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On Left ϕ-Biflat and Left ϕ-Biprojectivity of θ-Lau Product Algebras

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Abstract *Monfared* defined θ-Lau product structure $A \times_\theta B$ for two Banach algebras A and B, where $\theta : B \to C$ is a multiplicative linear functional. In this paper, we study the notion of left ϕ-biflatness and left ϕ-biprojectivity for the θ Lau product structure $A \times_\theta B$. For a locally compact group G, we show that $M(G) \times_\theta M(G)$ is left character biflat (left character biprojective) if and only if G is discrete and amenable (G is finite), respectively. Also we prove that $\ell^1(N_\vee) \times_\theta \ell^1(N_\vee)$ is neither $(\phi_{N_\vee}, \theta)$-biprojective nor $(0, \phi_{N_\vee})$-biprojective, where $\phi_{N_\vee}$ is the augmentation character on $\ell^1(N_\vee)$. Finally, we give an example among the Lau product structure of matrix algebras which is not left ϕ-biflat.

Keywords Left ϕ-amenability · Left ϕ-biflatnes · Left ϕ-biprojectivity · Left ϕ-contractibility · θ-Lau product

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1 Introduction

Johnson defined amenable Banach algebras thorough virtual diagonals [8]. In fact a Banach algebra A is amenable, if there exists an element $M \in (A\hat{\otimes}A)^{**}$ such that $a \cdot M = M \cdot a$ and $\pi_A^{**}(M)a = a$ for each $a \in A$, here π_A is given by $\pi_A(a \otimes b) = ab$ for each $a, b \in A$, see [14].

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There are two homological notions parallel to amenability, namely biflatness and biprojectivity which were defined by Helemskii. In fact a Banach algebra A is called biflat (biprojective) if there exists a bounded A-bimodule morphism $\rho : A \to (A\hat{\otimes}A)^{**}(\rho : A \to A\hat{\otimes}A)$ such that $\pi_A^{**}\circ\rho(a) = a(\pi_A\circ\rho(a) = a)$, for all $a \in A$, respectively. It is well-known that a Banach algebra A is amenable if and only if A is biflat and A has a bounded approximate identity, see [14].

Recently some homological notions related to a multiplicative linear functional were given for Banach algebras. The notions like left ϕ-amenability, left ϕ-contractibility, left ϕ-biflatness and left ϕ-biprojectivity studied for the group algebras, the measure algebras and the Fourier algebras, for more information about these notions see [1], [7], [9], [13], [15] [16] and [17].

In this paper, we study the notion of left ϕ-biflatness and left ϕ-biprojectivity for the θ-Lau product structure $A\times_\theta B$. For a locally compact group G, we show that $M(G)\times_\theta M(G)$ is left character biflat (left character biprojective) if and only if G is discrete and amenable (G is finite), respectively. Also we prove that $\ell^1(N_\vee)\times_\theta \ell^1(N_\vee)$ is neither $(\phi_{N_\vee},\theta)$-biprojective nor $(0,\phi_{N_\vee})$-biprojective, where ϕ_N is the augmentation character on $\ell^1(N)$. Finally, we give an example among the θ-Lau product structure of matrix algebras which is not left ϕ-biflat.

We remind some definitions and notations which we need in this paper. For an arbitrary Banach algebra A, the character space is denoted by $\sigma(A)$ consists of all non-zero multiplicative linear functionals on A and any element of $\sigma(A)$ is called a character. The $\theta-$Lau product was first introduced by Lau [10] for F-algebras. Monfared [12] introduced and investigated θ-Lau product space $A\times_\theta B$, for Banach algebras in general. Indeed for two Banach algebras A and B such that $\sigma(B)\neq\emptyset$ and θ be a non-zero character on B, the Cartesian product $A\times B$ by following multiplication and norm

$$(a,b)(a',b') = (aa' + \theta(b')a + \theta(b)a', bb'), \tag{1}$$

$$\|(a,b)\| = \|a\|_A + \|b\|_B \tag{2}$$

is a Banach algebra, for all $a,a'\in A$ and $b,b'\in B$. The Cartesian product $A\times B$ with the above properties called the $\theta-$Lau product of A and B which is denoted by $A\times_\theta B$. From [12] we identify $A\times\{0\}$ with A, and $\{0\}\times B$ with B. Thus, it is clear that A is a closed two-sided ideal while B is a closed subalgebra of $A\times_\theta B$, and $(A\times_\theta B)/A$ is isometrically isomorphic to B. If $\theta = 0$, then we obtain the usual direct product of A and B. Since direct products often exhibit different properties, we have excluded the possibility that $\theta = 0$. Moreover, if $B = C$, the complex numbers, and θ is the identity map on C, then $A\times_\theta B$ is the unitization $A^\sharp$ of A. Note that, by [12, Proposition 2.4], the character space $\sigma(A\times_\theta B)$ of $A\times_\theta B$ is equal to

$$\{(\phi,\theta):\ \phi\in\sigma(A)\}\bigcup\{(0,\psi):\ \psi\in\sigma(B)\}. \tag{3}$$

