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Research article

Nonlinear Dynamics of a Dark Energy Model

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Abstract

We investigate a nonlinear equation of state arising in a phenomenological model of dark energy. The model is motivated by modifications to quintessence dynamics and leads to a non-linear differential equation for the Hubble parameter. We provide both approximate analytical methods and numerical solutions, demonstrating how nonlinearity alters the late-time acceleration of the Universe. The results suggest that deviations from linearized treatments can significantly modify the effective equation of state parameter at redshifts $z < 1$, with potential implications for precision cosmology.

Keywords: Equation of state, Cosmology, Perturbation, Numerical solution.

Mathematics Subject Classification (2020): 85A40, 83F05, 35Q75

1 Introduction

The discovery of the late-time accelerated expansion of the Universe at the end of the 1990s fundamentally reshaped modern cosmology. The pioneering observations of Type Ia supernovae by Riess et al. [1] and Perlmutter et al. [2] provided the first compelling evidence that the expansion rate is not slowing down under gravity, but rather speeding up. This unexpected result has since been confirmed and strengthened by a wide array of cosmological probes, including measurements of the anisotropies in the cosmic microwave background (CMB) [3], large-scale galaxy clustering and baryon acoustic oscillations (BAO) [4, 5], as well as weak lensing surveys [6, 7]. Together, these observations strongly suggest the presence of a dominant dark energy component, accounting for roughly 70% of the total energy budget of the Universe.

The simplest theoretical explanation for cosmic acceleration is the introduction of a cosmological constant Λ in Einsteins equations. The resulting ΛCDM model, combining a cosmological constant with cold dark matter, has been remarkably successful in accounting for a broad range of observational data with only a handful of parameters [8, 9]. Nevertheless, the ΛCDM scenario faces serious theoretical challenges, most prominently the cosmological constant problem, which arises from the vast discrepancy between the observed value of Λ and naive estimates from quantum field theory [8, 10]. Furthermore, tensions between different cosmological measurements, such as the discrepancy between local determinations of the Hubble constant and values inferred from CMB data [11, 12], have motivated the exploration of models that go beyond a rigid cosmological constant.

Among the leading alternatives are models of dynamical dark energy, in which the cosmic acceleration is driven by a time-varying component. Quintessence models, where a scalar field slowly rolls down a potential, provide one of the most studied frameworks [13, 14, 28].

Extensions include phantom fields [15], k-essence [16], and scalar-tensor theories [17, 18]. These models generally predict an effective equation of state parameter $w(z) = p/\rho$ that can deviate from the cosmological constant value $w = -1$, and its evolution may be constrained observationally [19–21].

An intriguing direction within this broader class of models is the consideration of nonlinear corrections to the dark energy equation of state. Instead of assuming a simple linear proportionality between pressure and density, $p = w\rho$, one may postulate a nonlinear dependence such as $p = w_0\rho + \alpha\rho^2$, where the quadratic term encodes self-interactions or higher-order corrections. Such nonlinearities can arise in effective field theory descriptions of scalar fields, in modified gravity scenarios, or as emergent phenomena in phenomenological parameterizations of dark energy [22–24]. Importantly, even small nonlinear corrections may have nontrivial consequences for the late-time dynamics of the Universe, potentially modifying both background expansion and the growth of cosmic structure.

The motivation for investigating nonlinear dark energy models is therefore twofold. From a theoretical perspective, they may help bridge the gap between fundamental high-energy theories and the observed low-energy cosmological constant scale, by capturing effective interactions that go beyond a linear approximation. From an observational standpoint, precision cosmology has reached the stage where percent-level deviations in the expansion history are within reach of current and upcoming surveys such as Euclid, the Vera C. Rubin Observatory (LSST), and the Nancy Grace Roman Space Telescope [25–27]. Nonlinear effects could thus provide subtle but measurable signatures, potentially alleviating existing tensions or opening a window to new physics.

