**Global Analysis and Discrete Mathematics**
Volume 6, Issue 1, pp. 25–31
ISSN: 2476-5341 *Research Article*

# Upper and Lower Central Series in A Pair of Lie Algebras

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Received: 10 October 2020 / Accepted: : 29 November 2020

**Abstract** The Baer's theorem in the termes of the Lie algebras states that for a Lie algebra $L$ the finiteness of $\dim(L/Z_i(L))$ implies the finiteness of $\dim(\gamma_{i+1}(L))$ for all non negative integers $i$. Let $(N, L)$ denote a pair of Lie algebras, where $N$ is an ideal of $L$, and $d_i = d_i(L)$ denote the minimal number of generators of $L/Z_i(N, L)$ for all non negative integers $i$. In this paper, we consider the pair $(N, L)$ and show that if $d_n$ is finite, then the converse of Baer's theorem is true. In fact, we shall show that if for all $i \geq n$, $d_n$ and $\dim(\gamma_{i+1}(N, L))$ are finite, then $N/Z_i(N, L))$ is finite. In particular, we give an upper bound as following,

$$\dim(\frac{N}{Z_i(N,L)}) \leq ((d_n)^n d_n d_{n+1} \dots d_{i-1})\dim(\gamma_{i+1}(N,L))$$

$$\leq (d_n)^i (\dim\gamma_{i+1}(N,L))$$

for all non negative integers $i$.

**Keywords** Lie algebra · Baer's Theorem · Schur's Theorem

**Mathematics Subject Classification (2010)** 17B30 · 20F14

## 1 Introduction

Let $G$ be an arbitrary group and let $Z_n(G)$ denotes the $(n+1)$-th term of the upper central series of $G$, and $\gamma_n(G)$ denotes the $n$-th term of the lower central series of $G$. A basic theorem of Schur (see [8, 10.1.4]) asserts that if the

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center of the group $G$ has finite index, then the derived subgroup of $G$ is finite. Moreover, some bounds for the order of the derived subgroup in terms of the index of the center were given by several authors in [1–3, 6, 7, 9, 10]. Thence finding some conditions under which the converse of Schur's theorem holds, have been interesting in several papers. For example B. H. Neumann [6] proved that $G/Z(G)$ is finite if $\gamma_2(G)$ is finite and $G$ is finitely generated. This result is generalized in [7] by proving that $G/Z(G)$ is finite if $\gamma_2(G)$ is finite and $G/Z(G)$ is finitely generated. One of the naturally raised questions concerns the existence of a generalization to higher terms of the upper and lower central series. Concerning this question, R. Baer (see for example [8, 14.5.1]) proved that, if $G/Z_i(G)$ is finite, then $\gamma_{i+1}(G)$ is finite for all non negative integers $i$. Now it is interesting that, whether the converse of Baer's theorem is true? P. Hall [4] presented a partial answer to the last question. He proved that: if $\gamma_{i+1}(G)$ is finite, then $G/Z_{2i}(G)$ is finite, More precisely, he proved that finite-by-nilpotent groups are nilpotent-by-finite. Recently, it has been shown that the converse of Baer's theorem is true provided that $G/Z_i(G)$ is finitely generated in [3]. Let $L$ be a Lie algebra over a fixed field $\Lambda$ and $Z_c(L)$ and $\gamma_{c+1}(L)$ be the $c$-th term of upper central series and $(c+1)$-th term of lower central series of $L$, respectively. Analogue to Schur's theorem, Maneyhum [5] in 1994 proved that if $\dim(L/Z(L)) = n$, then $\dim(L^2) \leq n(n-1)/2$. Also, Baer's theorem was experessed by Salemkar and et al. intermes of Lie algebras [9]. Indeed, they proved that if $L/Z_c(L)$ is finite dimensional, then $\gamma_{c+1}(L)$ is finite dimensional too, we will menthion to it as Baer's theorem. Arabyani and saeedi [2] showed that when $\gamma_2(L) = L^2$ is finite dimensional and $L/Z(L)$ is finitely generated, then $L/Z(L)$ is finite dimensional. Also Arabyani et al.[1] established an upper bound for the dimension of $L/Z_n(L)$, when $\gamma_{c+1}(L)$ is finite dimensional and $L/Z_n(L)$ is finitely generated. They showed that

$$\dim(L/Z_n(L)) \leq d^n(\dim(\gamma_{c+1}(L))),$$

where $d$ is the minimal number of generators of $L/Z_n(L)$. Let $(N, L)$ be a pair of Lie algebras, where $N$ is an ideal of $L$. Consider a series of ideals of $N$ as follows:

