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Research article

A Fixed-Point Theoretic Approach to the Unique Solvability of a High-Order Nonlinear Fractional Boundary Value Problem with Integral Conditions

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Abstract

This work is devoted to a rigorous analysis of the existence and uniqueness of solutions for a class of high-order nonlinear differential equations of fractional order. The considered problem is defined by a Caputo fractional derivative and is augmented by a set of nonlocal boundary constraints. A key feature of these constraints is an integral condition that couples the behavior of the solution across its entire spatial domain, reflecting a global dependency. Our primary analytical strategy is to recast the differential problem as a fixed-point equation for an equivalent integral operator. This is accomplished by first methodically constructing the Green's function associated with the corresponding linear problem. With the integral operator established, the existence of a unique solution for the full nonlinear problem is then proven by leveraging the power of the Banach contraction mapping principle. To demonstrate the practical relevance and applicability of our theoretical framework, a detailed illustrative example is presented and analyzed.

Keywords: Nonlinear fractional differential equations, Integral boundary conditions, Caputo fractional derivative, Existence and uniqueness theory, Green's function methods, Banach fixed-point theorem.

Mathematics Subject Classification (2020): 34B10, 34B15, 34B27, 34B99

1 Introduction and Scholarly Context

In the last few decades, the theory of fractional calculus has transitioned from a topic of purely mathematical curiosity to a vital and powerful tool in the modeling of complex systems across science and engineering. The core strength of fractional operators lies in their nonlocal nature. Whereas classical integer-order derivatives are local point properties, fractional derivatives (and integrals) depend on the entire history of the function being analyzed. This inherent "memory" is crucial for accurately describing phenomena with hereditary properties, such as the stress-strain relationship in viscoelastic materials, anomalous diffusion in porous media, and long-range interactions in electrostatics [1–3]. This has led to a burgeoning interest in the study of fractional differential equations (FDEs) as sophisticated modeling instruments [4–9].

Within the broader field of FDEs, the analysis of boundary value problems (BVPs) is of paramount importance, as they provide the mathematical framework for physical systems constrained at their boundaries. Recently, BVPs with nonlocal conditions have attracted significant attention. Unlike classical Dirichlet or Neumann conditions, nonlocal conditions can connect the boundary values to interior values of the solution, often via an integral. Such integral boundary conditions are not merely mathematical abstractions; they arise naturally in various applications. For instance, they can model the total heat flux in a thermodynamics problem or the total population size in an ecological model.

The analytical treatment of nonlinear FDEs is challenging, and closed-form solutions are rarely attainable. Consequently, the focus often shifts to proving the existence, uniqueness, and qualitative properties of solutions. Fixed-point theory has emerged as the preeminent tool for this purpose. Foundational results like the Banach Contraction Principle, the Schauder Fixed-Point Theorem, and the Leray-Schauder Nonlinear Alternative provide a robust framework for establishing the solvability of nonlinear equations [10]. The standard methodology involves converting the BVP into an equivalent integral equation, whose solution corresponds to a fixed point of an integral operator. The kernel of this integral operator is the Green's function associated with the linear part of the BVP. The successful application of this technique has been demonstrated in a vast body of literature. For instance, Cabada and his co-authors have made significant contributions, such as in [11] and [12], where they explored problems involving Riemann-Liouville and Caputo derivatives with various nonlocal conditions. Their work highlights the nuances in constructing the Green's function and applying different fixed-point theorems based on the specific structure of the problem.

