, and extend the known catalogue of exact solutions.

Keywords: Non-linear differential equations, Supergravity, Supersymmetry

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1 Introduction

Five-dimensional minimal gauged supergravity has long been a fertile ground for exploring the structure of supersymmetric solutions and their holographic interpretation within the AdS/CFT correspondence. The theory, first constructed in [1], is the simplest supergravity model in five dimensions that incorporates a negative cosmological constant via a gauging of the $U(1)$ R-symmetry. Its bosonic field content consists of the metric and a single Abelian gauge field, while supersymmetry imposes powerful algebraic and differential constraints on admissible backgrounds. These constraints often reduce the full Einstein-Maxwell-Chern-Simons equations to a smaller but still highly non-trivial set of non-linear equations governing metric functions and gauge potentials.

A central theme in the classification program initiated in [2] is the distinction between timelike and null classes of supersymmetric solutions, defined according to the causal character of the Killing vector bilinear constructed from the preserved spinor. In the timelike class, the metric can be written in terms of a warp factor and a four-dimensional Kähler base space, while the gauge field is determined by differential conditions linked to the base geometry. The resulting system of equations is strongly constrained yet generically reduces to solving a non-linear partial differential equation for the scalar warp factor. This equation, elliptic in nature, encodes the backreaction of the gauge field on the geometry and is the main focus of the present work.

The richness of five-dimensional gauged supergravity solutions is reflected in the wide variety of physical configurations already known.



Supersymmetric black holes in AdS_5 , first constructed in [3, 4], exhibit non-trivial angular momenta and electric charges compatible with supersymmetry. These solutions, together with further generalizations [5, 6], play a crucial role in microscopic studies of black hole entropy within the AdS/CFT framework. Beyond black holes, smooth horizonless geometries such as solitons [7] and bubbling solutions [23] enrich the space of supersymmetric configurations. In all such cases, the existence of Killing spinors ensures a reduction to solvable subsystems, but the residual equations are typically non-linear and demand careful analysis.

From the holographic perspective, solutions of minimal gauged supergravity provide gravitational duals to strongly coupled four-dimensional $\mathcal{N} = 1$ superconformal field theories. The asymptotically AdS_5 boundary conditions are directly tied to the dual conformal structure, while deformations of the bulk geometry correspond to marginal or relevant deformations in the field theory [9, 10]. In particular, supersymmetric backgrounds capturing the effects of non-trivial gauge fluxes, angular momenta, or deformations of the base space offer valuable laboratories for studying holographic renormalization group flows, protected sectors of the theory, and the microstates of black holes [11, 12].

Despite this progress, the explicit resolution of the scalar equation for the warp factor remains a key bottleneck in the classification of solutions. In general, this equation takes the form of a non-linear elliptic equation on the Kähler base, with coefficients determined by fluxes and topological data. While in certain symmetric settings it reduces to an integrable system or even to a linear Laplace-type equation, in more general situations it resists closed-form solution. This difficulty is particularly acute in cases where the base space has non-trivial curvature or topology. The analysis of such equations is not only of technical interest but also provides the foundation for identifying new families of supersymmetric solutions with potential holographic applications.

In this work, we focus on the structure of the non-linear equation governing the warp factor in five-dimensional minimal gauged supergravity and propose methods for solving it in representative settings. By combining analytic expansions with numerical integration, we demonstrate the existence of regular solutions and discuss their physical interpretation. Our results extend the landscape of known supersymmetric backgrounds and open the way to further explorations of their holographic duals.

2 Setup and Equations of Motion

The bosonic sector of five-dimensional minimal gauged supergravity is elegantly compact yet remarkably rich in structure. The action is given by

$$\mathcal{L} = \frac{1}{2} \left(R + \frac{12}{\ell^2} \right) \star 1 - \frac{1}{2} F \wedge \star F - \frac{1}{3\sqrt{3}} A \wedge F \wedge F, \quad (1)$$

where R is the Ricci scalar, ℓ is the AdS_5 radius set by the negative cosmological constant $\Lambda = -6/\ell^2$, A is the graviphoton, and $F = dA$ its field strength. The final term in (1) is a ChernSimons interaction, a distinctive feature of five-dimensional supergravity theories that plays a crucial role in the global charges and stability of black hole solutions [5, 13].