Also, the dual space $(A \times_\theta B)^*$ of $A \times_\theta B$ is identified with $A^* \times B^*$ such that for each $(a, b) \in A \times_\theta B$, $\phi \in \sigma(A)$ and $\psi \in \sigma(B)$ we have

$$\langle(\phi, \psi), (a, b)\rangle = \phi(a) + \psi(b). \tag{4}$$

Now, suppose that A^{**}, B^{**} and $(A \times_\theta B)^{**}$ are equipped with their first Arens products. Then $(A \times_\theta B)^{**}$ is isometrically isomorphic with $A^{**} \times_\theta B^{**}$. Also, for all $(m, n), (p, q) \in (A \times_\theta B)^{**}$ the first Arens product is defined by

$$(m, n)(p, q) = (mp + n(\theta)p + q(\theta)m, nq); \tag{5}$$

see [12, Proposition 2.12]. Note that every $\phi \in \sigma(A)$ has a unique extension to a character on A^{**} is given by $\tilde{\phi}$ where $\tilde{\phi}(m) = m(\phi)$, for all $m \in A^{**}$.

Note that A and B are closed two-sided ideal and closed subalgebra of $L := A \times_\theta B$, respectively. So, we can write $a = (a, 0)$ and $b = (0, b)$ for all $a \in A$ and $b \in B$. Therefore, $L = A \times_\theta B$ is a Banach $A-$bimodule and also is a Banach $B-$bimodule. It has worth to mention that some generalizations of twisted product related to a homomorphism are given recently but by [3] it seems those products are trivial.

We recall that if X is a Banach A-bimodule, then with the following actions X^* is also a Banach A-bimodule:

$$a \cdot f(x) = f(x \cdot a), \quad f \cdot a(x) = f(a \cdot x) \qquad (a \in A, x \in X, f \in X^*). \tag{6}$$

The projective tensor product of A with A is denoted by $A\widehat{\otimes}A$. The Banach algebra $A\widehat{\otimes}A$ is a Banach A-bimodule with the following actions

$$a \cdot (b \otimes c) = ab \otimes c, \quad (b \otimes c) \cdot a = b \otimes ca \quad (a, b, c \in A). \tag{7}$$

2 Left $\phi-$biflatness and left $\phi-$biprojectivity

In this note $p_A : L \longrightarrow A$ and $p_B : L \longrightarrow B$ are denoted for the usual projections given by $p_A(a, b) = a$ and $p_B(a, b) = b$. Suppose that $q_A : A \longrightarrow L$ and $q_B : B \longrightarrow L$ are injections defined by $q_A(a) = (a, 0)$ and $q_B(b) = (0, b)$. So q_A and p_B give

$$q_A \otimes q_A : A\widehat{\otimes}A \longrightarrow L\widehat{\otimes}L \tag{8}$$

and

$$p_B \otimes p_B : L\widehat{\otimes}L \longrightarrow B\widehat{\otimes}B \tag{9}$$

with

$$(q_A \otimes q_A)(a \otimes c) = (a, 0) \otimes (c, 0) \tag{10}$$

and

$$(p_B \otimes p_B)((a, b) \otimes (c, d)) = b \otimes d, \tag{11}$$

for all $a, c \in A$ and $b, d \in B$ respectively. It is easy to see that q_A and $q_A \otimes q_A$ are A-bimodule morphisms and p_B, q_B and $p_B \otimes p_B$ are B-bimodule morphisms.

The notion of left $\phi-$biprojectivity for Banach algebras first introduced by Sahami [17]. For a non-zero multiplicative linear functional ϕ on A, the Banach algebras A is called left $\phi-$biprojective if there exists a bounded linear map $\rho : A \longrightarrow A\widehat{\otimes}A$ such that

$$\rho(ab) = a \cdot \rho(b) = \phi(b)\rho(a), \quad \phi \circ \pi_A \circ \rho(a) = \phi(a), \quad (a, b \in A). \tag{12}$$

Proposition 1 *Let A and B be two Banach algebras which A has unit e. Also let $\phi \in \sigma(A)$ and $\theta \in \sigma(B)$. If L is left $(\phi, \theta)-$biprojective. Then A is left $\phi-$biprojective.*