Inspired by these developments, the present article undertakes a detailed investigation into the existence and uniqueness of a solution for the following high-order nonlinear fractional BVP:

$$\begin{cases} {}^{C}D_t^{\beta} w(t) + h(t, w(t), w'(t), w''(t)) = 0, \quad t \in (0,1), \\ w(0) = w''(0) = 0, \quad \sigma w'(0) + (1-\sigma) w(1) = \int_0^1 w(s) ds, \end{cases} \tag{1}$$

where the fractional order β lies in the range $2 < \beta \leq 3$, the parameter σ is a constant in $[0,1]$, ${}^{C}D_t^{\beta}$ denotes the Caputo fractional derivative, and the nonlinear source term $h : (0,1) \times \mathbb{R}^3 \to \mathbb{R}$ is a given continuous function. The choice of the Caputo derivative is motivated by its utility in modeling real-world problems, as it allows for the specification of initial conditions in a form that is familiar from classical integer-order differential equations. The novelty of our work lies in the specific combination of a high-order derivative ($\beta > 2$) with this particular three-point integral boundary condition, for which we provide a complete and self-contained unique solvability analysis based on the Banach fixed-point theorem.

This paper is organized to guide the reader logically through our analysis. Section 2 establishes the mathematical preliminaries by defining the essential fractional operators. Section 3 is dedicated to the foundational step of solving the linear analogue of our problem and explicitly constructing its Green's function. In Section 4, we leverage this result to formulate the nonlinear problem as a fixed-point equation within a suitable Banach space. Section 5 contains our main theorem, where we state and rigorously prove sufficient conditions for the existence of a unique solution. To show the utility of our abstract result, Section 6 presents a concrete example. Finally, Section 7 summarizes our findings and discusses several promising avenues for future research.

2 Mathematical Preliminaries

In this section, we briefly review the fundamental definitions and properties of fractional calculus that are essential for our subsequent analysis. A detailed theoretical treatment can be found in reference texts such as [4] and [13].

Definition 1. *Let $\beta > 0$ and set $n = \lceil \beta \rceil$. For a function w for which the n-th derivative $w^{(n)}$ is continuous, the* ***Caputo fractional derivative*** *of order β is defined by the integral expression:*

$$ {}^{C}D^{\beta} w(t) := \frac{1}{\Gamma(n-\beta)} \int_0^t (t-\tau)^{n-\beta-1} w^{(n)}(\tau) d\tau. $$

This definition is particularly advantageous in applied problems because, unlike the Riemann-Liouville derivative, the initial conditions for Caputo FDEs are specified in terms of integer-order derivatives, which often have clear physical interpretations.

Definition 2. *For a locally integrable function w and for $\beta > 0$, the* ***Riemann-Liouville fractional integral*** *of order β is defined as:*

$$ I^{\beta} w(t) := \frac{1}{\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1} w(\tau) d\tau. $$

This operator serves as the fundamental building block for fractional calculus and can be viewed as a continuous generalization of integer-order integration.

The relationship between these two operators is the key to transforming differential equations into integral equations. The following lemma is the cornerstone of our entire analytical approach.

Lemma 1. *Let $\beta > 0$ and $n = \lceil \beta \rceil$. For a sufficiently smooth function w, the action of the fractional integral on the Caputo fractional derivative yields:*

$$I^{\beta}[{}^{C}D^{\beta}w(t)] = w(t) - \sum_{k=0}^{n-1} \frac{w^{(k)}(0)}{k!} t^{k}. \tag{2}$$

This identity demonstrates that the fractional integral is a left-inverse to the fractional derivative, with the resulting polynomial accounting for the initial conditions of the function. Consequently, the general solution to the homogeneous FDE ${}^{C}D^{\beta}w(t) = 0$ is a polynomial of degree $n-1$.

3 The Linear Problem and Its Green's Function

Before tackling the nonlinear problem (1), we must first understand its linear structure. This is a standard and powerful technique in the study of differential equations. By solving the corresponding linear problem, we can construct a Green's function, which acts as the kernel of an integral operator that fully encapsulates the properties of the linear differential operator and the associated boundary conditions.