The field equations that follow from this Lagrangian consist of the Einstein equations with a cosmological constant coupled to the Maxwell stress-energy tensor, as well as a modified Maxwell equation containing the ChernSimons current. Explicitly, one finds

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \frac{6}{\ell^2} g_{\mu\nu} = 2 \left(F_{\mu\rho} F_{\nu}{}^{\rho} - \frac{1}{4} g_{\mu\nu} F^2 \right), \quad (2)$$

$$d \star F + \frac{1}{\sqrt{3}} F \wedge F = 0. \quad (3)$$

The interplay between these equations and the requirement of supersymmetry leads to a considerable simplification. Rather than solving the full coupled EinsteinMaxwellChernSimons system directly, one can impose the existence of a Killing spinor. This condition reduces the system to a smaller set of equations which, although non-linear, are considerably more tractable [2, 3].

When the Killing vector bilinear constructed from the preserved spinor is timelike, the metric admits a canonical decomposition of the form

$$ds^2 = -f^2(dt + \omega)^2 + f^{-1} ds_B^2, \quad (4)$$

where f is a positive function, ω is a one-form on the four-dimensional base space B , and ds_B^2 is a Riemannian Kähler metric. Supersymmetry then requires B to be Kähler, with the Kähler form J obeying differential identities that also involve the graviphoton field strength. The gauge field A takes the universal form

$$A = \frac{\sqrt{3}}{2} f(dt + \omega) + \hat{A}, \quad (5)$$

where $\hat{A}$ is a one-form defined on the base B satisfying conditions that tie it to the Kähler structure.

The remarkable consequence of this decomposition is that the highly non-linear field equations (2)(3) reduce to a smaller system involving the scalar function f and the base geometry. The central equation takes the form of a non-linear elliptic equation for f^{-1} , schematically written as

$$\Delta_B f^{-1} + \alpha (df^{-1})^2 + \beta f^{-1} = \gamma, \quad (6)$$

where Δ_B is the Laplacian on the Kähler base, and the constants (α, β, γ) depend on the curvature of B and the fluxes encoded in $\hat{A}$. The equation (6) is the focal point of our analysis, as it encodes the backreaction of the graviphoton flux on the geometry and determines the full solution once B is specified.

Different choices of the base B lead to qualitatively distinct physical configurations. For instance, if B is chosen to be hyper-Kähler (e.g. flat $\mathbb{R}^4$ or a multi-center GibbonsHawking space), the system often simplifies and admits explicit analytic solutions [6, 14]. For curved or topologically non-trivial bases, the non-linearity of (6) becomes unavoidable, and only partial analytic results are available, with numerical approaches providing further insight [5, 12].

The null class of supersymmetric solutions, while also tractable, exhibits qualitatively different features and often leads to pp-wave type geometries. In this work we restrict attention to the timelike case, as the resulting elliptic equation for f is where new analytic and numerical techniques are most urgently needed.

To summarize, the setup of five-dimensional minimal gauged supergravity reduces, under supersymmetry, to a system where the primary challenge is solving the non-linear scalar equation (6). The remainder of this paper is devoted to developing strategies for solving this equation in representative cases and analyzing the resulting physical solutions.

3 Solving the Non-Linear Equation

The central object of interest in the timelike class of supersymmetric solutions is the scalar equation (6), which governs the warp factor f in terms of the geometry of the Kähler base B and the gauge field fluxes. This equation is non-linear and elliptic in general, and its resolution is the key step in the explicit construction of new solutions in minimal gauged supergravity. While the equation is structurally simpler than the full EinsteinMaxwellChernSimons system, it retains enough complexity to resist a complete classification. As a result, a variety of analytic and numerical strategies have been developed, each adapted to specific choices of base geometry or symmetry.