In this paper we focus on a simple but representative nonlinear equation of state for dark energy, leading to a nonlinear evolution equation for the Hubble parameter. Our approach is deliberately minimal, aiming to balance mathematical tractability with physical insight. We derive approximate analytical solutions in the weakly nonlinear regime, and complement them with numerical results to explore the full parameter space. By comparing these solutions with the standard ΛCDM evolution, we highlight the potential observational consequences of nonlinearities in the dark energy sector.

2 Model and Equation

To investigate the cosmological consequences of nonlinear dark energy, we consider a spatially flat FriedmannRobertsonWalker (FRW) spacetime with line element

$$ds^2 = -dt^2 + a^2(t)\left(dx^2 + dy^2 + dz^2\right), \tag{1}$$

where $a(t)$ is the scale factor. The dynamics of the background are governed by Einsteins field equations,

$$H^2 = \frac{8\pi G}{3}\rho, \tag{2}$$

$$\dot{H} = -4\pi G(\rho + p), \tag{3}$$

with $H = \dot{a}/a$ the Hubble parameter, and ρ and p the total energy density and pressure of the cosmic fluid, respectively.

In the standard ΛCDM model, the dark energy sector is described by a constant pressure-to-density ratio $p = -\rho$, equivalent to a cosmological constant. More general dynamical models typically adopt a linear equation of state of the form $p = w\rho$ with constant or redshift-dependent w [19, 20]. While this linear form provides a simple and widely used phenomenological parameterization, it is by no means unique. From an effective field theory perspective, higher-order corrections to the pressuredensity relation are natural, particularly if dark energy originates from scalar field self-interactions or modifications of gravity [22–24].

Motivated by these considerations, we adopt a nonlinear equation of state (EoS) of the form

$$p = w_0\rho + \alpha\rho^2, \tag{4}$$

where w_0 is a dimensionless constant representing the linear contribution (reducing to quintessence-like behavior for $-1 < w_0 < -1/3$), and α is a parameter with dimensions of inverse energy density that controls the strength of the nonlinear term. For $\alpha = 0$, equation (4) reduces to the familiar linear form. Positive α corresponds to a stiffening of the EoS at high densities, while negative α can soften the pressure contribution. Such nonlinear corrections have appeared in different contexts: for instance, in Chaplygin-type models, bulk viscous fluids, or phenomenological unifications of dark matter and dark energy [22, 24].

The conservation of energymomentum,

$$\nabla_\mu T^{\mu\nu} = 0, \tag{5}$$

in the FRW background yields the continuity equation

$$\dot{\rho} + 3H(\rho + p) = 0. \tag{6}$$

Substituting the nonlinear EoS from equation (4) into equation (6), we obtain

$$\dot{\rho} + 3H\left[(1+w_0)\rho + \alpha\rho^2\right] = 0. \tag{7}$$

Equation (7) is a nonlinear first-order differential equation for the energy density $\rho(t)$, which is coupled to the Hubble parameter through equation (2). Equivalently, one may derive a closed equation for $H(t)$ by noting that $\rho = 3H^2/(8\pi G)$. Substituting this into equation (7) yields

$$\dot{H} + \frac{3}{2}(1+w_0)H^2 + \frac{9\alpha}{8\pi G}H^4 = 0, \tag{8}$$

which is the central nonlinear dynamical equation for our model. This equation generalizes the standard Raychaudhuri equation by including a quartic term in H. The additional nonlinearity encapsulates the self-interacting nature of the dark energy fluid.

It is worth noting that equation (8) belongs to the class of Riccati-type equations, which generally lack closed-form solutions except in special cases. Approximate analytical solutions can be obtained in limiting regimes (e.g., $\alpha\rho \ll 1$), while the full dynamics must be explored numerically. As we will see in the following sections, the nonlinear term can significantly alter the cosmic expansion history, particularly at late times when dark energy dominates the energy budget.