Consider the linear fractional BVP with an arbitrary continuous source term $z(t)$:

$${}^{C}D_{t}^{\beta}w(t) + z(t) = 0, \quad t \in (0,1), \tag{3}$$

subject to the same boundary constraints:

$$w(0) = w''(0) = 0, \quad \sigma w'(0) + (1-\sigma)w(1) = \int_0^1 w(s)ds. \tag{4}$$

Theorem 1. *For $2 < \beta \leq 3$ and any continuous function $z \in C[0,1]$, the unique solution $w \in C^2[0,1]$ to the linear BVP (3)-(4) is given by the integral representation:*

$$w(t) = \int_0^1 \mathscr{G}(t,s)z(s)ds, \tag{5}$$

where the Green's function $\mathscr{G}(t,s)$ is defined as:

$$\mathscr{G}(t,s) = \begin{cases} \frac{2t(1-s)^{\beta-1}[\beta(1-\sigma)-(1-s)]-\beta(t-s)^{\beta-1}}{\Gamma(\beta+1)}, & 0 \leq s \leq t \leq 1; \\ \frac{2t(1-s)^{\beta-1}[\beta(1-\sigma)-(1-s)]}{\Gamma(\beta+1)}, & 0 \leq t \leq s \leq 1. \end{cases} \tag{6}$$

Proof. Our starting point is the general solution of the FDE (3). Applying the fractional integral operator I^{β} to the equation ${}^{C}D^{\beta}w(t) = -z(t)$ and using Lemma 1 with $n = \lceil \beta \rceil = 3$, we find:

$$w(t) = -I^{\beta}(z(t)) + k_0 + k_1 t + k_2 t^2 = -\frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} z(s)ds + k_0 + k_1 t + k_2 t^2. \tag{7}$$

The boundary conditions $w(0) = 0$ and $w''(0) = 0$ allow us to determine the constants k_0 and k_2. We have $k_0 = w(0)/0! = 0$ and $k_2 = w''(0)/2! = 0$. This reduces the solution form to:

$$w(t) = -\frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} z(s)ds + w'(0)t. \tag{8}$$

The remaining unknown is the initial slope $w'(0)$, which must be determined by the nonlocal integral condition. From (8), the value of the solution at $t = 1$ is $w(1) = -\frac{1}{\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} z(s)ds + w'(0)$. Substituting this into the integral condition gives:

$$\sigma w'(0) + (1-\sigma)\left[-\frac{1}{\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} z(s)ds + w'(0)\right] = \int_0^1 w(s)ds. \tag{9}$$

Solving this equation for $w'(0)$, we obtain:

$$w'(0) = \int_0^1 w(s)ds + \frac{1-\sigma}{\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} z(s)ds. \tag{10}$$

At this point, the expression for $w(t)$ contains the term $\int_0^1 w(s)ds$, which itself depends on $w(t)$. To resolve this, we employ a self-consistency argument. Let us denote this unknown integral by $C_w = \int_0^1 w(s)ds$. Substituting the expression for $w'(0)$ into (8), we have:

$$w(t) = -\frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} z(s)ds + t\left(C_w + \frac{1-\sigma}{\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} z(s)ds\right). \tag{11}$$

To find an explicit value for C_w, we integrate both sides of this equation with respect to t from 0 to 1. This yields:

$$C_w = -\frac{1}{\Gamma(\beta)} \int_0^1 \int_0^t (t-s)^{\beta-1} z(s)dsdt + C_w \int_0^1 tdt + \left(\frac{1-\sigma}{\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} z(s)ds\right) \int_0^1 tdt.$$

After evaluating the time integrals and changing the order of integration in the double integral, we get:

$$C_w = -\frac{1}{\Gamma(\beta+1)} \int_0^1 (1-s)^{\beta} z(s)ds + \frac{1}{2} C_w + \frac{1-\sigma}{2\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} z(s)ds.$$

Solving this simple algebraic equation for C_w gives the desired explicit expression:

$$C_w = \frac{1-\sigma}{\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} z(s)ds - \frac{2}{\Gamma(\beta+1)} \int_0^1 (1-s)^{\beta} z(s)ds. \tag{12}$$