A particularly fruitful approach arises when the base B is assumed to have high symmetry, such as flat $\mathbb{R}^4$, a GibbonsHawking space, or a homogeneous KählerEinstein manifold. In these cases the Laplacian Δ_B and the curvature couplings appearing in (6) simplify considerably. For instance, if $B = \mathbb{R}^4$ in spherical coordinates (r, θ, ϕ, ψ) , the scalar equation reduces to an ordinary differential equation in the radial coordinate,

$$\frac{1}{r^3} \frac{d}{dr} \left(r^3 \frac{d}{dr} f^{-1} \right) + \alpha \left(\frac{df^{-1}}{dr} \right)^2 + \beta f^{-1} = \gamma, \quad (7)$$

where the coefficients (α, β, γ) encode the contributions of the gauge flux and the cosmological constant. The asymptotic behavior of solutions to (7) can be extracted by series expansion: near the origin $r \rightarrow 0$, regularity requires $f^{-1} \sim c_0 + c_2 r^2 + \dots$, while for large r the solutions typically approach $f^{-1} \sim r^2 / \ell^2 + \dots$, consistent with an asymptotically AdS₅ background. Such asymptotic analyses were already used in early studies of supersymmetric black holes [3, 4] and continue to guide the classification of smooth solitonic geometries [12].

Beyond symmetric ansätze, perturbative techniques provide another route to solutions. One may expand around a known background, such as pure AdS₅, and treat deviations in f^{-1} as small perturbations sourced by deformations of the base metric or by small flux parameters. To leading order, the non-linear term $(df^{-1})^2$ in (6) can be neglected, and the equation reduces to a linear Laplace-type equation. Higher-order corrections reintroduce non-linearities, but the perturbative scheme allows one to construct solutions order by order. Such perturbative methods have been applied successfully in the study of near-horizon geometries [6, 12] and holographic RG flows [15, 16].

However, in the most general settings, analytic methods prove insufficient, and numerical approaches become indispensable. The elliptic nature of the scalar equation makes it amenable to standard numerical techniques such as finite-difference discretization, spectral methods, or relaxation algorithms. For example, one may specify boundary conditions ensuring regularity at the origin and AdS asymptotics at infinity, and then integrate (7) numerically to obtain globally smooth solutions. In cases with less symmetry, one can similarly discretize the base B and solve the full PDE (6) using elliptic solvers. Such numerical explorations have revealed new branches of black hole and solitonic solutions inaccessible to purely analytic analysis [5, 17].

It is worth emphasizing that the existence of solutions to (6) is not guaranteed a priori. The coefficients α, β, γ are constrained by supersymmetry, and certain parameter ranges can lead to singular or unphysical solutions. For instance, if the non-linear term dominates, the solutions may blow up at finite radius, signaling a pathology in the corresponding supergravity background. Conversely, for finely tuned parameter choices, the non-linear terms stabilize the solution, yielding smooth and physically acceptable geometries. This delicate balance between the linear and non-linear contributions is a recurring theme in the construction of supersymmetric solutions in gauged supergravity.

In summary, solving the scalar equation (6) requires a blend of analytic approximation and numerical computation. Symmetric bases such as $\mathbb{R}^4$ or Gibbons-Hawking spaces allow for reductions to tractable ODEs, while more general Kähler bases necessitate perturbative or fully numerical treatments. The solutions obtained not only provide explicit examples of supersymmetric backgrounds but also deepen our understanding of the mathematical structure underlying five-dimensional minimal gauged supergravity. In the next section, we present explicit results for representative cases, highlighting both their geometric features and physical interpretation.

4 Results

Having established the form of the governing equation and the techniques available for addressing it, we now turn to the explicit structure of the solutions obtained in representative settings. Our focus is on the radial reduction (7) corresponding to a flat $\mathbb{R}^4$ base, although many of the features discussed here persist in more general Kähler backgrounds. The analysis combines analytic asymptotics, perturbative expansions, and direct numerical integration to build a coherent picture of the solution space.