$$N = [N,_0 L] = \gamma_1(N, L) \supseteq \gamma_2(N, L) \supseteq \gamma_3(N, L) \supseteq \cdots \supseteq \gamma_{n+1}(N, L) \supseteq \cdots,$$

where $\gamma_{n+1}(N, L) = [N,_n L] = [[N,_{n-1} L], L]$, for a positive integer $n$. We call such a series the lower central series of $N$ in $L$. We say that a pair $(N, L)$ of Lie algebras is nilpotent if it has a finite lower central series. The shortest length of such series is called the nilpotency class of the pair $(N, L)$.
Similarly, we may define the upper central series of $N$ in $L$ as follows:

$$0 = Z_0(N, L) \subseteq Z_1(N, L) \subseteq \cdots \subseteq Z_m(N, L) \subseteq \cdots,$$

where $Z(N, L) = Z_1(N, L) = \{n \in N | [n, l] = 0, \forall l \in L\}$ and for a positive integer $m$,

$$\frac{Z_m(N, L)}{Z_{m-1}(N, L)} = Z\left(\frac{N}{Z_{m-1}(N, L)}, \frac{L}{Z_{m-1}(N, L)}\right).$$

We denote $[x_1, x_2, \ldots, x_m] = [[x_1, x_2, \ldots, x_{m-1}], x_m]$, where $x_1, x_2, \ldots, x_m$ are elements of a Lie algebra. Now, as a simple consequences we mention that:

$$Z_m(N, L) = \{n \in N | [n, l_1, \cdots, l_m] = 0, l_1, \cdots, l_m \in L\},$$

and $(N, L)$ is nilpotent of class at most $c$ if and only if $Z_c(N, L) = N$. It is easy to see that $\gamma_{n+1}(N, L) = \gamma_{n+1}(L)$ and $Z_n(N, L) = Z_n(L)$, when $N = L$. In this paper, we shall consider a pair of Lie algebras $(N, L)$ and give an affirmative answer to the mentioned questions under some conditions, which generalize some important results of [1,2,10] and give a partial converse of Baer's theorem in terms of Lie algebra [9]. Let we denote by $d_0 = d_0(L) = d(L)$, the minimal number of the generators of $L$, and let $d_i = d_i(L) = d(L/Z_i(N, L))$ for all non negative integers $i$. Our main result is Theorem 1, which states that the finiteness of $d_n(L)$ and $\dim(\gamma_{i+1}(N, L))$ implies the finiteness of $\dim(\frac{N}{Z_i(N,L)})$, where $i \geq n$. Furthermore, it provides an upper bound for $\dim(\frac{N}{Z_i(N,L)})$ in terms of $\dim(\gamma_{i+1}(N, L))$. We may use this method to class of groups, which generalize the main results of [3,4,6,7].

## 2 Some preliminary results on pair of Lie algebras

We start by some preliminaries results which states some relations between the terms of the upper and lower central series of the pair$(N, L)$.

**Lemma 1** *Let $(N, L)$ be a pair of Lie algebras. Then $Z_n(N, L) = N \cap Z_n(L)$.*

*Proof* Let $x$ be an element of $L$, then $x$ is an element of $Z_n(N, L)$ if and only if $x \in N$ and for all elements $t_1, \cdots, t_n$ of $L$, $[x, t_1, \cdots, t_n] = 0$ in turns if and only if $x \in N \cap Z_n(N)$.

**Proposition 1** *Let $(N, L)$ be a pair of Lie algebras and let $M$ be an ideal of $L$ with $M \subseteq N$ and $i, j \geq 1$ are integers. Then we have*

*i)* $\gamma_i(\gamma_j(N, L), L) \leq \gamma_{i+j-1}(N, L)$;
*ii)* $\gamma_i(\frac{N}{M}, \frac{L}{M}) = \frac{\gamma_i(N,L)+M}{M}$.

*Proof* The results follow by using induction on $i$.

**Lemma 2** *Let $(N, L)$ be a pair of Lie algebras. Then the following statements hold.*

*(i)* $\gamma_n(\frac{N}{Z_i(N,L)}, \frac{L}{Z_i(N,L)}) \cong \frac{\gamma_n(N,L)+Z_i(N,L)}{Z_i(N,L)}$;
*(ii)* $\gamma_n(N, L) \leq \gamma_n(L) \leq C_L(Z_n(L))$;
*(iii)* $\frac{N+Z_i(L)}{(N+Z_i(L))\cap Z_n(L)} \cong \frac{N}{N\cap Z_n(L)}$,

*where $i \leq n$.*

*Proof* $(i)$ is obtained by Lemma 1 and Proposition 1. Statements $(ii)$ and $(iii)$ are straightforward.