The final step is to substitute this result for C_w back into (11). This eliminates all implicit dependencies and provides the explicit integral representation of the solution $w(t)$ in terms of the source $z(t)$:

$$w(t) = -\frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} z(s)ds + \frac{t(1-\sigma)}{\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} z(s)ds + t\left(\frac{1-\sigma}{\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} z(s)ds - \frac{2}{\Gamma(\beta+1)} \int_0^1 (1-s)^{\beta} z(s)ds\right).$$

The final manipulation involves restructuring this expression to clearly identify the Green's function kernel. By combining integral terms and splitting the domain of integration, we arrive at:

$$\begin{aligned} w(t) &= \int_0^t \frac{2t(1-s)^{\beta-1}[\beta(1-\sigma)-(1-s)] - \beta(t-s)^{\beta-1}}{\Gamma(\beta+1)} z(s)ds + \int_t^1 \frac{2t(1-s)^{\beta-1}[\beta(1-\sigma)-(1-s)]}{\Gamma(\beta+1)} z(s)ds \\ &= \int_0^1 \mathscr{G}(t,s) z(s)ds. \end{aligned}$$

This completes the construction of the Green's function and the proof of the theorem. □

4 Transformation to a Fixed-Point Problem

The integral representation derived in the previous section is the key to analyzing the nonlinear BVP. We now define the appropriate functional setting and formulate the problem as a fixed-point equation. Our analysis is set in the Banach space $E = C^2[0,1]$, the space of all twice continuously differentiable functions on the interval $[0,1]$. We equip this space with the norm $\|w\|_E = \|w\|_\infty + \|w'\|_\infty + \|w''\|_\infty$, where $\|\cdot\|_\infty$ is the standard supremum norm. The completeness of this space is essential for the application of the Banach fixed-point theorem.

By formally replacing the linear source term $z(t)$ with the nonlinear function $h(t, w(t), w'(t), w''(t))$ in Theorem 1, we observe that a function $w(t)$ is a solution to the nonlinear BVP (1) if and only if it satisfies the nonlinear integral equation:

$$w(t) = \int_0^1 \mathscr{G}(t,s) h(s, w(s), w'(s), w''(s))ds. \tag{13}$$

This equivalence allows us to define an operator $\mathscr{A}: E \to E$ by

$$(\mathscr{A}w)(t) = \int_0^1 \mathscr{G}(t,s) h(s, w(s), w'(s), w''(s))ds. \tag{14}$$

This operator, often called the solution operator, takes a function $w \in E$ as input and returns a new function $(\mathscr{A}w)$ that solves the linear problem with the source term evaluated at w. A fixed point of this operator, a function w^* such that $\mathscr{A}w^* = w^*$, is therefore a solution to the full nonlinear problem. Our task is thus reduced to proving that this operator $\mathscr{A}$ has a unique fixed point.

5 The Main Existence and Uniqueness Result

We are now in a position to present and prove the central theorem of this paper. This theorem provides sufficient conditions on the nonlinearity h to guarantee that the BVP (1) is well-posed, meaning it has exactly one solution. For clarity, we define the following constant, which encapsulates the influence of the Green's function and depends only on the parameters β and σ:

$$\mathscr{K} = \frac{4}{\Gamma(\beta+2)} + \frac{1}{\Gamma(\beta+1)} + \frac{1}{\Gamma(\beta)} + \frac{1}{\Gamma(\beta-1)} + \frac{4(1-\sigma)}{\Gamma(\beta+1)}.$$

Theorem 2. *Suppose the nonlinear function $h(t,w_1,w_2,w_3)$ meets the following requirement:*