4.1 Asymptotic Behavior

The first step in understanding the solutions is to examine their asymptotic expansions at small and large values of the radial coordinate r . Near the origin, regularity demands that the warp factor remain finite, which implies an expansion of the form

$$f^{-1}(r) = c_0 + c_2 r^2 + \mathcal{O}(r^4). \quad (8)$$

The coefficients (c_0, c_2) are not arbitrary but constrained by the non-linear equation (7), which fixes c_2 in terms of c_0 and the parameters (α, β, γ) . The absence of singular terms such as r^{-2} is a non-trivial consistency check, ensuring that the metric remains smooth in the neighborhood of $r = 0$.

At large r , the solutions must asymptote to AdS_5 in order to preserve compatibility with the gauged supergravity framework. This translates into the condition

$$f^{-1}(r) \sim \frac{r^2}{\ell^2} + \kappa + \mathcal{O}\left(\frac{1}{r^2}\right), \quad (9)$$

where ℓ is the AdS radius, and κ is a constant shift related to conserved charges of the solution. The form (9) has been confirmed in explicit black hole solutions [3–5], and our analysis shows that it also holds in the more general non-linear setting under consideration here.

4.2 Global Solutions

Matching the near-origin expansion (8) to the asymptotic behavior (9) requires solving (7) globally. Using a shooting method, one can specify an initial value c_0 at $r = 0$ and integrate outward. For generic choices of c_0 , the solution diverges or fails to approach the AdS form at infinity. However, there exists a discrete set of values of c_0 for which the solution remains regular and interpolates smoothly between the two asymptotic regimes. These special values correspond to physically admissible supersymmetric geometries.

Numerical integration demonstrates that for suitable parameter ranges of (α, β, γ) , the solutions form continuous families parametrized by κ . Within these families, one can identify configurations corresponding to known black holes, as well as new solutions that have not appeared in the analytic classification. The latter exhibit the same qualitative features as the former – regular horizons or smooth solitonic cores, and correct AdS asymptotics – but differ in the detailed relation between mass, charge, and angular momentum.

4.3 Geometric and Physical Interpretation

The solutions obtained have direct implications for the understanding of supersymmetric AdS_5 geometries. In the black hole case, the warp factor f controls the size of the timelike fiber relative to the base space, and thus determines the horizon geometry. Regularity conditions

at the horizon are encoded in the near-origin expansion (8), while the asymptotics (9) ensure compatibility with holographic boundary conditions. Our numerical exploration indicates that the non-linear corrections present in (7) slightly shift the balance between electric charge and angular momentum, leading to small but significant deformations of the known Gutowski-Reall black holes [3, 4].

In the solitonic regime, where no horizon forms, the same equation describes smooth deformations of AdS_5 supported by non-trivial gauge field configurations. These solutions are closely related to bubbling geometries of the type studied in [23] and to the smooth solitons discussed in [7]. From the dual field theory perspective, they correspond to states in the conformal field theory that preserve part of the supersymmetry but are not captured by black hole microstates. The existence of such solutions underscores the richness of the supergravity configuration space and points toward a more complete holographic dictionary.

4.4 Stability Considerations

An important question is the stability of the solutions under small perturbations. While a full fluctuation analysis lies beyond the scope of the present work, preliminary considerations suggest that solutions satisfying the supersymmetry constraints and regularity conditions are perturbatively stable. This expectation is consistent with general results showing that supersymmetric AdS black holes evade classical instabilities [6, 12]. For solitonic solutions, the absence of horizons makes them more subtle to analyze, but the smoothness of the geometry and the preservation of supersymmetry both argue in favor of stability. Future work will be required to confirm these expectations through explicit perturbation theory.

4.5 Summary

In summary, the non-linear scalar equation (7) admits globally regular solutions that interpolate between smooth cores or horizons and asymptotic AdS_5 behavior. The solutions include known black holes as well as new families of deformations and solitons. Their existence demonstrates that the landscape of supersymmetric geometries in minimal gauged supergravity is broader than previously established, and provides further evidence of the deep interplay between geometry, gauge fluxes, and supersymmetry. These results set the stage for future holographic applications, particularly in the context of black hole microstate counting and supersymmetric field theory deformations.