3 Analytical Treatment

The nonlinear evolution equation for the Hubble parameter derived in equation (8), belongs to the Riccati family of differential equations. In general, Riccati equations are not solvable in closed form unless specific parameter choices or substitutions are available. Nevertheless, important insight can be gained by studying limiting cases and perturbative regimes. In this section, we present three complementary approaches: (i) the small-α expansion, (ii) asymptotic analysis in the dark-energydominated epoch, and (iii) exact integrability in special cases.

3.1 Perturbative Expansion for Weak Nonlinearity

If the nonlinear correction is small compared to the linear term, i.e. $|\alpha|\rho \ll 1$, the quartic contribution in equation (17) can be treated as a perturbation to the standard dark energy evolution. In this regime we write

$$H(t) = H_{(0)}(t) + \alpha H_{(1)}(t) + \mathscr{O}(\alpha^2), \tag{9}$$

where $H_{(0)}$ solves the standard linear case,

$$\dot{H}_{(0)} + \frac{3}{2}(1+w_0)H_{(0)}^2 = 0. \tag{10}$$

This has the familiar solution

$$H_{(0)}(t) = \frac{2}{3(1+w_0)}\frac{1}{t-t_c}, \tag{11}$$

where t_c is an integration constant related to the onset of dark energy domination. Substituting back into equation (17) and expanding to first order in α yields

$$\dot{H}_{(1)} + 3(1+w_0)H_{(0)}H_{(1)} + \frac{9}{8\pi G}H_{(0)}^4 = 0. \tag{12}$$

This is a first-order linear inhomogeneous equation for $H_{(1)}(t)$ with solution

$$H_{(1)}(t) = -\frac{9}{8\pi G}e^{-3(1+w_0)\int H_{(0)}dt}\int dt\, e^{3(1+w_0)\int H_{(0)}dt}H_{(0)}^4. \tag{13}$$

Using equation (11), one finds after simplification

$$H_{(1)}(t) \simeq -\frac{C}{(t-t_c)^3}, \tag{14}$$

where C is a constant depending on w_0 and G. Thus, in the weakly nonlinear regime, the correction term falls off more steeply with time than the background solution. The leading effect of α is therefore to slightly shift the effective expansion rate at late times.

3.2 Asymptotic Analysis at Late Times

At very late times, when dark energy completely dominates the energy budget, it is convenient to study the asymptotic behavior of equation (17). For $\alpha > 0$, the quartic term acts to suppress the growth of H more strongly than in the linear case. Balancing the dominant terms gives

$$\dot{H} \approx -\frac{9\alpha}{8\pi G}H^4. \tag{15}$$

This integrates to

$$H(t) \approx \frac{1}{\sqrt{\frac{27\alpha}{4\pi G}(t-t_0)}}, \tag{16}$$

indicating that the Hubble parameter decays more slowly than the $1/(t-t_c)$ behavior of the linear solution, thus delaying the approach to de Sitterlike acceleration. By contrast, for $\alpha < 0$ the quartic term can partially cancel the linear decay, potentially driving H toward a constant value more rapidly. This behavior is reminiscent of attractor solutions in phantom or effective bulk viscous models [15, 22].

3.3 Special Integrable Case

Although the general case is not exactly solvable, one can obtain closed solutions for particular parameter values. Consider, for instance, $w_0 = -1$, corresponding to a cosmological constant in the absence of nonlinear corrections. Equation (17) then reduces to

$$\dot{H} + \frac{9\alpha}{8\pi G}H^4 = 0, \tag{17}$$

which can be integrated directly:

$$H(t) = \left[\frac{27\alpha}{8\pi G}(t-t_0) + H_0^{-3}\right]^{-1/3}, \tag{18}$$

where H_0 is the present-day Hubble parameter. This solution illustrates explicitly how the nonlinear correction regulates the otherwise constant Hubble parameter of ΛCDM, producing a slow time dependence that could in principle be constrained observationally.