Proof bounded linear map $\rho_L : L \longrightarrow L\widehat{\otimes}L$ such that $\rho_L(ab) = a \cdot \rho_L(b) = \phi(b)\rho_L(a)$ and $(\phi, \theta) \circ \pi_L \circ \rho_L = (\phi, \theta)$. We know that

$$r_{_A} \circ \pi_{_L} = \pi_A \circ (r_{_A} \otimes r_{_A}), \qquad \phi \circ r_{_A} = (\phi, \theta). \tag{13}$$

Define $\rho_A : A \longrightarrow A\widehat{\otimes}A$ by $\rho_A = (r_{_A} \otimes r_{_A}) \circ \rho_L \circ q_{_A}$. Consider

$$\begin{aligned} \rho_A(a_1a_2) &= (r_{_A} \otimes r_{_A}) \circ \rho_L \circ q_{_A}(a_1a_2) \\ &= (r_{_A} \otimes r_{_A}) \circ \rho_L(a_1 \cdot q_{_A}(a_2)) \\ &= a_1 \cdot (r_{_A} \otimes r_{_A}) \circ \rho_L(q_{_A}(a_2)) \\ &= a_1 \cdot \rho_A(a_2) \end{aligned}$$

and

$$\begin{aligned} \rho_A(a_1a_2) &= (r_{_A} \otimes r_{_A}) \circ \rho_L \circ q_{_A}(a_1a_2) \\ &= (r_{_A} \otimes r_{_A}) \circ \rho_L(q_{_A}(a_1) \cdot a_2) \\ &= \phi(a_2)(r_{_A} \otimes r_{_A}) \circ \rho_L(q_{_A}(a_1)) \\ &= \phi(a_2) \cdot \rho_A(a_1) \end{aligned}$$

for every a_1 and a_2 in A. So these facts follow that

$$\rho_A(a_1a_2) = a_1 \cdot \rho_A(a_2) = \phi(a_2)\rho_A(a_1). \tag{14}$$

Moreover we have

$$\begin{aligned} \phi \circ \pi_A \circ \rho_A(a) &= \phi \circ \pi_A \circ (r_{_A} \otimes r_{_A}) \circ \rho_L \circ q_{_A}(a) \\ &= (\phi \circ r_{_A} \circ \pi_L \circ \rho_L)(a, 0) \\ &= ((\phi, \theta) \circ \pi_L \circ \rho_L)(a, 0) \\ &= (\phi, \theta)(a, 0) \\ &= \phi(a), \end{aligned}$$

for all $a \in A$. Hence $\phi \circ \pi_A \circ \rho_A = \phi$. Therefore A is left ϕ-biprojective.

Proposition 2 *Let A and B be two Banach algebras $\psi \in \sigma(B)$. If L is left $(0, \psi)-$biprojective, then B is left $\psi-$biprojective. Converse holds whenever A is unital.*

Proof Suppose that L is left $(0,\psi)-$biprojective. Then there exists a bounded linear map $\rho_L : L \longrightarrow L\widehat{\otimes}L$ such that $(0,\psi)\circ\pi_L\circ\rho_L = (0,\psi)$. Define $\rho_B : B \longrightarrow B\widehat{\otimes}B$ by $\rho_B = (p_{_B}\otimes p_{_B})\circ\rho_L\circ q_{_B}$. Clearly

$$\pi_B\circ(p_{_B}\otimes p_{_B}) = p_{_B}\circ\pi_L, \qquad \psi\circ p_{_B} = (0,\psi). \tag{15}$$

Note that

$$\rho_B(b_1b_2) = b_1\cdot\rho_B(b_2) = \psi(b_2)\rho_B(b_1), \quad (b_1,b_2\in B). \tag{16}$$

Also $\psi\circ\pi_B\circ\lambda_B = \psi$. To see these facts, consider

$$\begin{aligned}\rho_B(b_1b_2) = (p_{_B}\otimes p_{_B})\circ\rho_L\circ q_{_B}(b_1b_2) &= (p_{_B}\otimes p_{_B})\circ\rho_L(q_{_B}(b_1)\cdot b_2)\\ &= \psi(b_2)(p_{_B}\otimes p_{_B})\circ\rho_L(q_{_B}(b_1)\\ &= \psi(b_2)\rho_B(a_1)\end{aligned}$$

and

$$\begin{aligned}\rho_B(b_1b_2) = (p_{_B}\otimes p_{_B})\circ\rho_L\circ q_{_B}(b_1b_2) &= (p_{_B}\otimes p_{_B})\circ\rho_L(b_1\cdot q_{_B}(b_2))\\ &= b_1\cdot(p_{_B}\otimes p_{_B})\circ\rho_L(q_{_B}(b_2),\\ &= b_1\cdot\rho_B(b_2),\end{aligned}$$