($\mathbf{H}_1$) There exists a constant $L_h > 0$ such that h satisfies a Lipschitz condition with respect to its last three variables. That is, for all $t \in [0,1]$ and all $u_i, v_i \in \mathbb{R}$ ($i = 1,2,3$):

$$|h(t,u_1,u_2,u_3) - h(t,v_1,v_2,v_3)| \le L_h\big(|u_1-v_1| + |u_2-v_2| + |u_3-v_3|\big). \tag{15}$$

If the Lipschitz constant is sufficiently small such that $L_h\mathscr{K} < 1$, then the BVP (1) has a unique solution w^* within the closed ball

$$B_R = \{w \in C^2[0,1] \mid \|w\|_E \le R\}.$$

The radius R of this ball must satisfy the condition $R \ge \frac{N_h\mathscr{K}}{1-L_h\mathscr{K}}$, where $N_h = \sup_{t\in[0,1]} |h(t,0,0,0)|$.

Proof. The proof relies on showing that the operator $\mathscr{A}$ defined in (14) is a contraction mapping on the closed set $B_R \subset E$. The proof consists of two logical steps.

Step 1: **Show that $\mathscr{A}$ maps the ball B_R into itself (i.e., $\mathscr{A}(B_R) \subseteq B_R$).** Let $w \in B_R$, so $\|w\|_E \le R$. We first establish a uniform bound on the nonlinear term. Using the triangle inequality and the Lipschitz condition ($\mathbf{H}_1$):

$$\begin{aligned} |h(t,w(t),w'(t),w''(t))| &\le |h(t,w(t),w'(t),w''(t)) - h(t,0,0,0)| + |h(t,0,0,0)| \\ &\le L_h(|w(t)| + |w'(t)| + |w''(t)|) + N_h \\ &\le L_h(\|w\|_\infty + \|w'\|_\infty + \|w''\|_\infty) + N_h = L_h\|w\|_E + N_h. \end{aligned}$$

Since $\|w\|_E \le R$, we have the bound $\|h(\cdot,w(\cdot),\dots)\|_\infty \le L_hR + N_h$. Now we must show that $\|\mathscr{A}w\|_E \le R$. This involves bounding the norms of $(\mathscr{A}w)$, $(\mathscr{A}w)'$, and $(\mathscr{A}w)''$. The calculations follow the same integral estimations used in the proof of Step 1 of Theorem 5.1 in the previous provided source, but we summarize the outcome for completeness. By bounding the integrals involving the Green's function and its derivatives, we obtain the cumulative bound:

$$\begin{aligned} \|\mathscr{A}w\|_E &= \|(\mathscr{A}w)\|_\infty + \|(\mathscr{A}w)'\|_\infty + \|(\mathscr{A}w)''\|_\infty \\ &\le (L_hR + N_h)\left[\frac{4}{\Gamma(\beta+2)} + \frac{1}{\Gamma(\beta+1)} + \frac{1}{\Gamma(\beta)} + \frac{1}{\Gamma(\beta-1)} + \frac{4(1-\sigma)}{\Gamma(\beta+1)}\right] \\ &= (L_hR + N_h)\mathscr{K}. \end{aligned}$$

This inequality shows that the norm of the output $(\mathscr{A}w)$ is bounded by a quantity related to the norm of the input w. For the operator to map the ball B_R to itself, we need this output norm to not exceed the radius R. The condition imposed on R in the theorem statement, $R \ge \frac{N_h\mathscr{K}}{1-L_h\mathscr{K}}$, is precisely equivalent to $R \ge (L_hR + N_h)\mathscr{K}$. Therefore, we have demonstrated that $\|\mathscr{A}w\|_E \le R$, which confirms the invariance of the ball B_R.