Now we may mention a key lemma which will be used in proving the Main Theorem. We denote $C_M(x) = \{y \in M | [x, y] = 0\}$, where $M$ is an ideal of $L$ and $x \in L$.

**Lemma 3** *Let $(N, L)$ be a pair of Lie algebras. If $d_n$ and $\dim(\gamma_{i+1}(N, L))$ are finite, where $i \geq n$. Then*

$$\dim(\gamma_i(\frac{N}{Z(N,L)}, \frac{L}{Z(N,L)})) \leq d_n(\dim(\gamma_{i+1}(N, L))).$$

*Proof* Let $m = d_n(L)$ and $L = \langle x_1, x_2, \ldots, x_m, Z_n(N, L)\rangle$. Since $\gamma_{i+1}(N, L) = [\gamma_i(N, L), L]$ is a finite dimensional Lie algebra, we have

$$\dim(\frac{\gamma_i(N,L)}{C_{\gamma_i(N,L)}(x_j)}) \leq \dim(\gamma_{i+1}(N, L)) < \infty,$$

where $j = 1, 2, \ldots, m$. On the other hand, by using Lemma 2(ii), we have

$$\gamma_i(N, L) \cap Z(N, L) = \gamma_i(N, L) \cap N \cap Z(L)$$

$$= \bigcap_{j=1}^{m} (C_L(x_j) \cap \gamma_i(N, L)) = \bigcap_{j=1}^{m} C_{\gamma_i(N,L)}(x_j).$$

Now it follows from Lemma 2(i) that

$$\dim(\gamma_i(\frac{N}{Z(N,L)}, \frac{L}{Z(N,L)})) = \dim(\frac{\gamma_i(N,L) + Z(N,L)}{Z(N,L)})$$

$$= \dim(\frac{\gamma_i(N,L)}{\gamma_i(N,L) \cap Z(N,L)}) = \dim(\frac{\gamma_i(N,L)}{\bigcap_{j=1}^{m} C_{\gamma_i(N,L)}(x_j)})$$

$$\leq \sum_{j=1}^{m} \dim(\frac{\gamma_i(N,L)}{C_{\gamma_i(N,L)}(x_j)}) \leq m(\dim(\gamma_{i+1}(N, L))).$$

Let $(N, L)$ be a pair of Lie algebras such that $d_1$ and $\dim(\gamma_2(N, L))$ are finite, and $L = \langle l_1, l_2, \cdots, l_{d_1}, Z(N, L)\rangle$. Then define

$$f : \tfrac{N}{Z(N,L)} \longrightarrow \sum_{i=1}^{d_1} Img(ad_{l_i})$$
$$\overline{n} \longmapsto ([n, l_1], [n, l_2], \cdots, [n, l_{d_1}]),$$

where $ad_{l_i}$ is a derivation of $L$ which maps $x$ into $[x, l_i]$. Now, one can easily see that $f$ is a well defined injection map, which provides a proof for the following consequence of Lemma 3.

**Corollary 1** *If $d_1$ and $\dim(\gamma_2(N, L))$ are finite. Then $\dim(N/Z(N, L))$ is finite and $\dim(N/Z(N, L)) \leq d_1(\dim(\gamma_2(N, L)))$.*

We should point out that the case $N = \gamma_i(L)$ in the above corollary provides a partial converse of Baer's theorem in terms of Lie algebras [9], as following.

**Corollary 2** *If $d_1$ and $\dim(\gamma_{i+1}(L))$ are finite. Then $L/Z_i(L)$ is finite for all non negative integers $i$.*

Salemkar and Mirzaei [10] proved that if $L$ is a finitely generated Lie algebra, then $L/Z_n(L)$ is finite dimensional if and only if $\gamma_{n+1}(L)$ is finite dimensional. In the following result we generalize this result.

**Proposition 2** *Let $(N, L)$ be a pair of Lie algebras. If $\dim(\gamma_{n+1}(N, L))$ and $d_0$ are finite. Then $\dim(\frac{N}{N \cap Z_n(L)})$ is finite. Moreover*
$\dim(\frac{N}{N \cap Z_n(L)}) \le d_0 d_1 \cdots d_{n-1}(\dim(\gamma_{n+1}(N, L))) \le d_0^n(\dim(\gamma_{n+1}(N, L)))$.

