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Coupled Fixed Point Theorems on G-Metric Spaces Via α-Series

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Abstract The object of this paper is to study of the results of coupled fixed point in generalized metric spaces, as known as G-metric spaces. We will impose some conditions upon a self-mapping and a sequence of mappings via a kind of series, known as a-series. Also, an example is provided to illustrate the results.

Keywords G-metric space · α-series · Couple fixed point · Couple coincidence point · compatible mappings

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1 Introduction

The concept of G-metric spaces, as a generalization of the metric space (X, d), was introduced in [8] and [9]. In [2], Bhaskar and Lakshmikantham introduced coupled fixed point results for partially ordered metric spaces. Latter, many authors have acquired interesting important coupled fixed point theorems [4]-[6].

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In this paper, we investigate coupled fixed point theorems by imposing some conditions on a self-mapping g and a sequence of mappings $\{T_m\}_{m\in\mathbf{N}_0}$, for partially ordered G-metric spaces, via α-series. The α-series are wider than the convergent series. Throughout this article, $\mathbf{N}$ is a positive integer and $\mathbf{N}_0 = \mathbf{N} \cup \{0\}$, and "$\xrightarrow{G}$" will denote G-convergence. First, we present some basic definitions that are used throughout the paper.

Definition 1 [9] A G-metric space is a pair (X, G), where $X \neq \emptyset$ and $G : X^3 \to [0, +\infty)$ is map such that satisfies:

(G1) $G(x^1, x^2, x^3) = 0$ if $x^1 = x^2 = x^3$.

(G2) $G(x^1, x^1, x^2) > 0$ for all $x^1, x^2 \in X$ with $x^1 \neq x^2$.

(G3) $G(x^1, x^1, x^2) \leq G(x^1, x^2, x^3)$ for all $x^1, x^2, x^3 \in X$ with $x^3 \neq x^2$.

(G4) $G(x^1, x^2, x^3) = G(x^1, x^3, x^2) = G(x^2, x^3, x^1) = \ldots$ (symmetry in all three variables).

(G5) $G(x^1, x^2, x^3) \leq G(x^1, e, e) + G(e, x^2, x^3)$ for all $x^1, x^2, x^3, e \in X$, (rectangle inequality).

The map G is called a G-metric on X.

Proposition 1 *[9] Every G-metric space (X, G) defines a metric space (X, d_G) by $d_G(x^1, x^2) = G(x^1, x^2, x^2) + G(x^2, x^1, x^1)$, for all $x^1, x^2 \in X$.*

Definition 2 [9] Let (X, G) be a G-metric space. Then

1. a sequence $\{x^1_m\} \in X$ is G-convergent to x^1 if
$$\lim_{m,n\to+\infty} G(x^1, x^1_m, x^1_m) = 0,$$
that is, for each $\epsilon > 0$ there exists N such that $G(x^1, x^1_m, x^1_m) < \epsilon$ for all $m, n \geq N$.
2. If for each $\epsilon > 0$, there exists $N \in \mathbf{N}$ such that $G(x^1_m, x^1_m, x^1_m) < \epsilon$ for all $m, n \geq N$, then a sequence $\{x^1_m\}$ is called G-Cauchy.
3. If every G-Cauchy sequence in (X, G) be G-convergent in (X, G), then G-metric space is called G-complete.

Proposition 2 *[9] A G-metric space (X, G) is G-complete if and only if (X, d_G) is a complete metric space.*

Definition 3 [9] Let (X, G) be a G-metric space, and let a mapping $F : X^2 \to X$. F is called continuous if sequences $x^1_n \xrightarrow{G} x^1, x^2_n \xrightarrow{G} x^2$, then sequence $F(x^1_m, x^2_m) \xrightarrow{G} F(x^1, x^2)$.

Definition 4 [9] Let (X, G) and (X', G') be G-metric spaces, and let a function $f : (X, G) \to (X', G')$. Then f is called G-continuous at a point $e \in X$ iff for every $\epsilon > 0$, there exists $\delta > 0$ such that $x^1, x^2 \in X$ and $G(e, x^1, x^2) < \delta$ implies $G(f(e), f(x^1), f(x^2)) < \epsilon$.
A function f is G-continuous at X iff it is G-continuous at all $e \in X$.