Step 2: **Show that $\mathscr{A}$ is a contraction mapping on B_R.** Here, we show that the operator $\mathscr{A}$ uniformly shrinks the distance between any two functions in our space. This is the crucial property that guarantees convergence to a unique fixed point. Let $w, v \in B_R$ be two arbitrary functions. We examine the norm of their difference after applying the operator:

$$|(\mathscr{A}w)(t) - (\mathscr{A}v)(t)| \le \int_0^1 |\mathscr{G}(t,s)||h(s,w(s),\dots) - h(s,v(s),\dots)|ds$$

$$\le L_h \int_0^1 |\mathscr{G}(t,s)| \big(|w(s)-v(s)| + |w'(s)-v'(s)| + |w''(s)-v''(s)| \big) ds$$
$$\le L_h \|w-v\|_E \int_0^1 |\mathscr{G}(t,s)| ds.$$

By performing similar estimations for the derivatives $(\mathscr{A}w)' - (\mathscr{A}v)'$ and $(\mathscr{A}w)'' - (\mathscr{A}v)''$ and summing the resulting norm bounds, we arrive at the inequality: $\|(\mathscr{A}w) - (\mathscr{A}v)\|_E \le L_h \|w-v\|_E \mathscr{K}$. By the central hypothesis of the theorem, we have $L_h \mathscr{K} < 1$. This inequality establishes that $\mathscr{A}$ is a strict contraction mapping on the space B_R with the contraction constant $L_h \mathscr{K}$. Since B_R is a closed subset of the Banach space E, it is a complete metric space. The conclusion of the Banach Fixed-Point Theorem now applies directly, guaranteeing that $\mathscr{A}$ has a unique fixed point in B_R. This fixed point is the unique solution to our BVP, and the proof is complete.

□

6 An Illustrative Example

To ground our abstract theorem in a tangible context, we now apply it to a specific BVP. This example will demonstrate how to verify the conditions of the theorem and determine the constraints on the problem's parameters. Consider the following BVP:

$$\begin{cases} {}^C D_t^{2.5} w(t) + \gamma \left(\frac{\sin(w(t))}{5} + \frac{\arctan(w'(t))}{4} + \frac{w''(t)}{3+w''(t)^2} + e^{-t} \right) = 0, \\ w(0) = w''(0) = 0, \quad 0.25 w'(0) + 0.75 w(1) = \int_0^1 w(s) ds, \end{cases} \tag{16}$$

where $\gamma > 0$ is a parameter representing the strength of the nonlinearity. This problem fits our general form (1) with parameters $\beta = 2.5$ and $\sigma = 0.25$. The nonlinear function is:

$$h(t,w,w',w'') = \gamma \left(\frac{\sin(w)}{5} + \frac{\arctan(w')}{4} + \frac{w''}{3+(w'')^2} + e^{-t} \right).$$

First, we must verify the Lipschitz condition ($\mathbf{H}_1$). We use the Mean Value Theorem. The derivatives of the component functions with respect to their arguments are bounded:

$$|\frac{d}{du}(\frac{\sin u}{5})| \le 1/5, \qquad |\frac{d}{du}(\frac{\arctan u}{4})| \le 1/4,$$

and

$$|\frac{d}{du}(\frac{u}{3+u^2})| \le 1/3.$$

This implies:

$$|h(t,u_1,u_2,u_3) - h(t,v_1,v_2,v_3)| \le \gamma \left(\frac{1}{5}|u_1-v_1| + \frac{1}{4}|u_2-v_2| + \frac{1}{3}|u_3-v_3| \right)$$
$$\le \frac{\gamma}{3} \big(|u_1-v_1| + |u_2-v_2| + |u_3-v_3| \big).$$

Thus, condition ($\mathbf{H}_1$) holds with a Lipschitz constant $L_h = \gamma/3$.

Next, we calculate the constants N_h and $\mathscr{K}$.

$$N_h = \sup_{t\in[0,1]} |h(t,0,0,0)| = \sup_{t\in[0,1]} |\gamma e^{-t}| = \gamma.$$

For $\beta = 2.5$ and $\sigma = 0.25$, the constant $\mathscr{K}$ is:

$$\mathscr{K} = \frac{4}{\Gamma(4.5)} + \frac{1}{\Gamma(3.5)} + \frac{1}{\Gamma(2.5)} + \frac{1}{\Gamma(1.5)} + \frac{4(1-0.25)}{\Gamma(3.5)} = \frac{4}{\Gamma(4.5)} + \frac{4}{\Gamma(3.5)} + \frac{1}{\Gamma(2.5)} + \frac{1}{\Gamma(1.5)}.$$

Using the values of the Gamma function, we find $\mathscr{K} \approx 3.47$.