for all b_1 and b_2 in B. Moreover

$$\begin{aligned}\Big(\psi\circ\pi_B\circ\rho_B\Big)(b) &= \Big(\psi\circ\pi_B\circ(p_{_B}\otimes p_{_B})\rho_L\circ q_{_B}\Big)(b)\\ &= \Big(\psi\circ p_{_B}\circ\pi_L\circ\rho_L\Big)(0,b)\\ &= \Big((0,\psi)\circ\pi_L\circ\rho_L\Big)(0,b)\\ &= \psi(b),\end{aligned}$$

for all $b\in B$. For converse let B be left $\psi-$biprojective. Then there exists a bounded linear map $\rho_B : B \longrightarrow B\widehat{\otimes}B$ such that $\rho_B(ab) = a\cdot\rho_B(b) = \psi(b)\rho_B(a)$ and $\psi\circ\pi_B\circ\rho_B = \psi$. Define $\rho_L : L \longrightarrow L\widehat{\otimes}L$ via

$$\rho_L(a,b) := (S_{_B}\otimes S_{_B})\circ\rho_B(b),$$

for all $a\in A$ and $b\in B$. One can show that

$$\pi_L\circ(S_{_B}\otimes S_{_B}) = S_{_B}\circ\pi_B, \qquad (0,\psi)\circ S_{_B} = \psi, \quad ((S_{_B}\otimes S_{_B})\circ\rho_B(b))\cdot x = 0, \tag{17}$$

for all $b\in B$ and $x\in A$. Using these facts show that ρ_L is a bounded linear map such that

$$\rho_L(l_1l_2) = (0,\psi)(l_2)\rho_L(l_1) = l_1\cdot\rho_L(l_2), \tag{18}$$

for all $l_1,l_2\in L$. Also

$$(0,\psi)\circ\pi_L\circ\rho_L = (0,\psi). \tag{19}$$

It follows that L is left $(0,\psi)-$biprojective.

Remark 1 We claim that left $(\phi,\theta)-$biprojectivity of L gives that B is left $\theta-$biprojective. However it is easy but for the sake of completeness we give it here. We know that there exists a bounded linear map $\rho_L : L \longrightarrow L\widehat{\otimes}L$ such that

$$\rho_L(ab) = a\cdot\rho_L(b) = (\phi,\theta)(b)\rho_L(a), \quad (\phi,\theta)\circ\pi_L\circ\rho_L = (\phi,\theta), \qquad (a,b\in L). \tag{20}$$

On the other hand, one can see that

$$p_{_B}\circ\pi_L = \pi_B\circ(p_{_B}\otimes p_{_B}), \qquad r_A\circ\pi_L = \pi_A\circ(r_A\otimes r_A), \qquad \theta\circ p_{_B} = (0,\theta). \tag{21}$$

Let $\rho_B : B \longrightarrow B\widehat{\otimes}B$ be a map defined by $\rho_B := (p_{_B}\otimes p_{_B})\circ\rho_L\circ q_{_B}$. The fact $\Big((\phi,0)\circ\pi_L\circ\rho_L\Big)(0,b) = 0$ follows that

$$\begin{aligned}\Big(\theta\circ\pi_B\circ\rho_B\Big)(b) &= \langle(\phi,\theta),(0,b)\rangle - \Big((\phi,0)\circ\pi_L\circ\rho_L\Big)(0,b)\\ &= \theta(b),\end{aligned}$$

for every $b\in B$. Moreover

$$\rho(b_1b_2) = b_1\cdot\rho_B(b_2) = \theta(b_2)\rho_B(b_1), \quad (b_1,b_2\in B). \tag{22}$$

It implies that B is left $\theta-$biprojective.

Sahami in [17] introduced and studied the notion of left $\phi-$biflatness for Banach algebras. A Banach algebra A is called left $\phi-$biflat if there exists a bounded linear map $\rho_A : A \longrightarrow (A\widehat{\otimes}A)^{**}$ such that

$$\rho_A(ab) = a\cdot\rho_A(b) = \phi(b)\rho_A(a), \quad \tilde{\phi}\circ\pi_A^{**}\circ\rho_A = \phi, \quad (a,b\in A), \tag{23}$$

where $\tilde{\phi}(F) = F(\phi)$ for all $F\in A^{**}$.