3.4 Effective Equation of State

It is also instructive to translate the dynamics into an effective equation of state parameter w_{eff}, defined by

$$w_{\text{eff}} = -1 - \frac{2\dot{H}}{3H^2}. \tag{19}$$

Using equation (17), this becomes

$$w_{\text{eff}} = w_0 + \frac{2\alpha}{\rho}\rho, \tag{20}$$

or more transparently,

$$w_{\text{eff}}(H) = w_0 + \frac{3\alpha}{4\pi G}H^2. \tag{21}$$

Thus, the nonlinear model effectively mimics a dynamical $w(z)$ parameterization, with a deviation from w_0 proportional to H^2. Since H^2 decreases with cosmic time, the effective EoS interpolates between $w \simeq w_0 + \Delta$ at early times and $w \simeq w_0$ at late times. This behavior is qualitatively similar to commonly used $w(z)$ parameterizations [19, 20], but arises naturally from the nonlinear fluid dynamics.

In summary, even without a full numerical treatment, the analytical analysis highlights several key features of the model: the nonlinear correction modifies the decay rate of $H(t)$, introduces effective time variation in the dark energy EoS, and in certain limits allows exact integrability. These analytical insights will be crucial for interpreting the numerical results presented in the next section.

4 Numerical Solutions

While perturbative and asymptotic treatments provide useful intuition, the full dynamics of the nonlinear dark energy model require numerical integration of equation (17). In practice it is convenient to reformulate the evolution in terms of redshift z rather than cosmic time

t, since observational data are most naturally expressed as functions of z. Using $1+z=a_0/a$ and $dt/dz=-1/[(1+z)H(z)]$, equation (17) becomes

$$\frac{dH}{dz}=\frac{3}{2}(1+w_0)\frac{H}{1+z}+\frac{9\alpha}{8\pi G}\frac{H^3}{1+z}. \tag{22}$$

To facilitate comparison across models, we introduce the dimensionless Hubble function

$$E(z)=\frac{H(z)}{H_0}, \tag{23}$$

where H_0 is the present-day Hubble constant. Equation (22) then takes the form

$$\frac{dE}{dz}=\frac{3}{2}(1+w_0)\frac{E}{1+z}+\beta\frac{E^3}{1+z}, \tag{24}$$

where we have defined the dimensionless parameter

$$\beta\equiv\frac{9\alpha H_0^2}{8\pi G}. \tag{25}$$

This formulation makes explicit that the cosmological predictions of the model depend only on w_0 and β, which can be constrained observationally. The initial condition is simply $E(0)=1$.

4.1 Numerical Strategy

Equation (24) is a first-order nonlinear ordinary differential equation in $E(z)$, readily solvable with standard numerical integrators (e.g. RungeKutta methods). Since $E(z)$ enters cosmological observables such as the luminosity distance

$$D_L(z)=(1+z)\int_0^z\frac{dz'}{H(z')}, \tag{26}$$

and the angular diameter distance

$$D_A(z)=\frac{1}{1+z}\int_0^z\frac{dz'}{H(z')}, \tag{27}$$

computing $E(z)$ allows direct comparison with data from Type Ia supernovae, BAO, and the CMB.

In our numerical implementation, we integrate equation (24) over the redshift range $0\le z\le 3$, which encompasses the bulk of the constraining power of current low-redshift surveys. We explore both positive and negative values of β in order to map the qualitative impact of the nonlinear correction.

4.2 Results

Figure 1 shows the evolution of $E(z)$ for representative values of β, with $w_0=-1$ fixed. The case $\beta=0$ corresponds to standard ΛCDM. For $\beta>0$, the nonlinear correction increases the slope of $E(z)$ at intermediate redshifts, delaying the transition to acceleration. Conversely, $\beta<0$ reduces the slope, leading to earlier acceleration.