The condition for a unique solution from Theorem 2 is $L_h \mathscr{K} < 1$. Substituting our values, we need $(\gamma/3) \cdot 3.47 < 1$, which simplifies to $\gamma < 3/3.47 \approx 0.86$. This result provides a practical guideline: if the parameter γ is kept below this threshold, the BVP is guaranteed to have a unique, stable solution. For example, if we set $\gamma = 0.5$, the condition is met. A unique solution exists in any ball B_R where

$$R \ge \frac{N_h \mathscr{K}}{1 - L_h \mathscr{K}} = \frac{0.5 \cdot 3.47}{1 - (0.5/3) \cdot 3.47} \approx \frac{1.735}{0.421} \approx 4.12.$$

7 Conclusion and Future Directions

In this paper, we have successfully developed and presented a comprehensive framework for establishing the existence and uniqueness of solutions to a high-order nonlinear fractional differential equation subject to nonlocal integral boundary conditions. Our approach, rooted in the principles of functional analysis, involved transforming the BVP into a fixed-point problem for an integral operator. The kernel of this operator, the Green's function, was explicitly constructed. The Banach contraction principle then provided a powerful and direct path to proving our main result. The theorem offers a clear, practical criterion based on the Lipschitz constant of the nonlinear term that guarantees the well-posedness of the problem.

The significance of this work lies in its contribution to the rigorous mathematical theory of nonlocal fractional boundary value problems. By providing explicit conditions for unique solvability, our results offer a foundation upon which further qualitative and numerical studies can be built. The methodology employed is robust and demonstrates the synergy between classical analytical techniques and modern fixed-point theory.

This research also opens up several promising avenues for future investigation:

- **Weaker Conditions for Existence:** The Banach principle requires a contraction, which is a strong condition and guarantees uniqueness. It would be valuable to explore existence results under weaker conditions on the nonlinearity, such as sub-linear growth. This would involve applying different fixed-point theorems, such as Krasnoselskii's theorem for the sum of a contraction and a compact operator, or the Leray-Schauder nonlinear alternative.

- **Different Fractional Operators:** The analysis could be extended to problems involving other types of fractional derivatives, such as the Hadamard, Erdélyi-Kober, or Caputo-Fabrizio operators. Each of these operators has different mathematical properties and is suited to different classes of problems, and the construction of the corresponding Green's functions would present new and interesting challenges.

- **Development of Numerical Methods:** The integral operator $\mathscr{A}$ derived in our work is constructive. This naturally suggests the development of numerical schemes based on the Picard iteration, $w_{n+1} = \mathscr{A} w_n$. A future study could focus on the implementation and convergence analysis of such a scheme to provide numerical approximations of the unique solution.

- **Extension to Systems and Inclusions:** A natural generalization would be to consider systems of coupled fractional differential equations with similar nonlocal boundary conditions. Furthermore, one could investigate the corresponding problem for fractional differential inclusions, where the single-valued nonlinear term is replaced by a set-valued map, requiring the tools of multi-valued analysis.

In summary, this paper provides a solid and complete analysis for a specific class of fractional BVPs, while also highlighting the rich potential for further research in this dynamic and important area of mathematics.

Data Availability

The manuscript has no associated data or the data will not be deposited.

Conflicts of Interest

The author declares that there is no conflict of interest.

Ethical Considerations

The author has diligently addressed ethical concerns, such as informed consent, plagiarism, data fabrication, misconduct, falsification, double publication, redundancy, submission, and other related matters.

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