Proposition 3 *Let A and B be Banach algebras. Suppose that $\theta\in\sigma(B)$ and $\phi\in\sigma(A)$. If L is left $(\phi,\theta)-$biflat, then A is left $\phi-$biflat, provided that A is unital.*

Proof Since L is left $(\phi,\theta)-$biflat, there exists a bounded linear map $\rho_L : L \longrightarrow (L\widehat{\otimes}L)^{**}$ such that

$$\rho_L(l_1l_2) = l_1\cdot\rho_L(l_2) = (\phi,\theta)(l_2)\rho_L(l_1), \quad \widetilde{(\phi,\theta)}\circ\pi_L^{**}\circ\rho_L = (\phi,\theta), \qquad (l_1,l_2\in L). \tag{24}$$

We define $\rho_A : A \longrightarrow (A\widehat{\otimes}A)^{**}$ by $\rho_A := (r_{_A}\otimes r_{_A})^{**}\circ\rho_L\circ q_{_A}$. One can see that

$$(r_A\otimes r_A)^*(\phi\circ\pi_A) = (\phi,\theta)\circ\pi_L. \tag{25}$$

It gives that

$$\begin{aligned}\langle\tilde{\phi}\circ{\pi_A}^{**}\circ\rho_A, a\rangle &= \langle\rho_A(a), {\pi_A}^*(\phi)\rangle\\ &= \langle\rho_L(a,0), (r_A\otimes r_A)^*(\phi\circ\pi_A)\rangle\\ &= \phi(a),\end{aligned}$$

for all $a \in A$. Also

$$\rho_A(a_1a_2) = (r_{_A} \otimes r_{_A})^{**} \circ \rho_L(q_{_A}(a_1a_2)) = a_1 \cdot (r_{_A} \otimes r_{_A})^{**} \circ \rho_L(q_{_A}(a_2)) = a_1 \cdot \rho_A(a_2) \tag{26}$$

and

$$\begin{aligned} \rho_A(a_1a_2) = (r_{_A} \otimes r_{_A})^{**} \circ \rho_L \circ q_{_A}(a_1a_2) &= (r_{_A} \otimes r_{_A})^{**} \circ \rho_L(q_{_A}(a_1) \cdot a_2) \\ &= \phi(a_2)\rho_A(a_1), \end{aligned}$$

for all a_1 and a_2 in A. Hence A is left $\phi-$biflat.

Proposition 4 *Let A and B be Banach algebras. Also let A be unital and $\psi, \theta \in \sigma(B)$. Then L is left $(0, \psi)-$biflat if and only if B is left $\psi-$biflat.*

Proof Suppose that L is left $(0, \psi)-$biflat. Then there exists a bounded linear map $\rho_L : L \longrightarrow (L\widehat{\otimes}L)^{**}$ such that

$$\rho_L(l_1l_2) = l_1 \cdot \rho_L(l_2) = (0, \psi)(l_2)\rho_L(l_1), \quad \widetilde{(0, \psi)} \circ \pi_L^{**} \circ \rho_L = (0, \psi), \qquad (l_1, l_2 \in L). \tag{27}$$

We know that $\pi_B^*(\psi) = \psi \circ \pi_B$.

Define $\lambda_B : B \longrightarrow (B\widehat{\otimes}B)^{**}$ by

$$\rho_B := (p_{_B} \otimes p_{_B})^{**} \circ \rho_L \circ q_{_B}. \tag{28}$$

Clearly $\pi_L{}^*((0, \psi)) = (p_B \otimes p_B)^*(\psi \circ \pi_B)$. It follows that

$$\begin{aligned} \langle \tilde{\psi} \circ \pi_B{}^{**} \circ \rho_B, b \rangle &= \langle \pi_B{}^{**} \circ \rho_B(b), \psi \rangle \\ &= \langle \rho_B(b), \psi \circ \pi_B \rangle \\ &= \langle \rho_L((0, b)), (p_{_B} \otimes p_{_B})^*(\psi \circ \pi_B) \rangle \\ &= \psi(b), \end{aligned}$$

for all $b \in B$. Also we have

$$\rho_B(b_1b_2) = b_1 \cdot \rho_B(b_2) = \psi(b_2)\rho_B(b_1), \qquad (b_1, b_2 \in B). \tag{29}$$

It gives that B is left ψ-biflat.

To show the only if part, let B be left $\psi-$biflat. Then there exists a bounded linear map $\rho_B : B \longrightarrow (B\widehat{\otimes}B)^{**}$ such that

$$\rho_B(b_1b_2) = b_1 \cdot \rho_B(b_2) = \psi(b_2)\rho_B(b_1), \quad \tilde{\psi} \circ \pi_B^{**} \circ \lambda_B = \psi \qquad (b_1, b_2 \in B). \tag{30}$$

One can show that

$$(S_B \otimes S_B)^*((0, \psi) \circ \pi_L) = \pi_B{}^*(\psi). \tag{31}$$

Define $\rho_L : L \longrightarrow (L\widehat{\otimes}L)^{**}$ by

$$\rho_L := (S_{_B} \otimes S_{_B})^{**} \circ \rho_B \circ p_{_B}. \tag{32}$$

Clearly ρ_L is a bounded linear map which satisfies

$$\rho_L(l_1l_2) = l_1 \cdot \rho_L(l_2) = (\psi, 0)(l_2)\rho_L(l_1), \quad \widetilde{(0, \psi)} \circ \pi_L{}^{**} \circ \rho_L = \psi, \quad (l_1, l_2 \in L). \tag{33}$$

It follows that L is left $(0, \psi)-$biflat.