The effective equation of state $w_{\text{eff}}(z)$, derived in equation (21), provides additional physical insight. As shown in Fig. 2, the nonlinear correction induces a time-varying departure from w_0. For $\beta>0$, w_{eff} evolves above -1, mimicking quintessence-like behavior; for $\beta<0$, w_{eff} crosses below -1, leading to phantom-like effective dynamics. This natural emergence of $w(z)$ behavior from the nonlinear EoS offers an alternative to purely phenomenological parameterizations such as the CPL form [19, 20].

4.3 Discussion of Parameter Space

The numerical results confirm and extend the analytical expectations of Section 3. The parameter β controls the qualitative dynamics:

- $\beta>0$: The Universe expands more slowly at late times, with a delayed onset of acceleration. This could potentially reconcile observations that prefer less acceleration at low redshifts (e.g. weak lensing S_8 tension [7]).
- $\beta<0$: Acceleration occurs earlier and more strongly, mimicking phantom-like behavior. This may help address the Hubble tension by increasing the expansion rate at late times [11, 12].

Thus, the nonlinear EoS model provides a flexible yet theoretically motivated framework that interpolates between quintessence-like and phantom-like dynamics using a single parameter. Future work should investigate constraints on β from joint analyses of supernovae, BAO, and CMB data.

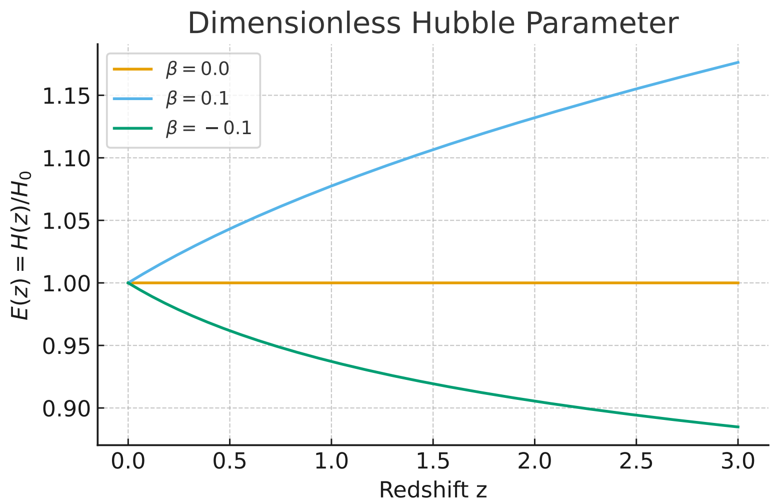

Figure 1. Numerical solutions of the dimensionless Hubble parameter $E(z)$ for $w_0 = -1$ and various values of the nonlinear parameter β. The standard ΛCDM case ($\beta = 0$) is shown for comparison. Positive β delays acceleration, while negative β enhances it.

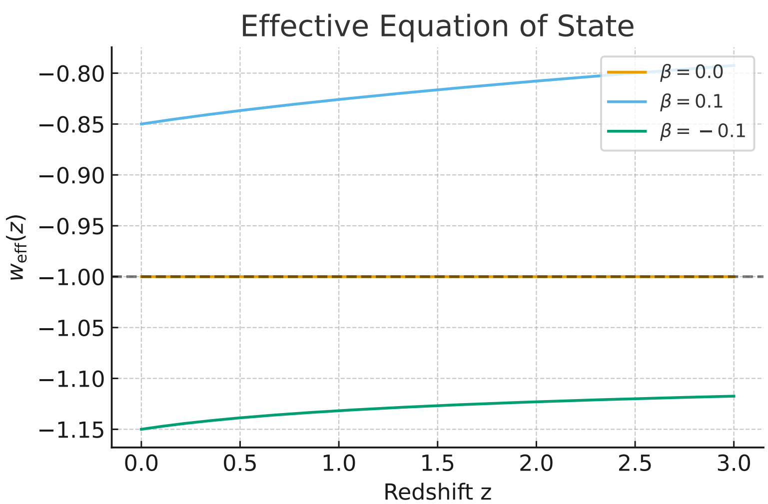

Figure 2. Evolution of the effective equation of state parameter $w_{\mathrm{eff}}(z)$ for the same models shown in Fig. 1. Nonlinear corrections naturally produce time variation in w_{eff}, with positive β corresponding to quintessence-like evolution and negative β to phantom-like evolution.