*Proof* It follows from Lemma 3 that

$$\dim(\gamma_n(\frac{N}{Z(N, L)}, \frac{L}{Z(N, L)})) \le d_0(\dim(\gamma_{n+1}(N, L))).$$

Since $d_1$ is finite, by applying Lemma 3 on the pair $(\frac{N}{Z(N,L)}, \frac{L}{Z(N,L)})$, one may easily see that

$$\dim(\gamma_{n-1}(\frac{N}{Z_2(N, L)}, \frac{L}{Z_2(N, L)})) \le d_0 d_1(\dim(\gamma_{n+1}(N, L))).$$

Continuing this process, we have,

$$\begin{aligned} \dim(\frac{N}{N \cap Z_n(L)}) &= \dim(\gamma_1(\frac{N}{Z_n(N, L)}, \frac{L}{Z_n(N, L)})) \\ &\le d_0 d_1 \cdots d_{n-1}(\dim(\gamma_{n+1}(N, L))). \end{aligned}$$

The last inequality will obtain from this fact that $d_0 \ge d_1 \ge \cdots \ge d_{n-1}$.

## 3 Main results

In the following result we generalize Proposition 2 to higher terms of central series of pair $(N, L)$.

**Theorem 1** *Let $(N, L)$ be a pair of Lie algebras such that $\dim(\gamma_{i+1}(N, L))$ and $d_n(L)$ are finite, where $i \ge n$. Then $\dim(N/Z_i(N, L))$ is finite and*

$$\dim(N/Z_i(N, L)) \le d_n^n d_n d_{n+1} \ldots d_{i-1} \dim\gamma_{i+1}(N, L) \le (d_n)^i \dim(\gamma_{i+1}(N, L)).$$

*Proof* It is evident that $d_n(L) = d_{n-1}(L/Z(N, L)) = \cdots = d_0(L/Z_n(N, L))$. So frequently applying of Lemma 3, implies that:

$$\begin{aligned} &\dim(\gamma_{i-n+1}(\frac{N}{Z_n(N, L)}, \frac{L}{Z_n(N, L)})) \\ &\le d_n(\dim(\gamma_{i-n+2}(\frac{N}{Z_{n-1}(N, L)}, \frac{L}{Z_{n-1}(N, L)}))) \end{aligned}$$

$$\leq \cdots \leq d_n^n(\dim(\gamma_{i+1}(N,L))).$$

Now since $d_0(L/Z_n(N,L))$ is finite, we have

$$\dim(\frac{\frac{N}{Z_n(N,L)}}{Z_{i-n}(\frac{N}{Z_n(N,L)},\frac{L}{Z_n(N,L)})}) = \dim(\frac{N}{Z_i(N,L)})$$

$$\leq t(\dim(\gamma_{i-n+1}(\frac{N}{Z_n(N,L)},\frac{L}{Z_n(N,L)}))),$$

where $t = d_o(\frac{L}{Z_n(N,L)})d_1(\frac{L}{Z_n(N,L)})\cdots d_{i-n-1}(\frac{L}{Z_n(N,L)})$, by Proposition 2. It is easy to see that $t = d_n(L)d_{n+1}(L)\ldots d_{i-1}(L)$. Therefore we have

$$\dim(\frac{N}{Z_i(N,L)}) \leq t(d_n)^n\dim(\gamma_{i+1}(N,L)) \leq (d_n)^i(\dim(\gamma_{i+1}(N,L))),$$

which completes the proof.

The following result is an immediate consequence of Theorem 1.

**Corollary 3** *Let $(N,L)$ be a pair of Lie algebras. If $\dim(Z_n(N,L))$ and $d_n$ are finite, then $\dim(Z_i(N,L))$ is finite, for all positive integers $i$.*

*Proof* There is nothing to prove when $i \leq n$. So we suppose that $i > n$ and $\dim(Z_{i-1}(N,L))$ is finite. Then $\dim(\gamma_2(Z_i(N,L),L))$ is finite and so is

$$\dim(\gamma_2(\frac{Z_i(N,L)}{Z_{n-1}(N,L)},\frac{L}{Z_{n-1}(N,L)})).$$

Now it is easy to check that $L/Z_{n-1}(N,L)$ satisfies all the assumptions of Theorem 1, therefore $\dim(Z_i(N,L)/Z_n(N,L))$ is finite, and so $\dim(Z_i(N,L))$ is finite, as required.

It is easy to see that all of our results are true for a pair $(N,G)$ of groups, where $N$ is a normal subgroup of $G$. The previous corollary has an interesting consequence in finitely generated nilpotent groups, which states that the center of a finitely generated nilpotent group controls the whole group.

**Corollary 4** *Let $G$ be a finitely generated nilpotent group. $G$ is finite if and only if $Z(G)$ is finite.*

Also one can easily see that the last corollary may be extended as following.

**Corollary 5** *Let $G$ be a nilpotent group of class $c$ and let $G/Z_n(G)$ be finitely generated, where $1 \leq n < c$. Then $G$ is finite if and only if $Z_n(G)$ is finite.*

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