By modifying the proof of Proposition 4 (if part), if we define

$$\rho_B = (p_{_B} \otimes p_{_B})^{**} \circ \rho_L \circ S_{_B}, \tag{34}$$

then we can show that B is left $\psi-$biflat.

3 Results

Suppose that A is a Banach algebra and $\phi \in \sigma(A)$. We remind that a Banach algebra A is left ϕ-amenable (left ϕ-contractible) if there exists an element m in A^{**} (an element m in A) such that $am = \phi(a)m$ ($am = \phi(a)m$) and $\tilde{\phi}(m) = 1$ ($\phi(m) = 1$) for all $a \in A$, respectively, see [9] and [13]. A Banach algebra A is called left character amenable (left character contractible), if A for all $\phi \in \sigma(A)$, is left ϕ-amenable (left ϕ-contractible) and A posses a bounded left approximate identity (left identity), respectively, see [13].

Example 1 We give a Lau product Banach algebra which is not left ϕ-biflat. To see this, let $C^1[0,1]$ be the set of all differentiable functions which its first derivation is continuous. Equip $C^1[0,1]$ with the point-wise multiplication and the sup-norm. Clearly $C^1[0,1]$ becomes a Banach algebra. It is known that $\sigma(C^1[0,1]) = \{\phi_t : t \in [0,1]\}$, where $\phi_t(f) = f(t)$ for all $t \in [0,1]$. We assume in contradiction that $C^1[0,1] \times_\theta C^1[0,1]$ is left $(\phi_t, \theta)-$biflat or left $(0, \phi_t)-$biflat, where $\phi_t(f) = f(t)$ for each $t \in [0,1]$. We know that the function 1 is an identity for $C^1[0,1]$. By Proposition 3 and Proposition 4 $C^1[0,1]$ is left ϕ_t-biflat. Therefore, there exists a bounded linear map $\rho : C^1[0,1] \longrightarrow (C^1[0,1]\hat{\otimes}C^1[0,1])^{**}$ such that

$$\rho_{C^1[0,1]}(fg) = f \cdot \rho_{C^1[0,1]}(g) = \phi_t(g)\rho_{C^1[0,1]}(f), \quad \tilde{\phi_t} \circ \pi^{**}_{C^1[0,1]} \circ \rho(f) = \phi_t(f) \tag{35}$$

for all $f, g \in C[0,1]$. Put $m = \pi^{**}_{C_{[0,1]}} \circ \rho(1) \in A^{**}$, we have

$$f \cdot m = f \cdot \pi^{**}_{C_{[0,1]}} \circ \rho(1) = \pi^{**}_{C_{[0,1]}} \circ \rho(f1) = \pi^{**}_{C_{[0,1]}} \circ \rho(1f) = \phi_t(f)m, \tag{36}$$

and

$$\tilde{\phi_t}(m) = \tilde{\phi_t} \circ \pi^{**}_{C_{[0,1]}} \circ \rho(1) = \phi_t(1) = 1, \tag{37}$$

for all $f \in C^1[0,1]$. It follows that $C^1[0,1]$ is left ϕ_t-amenable which is impossible by [9, Example 2.5].

The Banach algebra A is called left character biflat (left character biprojective) if A is left ϕ-biflat (left ϕ-biprojective) for each $\phi \in \sigma(A)$, respectively, see [17].

Proposition 5 *Let G be a locally compact group and let $M(G)$ be the measure algebra over G. Suppose that $\theta \in \sigma(M(G))$. Then $M(G) \times_\theta M(G)$ is left character biflat if and only if G is discrete and amenable.*

Proof Suppose that $M(G) \times_\theta M(G)$ is left character biflat. It is known that $M(G)$ has an identity. So Proposition 3 implies that $M(G)$ is left $\phi-$amenable for all $\phi \in \sigma(M(G))$ (By placing $m = \pi^{**}_{M(G)} \circ \rho(e)$, where e is the unit of $M(G)$). Since that $M(G)$ has an identity, $M(G)$ is left character amenable. Applying [11, Corollary 2.5] gives that G is discrete and amenable .
For converse, suppose that G is discrete and amenable. Then we have $M(G) = \ell^1(G)$. Thus by Johnson Theorem $\ell^1(G)$ is amenable. So [2, Corollary 2.1] finishes the proof.