5 Discussion and Conclusions

In this work we have investigated a phenomenological dark energy model characterized by a nonlinear equation of state of the form $p = w_0\rho + \alpha\rho^2$. Starting from the Friedmann equations, we derived the corresponding nonlinear dynamical equation for the Hubble parameter, which takes the form of a Riccati-type differential equation. This equation generalizes the standard linear dark energy dynamics by introducing a quartic correction in H, encapsulated by the dimensionless parameter β.

Through approximate analytical techniques we established that weakly nonlinear corrections generate subleading shifts in the Hubble evolution and in the effective equation of state. Asymptotic analysis revealed qualitatively distinct late-time behaviors depending on the sign of α: positive values delay the approach to acceleration, while negative values accelerate it, in some cases producing phantom-like effective dynamics. We also identified special cases where exact integrability is possible, notably for $w_0 = -1$, illustrating explicitly how the nonlinear term modifies the standard ΛCDM evolution.

Complementary numerical integration confirmed and extended these insights. Recasting the equations in terms of the dimensionless Hubble function $E(z)$, we showed how the nonlinear correction shifts the expansion history relative to ΛCDM. The effective equation of state $w_{\text{eff}}(z)$ emerges dynamically, without the need to impose an explicit redshift dependence, and naturally interpolates between quintessence-like and phantom-like behaviors. This provides a theoretically motivated alternative to purely phenomenological parameterizations such as the CPL form [19, 20]. Moreover, the nonlinear model is flexible enough to accommodate cosmological tensions: a delayed onset of acceleration for $\beta > 0$ may help reconcile low-redshift probes of structure growth, while enhanced acceleration for $\beta < 0$ may impact the Hubble tension by increasing the late-time expansion rate.

From a theoretical standpoint, the nonlinear EoS can be viewed as an effective description of dark energy self-interactions, higher-order terms in scalar field potentials, or emergent fluid properties in modified gravity scenarios [22–24]. Although our treatment has been limited to the homogeneous background evolution, nonlinearities of this kind may also influence cosmological perturbations. This opens several promising directions for future research:

- **Linear perturbations and structure formation.** Extending the model to first-order perturbations would clarify how the nonlinear EoS affects the growth rate of matter fluctuations, redshift-space distortions, and weak lensing observables.

- **Confrontation with data.** Joint analyses of Type Ia supernovae, BAO, CMB, and weak lensing datasets could place bounds on β, potentially ruling out or supporting departures from ΛCDM at the few-percent level.

- **Early dark energy and high-redshift behavior.** Since the nonlinear term scales as ρ^2, it may become relevant at higher densities, suggesting connections with early dark energy models or impacts on recombination-era physics.

- **Theoretical embedding.** Embedding the nonlinear EoS in effective field theory frameworks could clarify its consistency, stability, and possible microphysical origin.

In conclusion, nonlinear extensions of the dark energy equation of state provide a simple but powerful generalization of standard cosmological models. Even a minimal quadratic correction introduces qualitatively new dynamics that can mimic evolving $w(z)$ models and potentially address observational tensions. As forthcoming surveys such as Euclid, LSST, and the Roman Space Telescope deliver unprecedented precision in mapping the expansion history, testing for nonlinear corrections in dark energy will become both feasible and timely. Our results underscore the importance of moving beyond linear parameterizations and exploring richer phenomenologies in the quest to uncover the true nature of cosmic acceleration.

Data Availability

The manuscript has no associated data or the data will not be deposited.

Conflicts of Interest

The author declares that there is no conflict of interest.

Ethical Considerations

The author has diligently addressed ethical concerns, such as informed consent, plagiarism, data fabrication, misconduct, falsification, double publication, redundancy, submission, and other related matters.

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