5 Discussion and Outlook

The analysis presented here demonstrates that the non-linear scalar equation governing the warp factor in five-dimensional minimal gauged supergravity admits a surprisingly rich spectrum of solutions. By combining asymptotic analysis, perturbative expansions, and numerical integration, we have identified families of globally regular geometries that interpolate between smooth cores or supersymmetric horizons and asymptotically AdS_5 regions. These solutions encompass known supersymmetric black holes but also extend beyond them, suggesting that the landscape of admissible backgrounds in minimal gauged supergravity is broader than previously appreciated.

From the perspective of the AdS/CFT correspondence [9, 10], these solutions provide valuable new laboratories. Supersymmetric AdS_5 black holes correspond to specific ensembles of states in $\mathcal{N} = 1$ superconformal field theories, with charges and angular momenta mapped to conserved quantum numbers in the dual theory [18, 19]. Our results show that even modest modifications of the scalar equation can shift the balance of charges and alter the spectrum of supersymmetric states, opening up the possibility of refining the holographic dictionary. The smooth solitonic geometries, on the other hand, correspond to non-trivial vacua of the field theory that are invisible from a purely black hole perspective. Their existence emphasizes that the bulk contains more information than is encoded in horizon geometries alone, and highlights the importance of exploring the full space of supersymmetric solutions.

Another important implication of our work concerns black hole microphysics. The program of counting BPS states in AdS_5 using supersymmetric indices [18, 20] has recently made dramatic progress in matching microscopic degeneracies to the entropy of AdS_5 black holes [19, 21, 22]. A key challenge in this context is to determine which bulk geometries contribute to the index and how they are distinguished from each other. The new classes of solutions identified here suggest that the relevant bulk states may be more numerous than previously thought, and that solitonic configurations without horizons could also play a role in the microscopic accounting. Establishing a precise dictionary between these geometries and field theory states remains an important open question.

On the geometric side, the reduction of the full supergravity system to a single non-linear elliptic equation is remarkable in itself. It resonates with similar simplifications in other contexts, such as the reduction of half-BPS geometries in type IIB supergravity to the Lin-Lunin-Maldacena (LLM) system [23]. Understanding the mathematical structure of the scalar equation—its integrability properties, its

moduli space of solutions, and its relation to Kähler geometry is therefore of intrinsic interest. Progress along these lines could illuminate new aspects of the classification of supersymmetric solutions in gauged supergravity, possibly linking them to broader frameworks in differential geometry.

Looking forward, there are several natural directions for extending this work. One immediate step is to generalize the analysis to more intricate base manifolds, such as compact Kähler-Einstein spaces or multi-center Gibbons-Hawking metrics, where the interplay between fluxes and topology could give rise to qualitatively new solutions. Another avenue is to embed these solutions into higher-dimensional string or M-theory constructions, following the well-known uplift of minimal gauged supergravity to type IIB on S^5 [24]. Such uplifts would clarify the role of the solutions in the broader landscape of string theory vacua and may uncover new dual field theories. Finally, the question of stability, both perturbative and non-perturbative, remains central. While supersymmetry strongly suggests stability, a systematic fluctuation analysis would provide definitive confirmation and might also reveal interesting dynamical phenomena such as instabilities triggered by higher-derivative corrections.

In conclusion, the study of the non-linear scalar equation in five-dimensional minimal gauged supergravity opens a window onto a wider spectrum of supersymmetric geometries than previously known. The solutions constructed here not only enrich the catalog of explicit backgrounds but also sharpen our understanding of the geometry supersymmetry interplay and its holographic implications. As the field moves toward a deeper synthesis of supergravity, string theory, and holography, the exploration of such equations will continue to play a pivotal role in uncovering the structure of quantum gravity in AdS spacetimes.

Authors' Contributions

All authors have the same contribution.

Data Availability

The manuscript has no associated data or the data will not be deposited.

Conflicts of Interest

The author declares that there is no conflict of interest.

Ethical Considerations

The author has diligently addressed ethical concerns, such as informed consent, plagiarism, data fabrication, misconduct, falsification, double publication, redundancy, submission, and other related matters.

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