Proposition 6 *Suppose that G is a locally compact group. Then $M(G) \times_\theta M(G)$ is left character biprojective if and only if G is finite.*

Proof Suppose that $M(G) \times_\theta M(G)$ is left character biprojective. Then by Proposition 1, $M(G)$ is left character biprojective ($M(G)$ is unital). One can easily see that $M(G)$ is left $\phi-$contractible for all $\phi \in \sigma(M(G))$. Since $M(G)$ is unital, it follows that $M(G)$ is left character contractible. From [13, Corollary 6.2], we have G is a finite group.
Converse is clear.

It is well-known that the Fourier algebra $A(G)$ over a locally compact group G is a commutative Banach algebra. Also, $\sigma(A(G)) = \{\phi_g : g \in G\}$, where $\phi_g(f) = f(g)$, see [14].

Theorem 1 *Suppose that G is a locally compact group. Then $M(G) \times_\theta A(G)$ is left character biprojective if and only if G is a finite group.*

Proof Similar to the proof of previous Proposition.

Suppose that $N_\vee$ is the semigroup N (the natural numbers) with products $m \vee n = \max\{m, n\}$. Consider $\ell^1(N_\vee)$ with convolution product. We denote δ_n for the *point mass* at $\{n\}$. For every $n \in N$, we consider a homomorphism $\phi_n : \ell^1(N_\vee) \longrightarrow C$ with the formula $\phi_n \left(\sum_{i=1}^{\infty} \alpha_i \delta_i\right) = \sum_{i=1}^{n} \alpha_i$ for each $n \in N \cup \{\infty\}$. It is known that

$$\sigma(\ell^1(N_\vee)) = \{\phi_n : n \in N \cup \{\infty\}\} \tag{38}$$

We write $\phi_{N_\vee} = \phi_\infty$ for the *augmentation character*, see [4].

Theorem 2 *The Banach algebra $\ell^1(N_\vee) \times_\theta \ell^1(N_\vee)$ is neither $(\phi_{N_\vee}, \theta)$-biprojective nor $(0, \phi_{N_\vee})$-biprojective, where $\phi_{N_\vee}$ is the augmentation character on $\ell^1(N_\vee)$.*

Proof We assume conversely that $\ell^1(N_\vee) \times_\theta \ell^1(N_\vee)$ is either left $(\phi_{N_\vee}, \theta)$-biprojective or left $(0, \phi_{N_\vee})$-biprojective. Since $N_\vee$ is unital, $\ell^1(N_\vee)$ has an identity. By Proposition 1 and Proposition 2 $\ell^1(N_\vee)$ is left $\phi_{N_\vee}$-biprojective. The existence of a unit δ_1 implies that $\ell^1(N_\vee)$ is left $\phi_{N_\vee}$-contractible. Now we claim that $\ell^1(N_\vee)$ is left ϕ_n-contractible for all $n \in N$. To see this define

$m_n = \delta_n - \delta_{n+1} \in \ell^1(N_\vee)$. Let $a = \sum_{n=1}^{\infty} a_n\delta_n \in \ell^1(N_\vee)$, where (a_n) is a sequence in C such that $\sum_{n=1}^{\infty} |a_n| < \infty$. Consider

$$am_n = a(\delta_n - \delta_{n+1}) = \sum_{n=1}^{\infty} a_n\delta_n(\delta_n - \delta_{n+1}) = \phi_n(a)(\delta_n - \delta_{n+1}) = \phi_n(a)m_n \tag{39}$$

and

$$\phi_n(m_n) = \phi_n(\delta_n - \delta_{n+1}) = \phi_n(\delta_n) - \phi_n(\delta_{n+1}) = 1,$$

for every $a \in \ell^1(N_\vee)$. Thus $\ell^1(N_\vee)$ is character contractible. Applying [5, Corollary 2.2] follows that $\sigma(\ell^1(N_\vee)) = N_\vee \cup \{\infty\}$ is discrete with respect to the w^*-topology. Using the Gelfand representation theorem, we have $\sigma(\ell^1(N_\vee)) = N_\vee \cup \{\infty\}$ is compact, so is finite which is a contradiction.

Example 2 Suppose that $A = \{\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}: \ a, b, c \in C\}$ be a matrix algebra. With matrix operation and ℓ^1-norm A becomes a Banach algebra. Define $\phi : A \longrightarrow C$ by $\phi(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}) = c$. It is easy to see that is a character on A. We claim that $A \times_\theta A$ is neither (ϕ, θ)- biflat nor $(0, \phi)-$biflat, where $\theta \in \sigma(A)$. Suppose in contradiction that $A \times_\theta A$ is either (ϕ, θ)-biflat or $(0, \phi)$-biflat. Since A is unital, by Proposition 3 and Proposition 4 A is left ϕ-biflat. Since A is unital, it is easy to see that A is left ϕ-amenable. Set

$$J := \{\begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix}: \ b, d \in C\}$$

and $\phi_{|_J} \neq 0$. It is clear that J is a closed ideal of A. Since A is left ϕ-amenable, by [9, Lemma 3.1] we have that J is $\phi_{|_J}-$amenable. Now [9, Theorem 1.4] follows that, there exists a bounded net (u_α) in J such that $ju_\alpha - \phi(j)u_\alpha \longrightarrow 0$ and $\phi(u_\alpha) = 1$ for all $j \in J$. Let

$$j = \begin{pmatrix} 0 & j_1 \\ 0 & j_2 \end{pmatrix}$$

and

$$u_\alpha = \begin{pmatrix} 0 & w_\alpha \\ 0 & v_\alpha \end{pmatrix}$$

, for some $j_1, j_2, w_\alpha, v_\alpha \in C$. Thus,

$$ju_\alpha - \phi(j)u_\alpha = \begin{pmatrix} 0 & j_1w_\alpha \\ 0 & j_2v_\alpha \end{pmatrix} - \begin{pmatrix} 0 & j_2w_\alpha \\ 0 & j_2v_\alpha \end{pmatrix} \longrightarrow 0. \tag{40}$$

It gives that $j_1v_\alpha - j_2w_\alpha \longrightarrow 0$. If we put $j_1 = 1$ and $j_2 = 0$, then we have $v_\alpha \to 0$ which contradicts with $\phi(u_\alpha) = v_\alpha = 1$.

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References

1. M. Alaghmandan, R. Nasr-Isfahani, M. Nemati, Character amenability and contractibility of abstract Segal algebras, Bull. Aust. Math. Soc., 82, 274–281 (2010).
2. M. Askari-Sayah, A. Pourabbas, A. Sahami, Johnson pseudo-contractibility and pseudo-amenability of θ-Lau product, Krag. Jour. Math., 44, 593–601 (2020).
3. Y. Choi, Triviality of the generalised Lau product associated to a Banach algebra homomorphism, Bull. Aust. Math. Soc., 94, 286–289 (2016).
4. H. G. Dales, A. T. Lau, D. Strauss, Banach algebras on semigroups and their compactifications, Mem. Am. Math. Soc., 205, 1–165 (2010).
5. M. Dashti, R. Nasr-Isfahani, S. Soltani Renani, Character amenability of Lipschitz algebras, Canad. Math. Bull., 57, 37–41 (2014).
6. H. R. Ebrahimi Vishki, A. R. Khoddami, Biflatness and biprojectivity of Lau product of Banach algebras, Bull. Iran. Math. Soc., 39, 559–568 (2013).
7. E. Ghaderi, A. Sahami $\phi-$biflatness and $\phi-$biprojectivity for θ-Lau product with applications U.P.B. Sci. Bull. Series A., (To appear).
8. B. E. Johnson, Cohomology in Banach algebras, Mem. Amer. Math. Soc., 127 (1972).
9. E. Kaniuth, A. T. Lau, J. Pym, On $\phi-$amenability of Banach algebras, Math. Proc. Camb. Phil. Soc., 144, 85–96 (2008).
10. A. T. Lau, Analysis on a class of Banach algebras with application to harmonic analysis on locally compact groups and semigroups, Fund. Math., 118, 161–175 (1983).
11. M. S. Monfared, Character amenability of Banach algebras, Math. Proc. Camb. Philos. Soc., 144, 697–706 (2008).
12. M. S. Monfared, On certain products of Banach algebras with applications to harmonic analysis, Studia Math., 178, 277–294 (2007).
13. R. Nasr-Isfahani, S. Soltani Renani, Character contractibility of Banach algebras and homological properties of Banach modules, Studia Math., 202, 205–225 (2011).
14. V. Runde, Lectures on amenability, Springer, New York, (2002).
15. A. Sahami, M. Rostami, A. Pourabbas,On left ϕ-biflat Banach algebras, Comment. Math. Univ. Carolin., 61, (2020).
16. A. Sahami, M. Rostami, A. Pourabbas, Left ϕ-biprojectivity of some Banach algebras, Preprint.
17. A. Sahami, On left ϕ-biprojectivity and left ϕ-biflatness of certain Banach algebras, U.P.B. Sci. Bull. Series A., 81, 97–106 (2019).