Proposition 3 *[9] Let (X, G) be a G-metric space. Then the following are equivalent:*

1. *sequence* $x_n^1 \xrightarrow{G} x^1$*;*
2. $G(x_m^1, x_m^1, x^1) \to 0$ *as* $n \to +\infty$*;*
3. $G(x_m^1, x^1, x^1) \to 0$ *as* $n \to +\infty$*;*
4. $G(x_m^1, x_m^1, x^1) \to 0$ *as* $n, m \to +\infty$.

Definition 5 [3] let $F : X^2 \longrightarrow X$. An element $(x^1, x^2) \in X^2$ is called a *coupled fixed point* of F if

$$F(x^1, x^2) = x^1, \quad F(x^2, x^1) = x^2.$$

Definition 6 [7] Let $F : X^2 \longrightarrow X$ and $g : X \longrightarrow X$ are given. An element $(x^1, x^2) \in X^2$ is called a *coupled coincidence point* of the mappings F and g if $F(x^1, x^2) = gx^1$ and $F(x^2, x^1) = gx^2$. So, (gx^1, gx^2) is called a *coupled coincidence point.*

Definition 7 Let $(X, \preceq)$ be a poset (or partially ordered set) and $F : X^2 \to X$. We say that F has the *mixed monotone property* if for any $x^1, x^2 \in X$

$$\begin{aligned} x_1^1, x_2^1 \in X, \quad x_1^1 \preceq x_2^1 \quad &\Rightarrow \quad F(x_1^1, x^2) \preceq F(x_2^1, x^2), \\ x_1^2, x_2^2 \in X, \quad x_1^2 \preceq x_2^2 \quad &\Rightarrow \quad F(x^1, x_1^2) \succeq F(x^1, x_2^2) \end{aligned}$$

that is, $F(x^1, x^2)$ is monotone increasing in x^1 and is monotone decreasing in x^2.

Definition 8 [7] Let $(X, \preceq)$ be a poset, $g : X \longrightarrow X$ and $F : X^2 \longrightarrow X$ are given. We say that F has the *g-mixed monotone property* if for any $x^1, x^2 \in X$,

$$\begin{aligned} x_1^1, x_2^1 \in X, \quad gx_1^1 \preceq gx_2^1 \quad &\Rightarrow \quad F(x_1^1, x^2) \preceq F(x_2^1, x^2), \\ x_1^2, x_2^2 \in X, \quad gx_1^2 \preceq gx_2^2 \quad &\Rightarrow \quad F(x^1, x_1^2) \succeq F(x^1, x_2^2) \end{aligned}$$

that is, $F(x^1, x^2)$ is monotone increasing in x^1, and it is monotone decreasing in x^2.

Definition 9 [3] Let (X, d) be a metric space and let $g : X \longrightarrow X$ and $F : X^2 \longrightarrow X$. The mappings F and g are said to be *compatible* if

$$\begin{aligned} &\lim_{m \to +\infty} d(g(F(x_m^1, x_m^2)), F(gx_m^1, gx_m^2)) = 0, \\ &\lim_{m \to +\infty} d(g(F(x_m^2, x_m^1)), F(gx_m^2, gx_m^1)) = 0, \end{aligned}$$

whenever $\{x_m^1\}, \{x_m^2\}$ are sequences in X, such that

$$\lim_{m \to +\infty} F(x_m^1, x_m^2) = \lim_{m \to +\infty} gx_m^1 = x^1,$$

$$\lim_{m\to+\infty} F(x_m^2, x_m^1) = \lim_{m\to+\infty} gx_m^2 = x^2,$$

for some $x^1, x^2 \in X$.

Definition 10 [3] Let $g : X \longrightarrow X$ and $F : X^2 \longrightarrow X$ be mappings, if

$$\lim_{m\to+\infty} g(F(x_m^1, x_m^2)) = gx^1, and \lim_{m\to+\infty} F(gx_m^1, gx_m^2) = F(x^1, x^2)$$

$$\lim_{m\to+\infty} g(F(x_m^2, x_m^1)) = gx^2, and \lim_{m\to+\infty} F(gx_m^2, gx_m^1) = F(x^2, x^1)$$

whenever $\{x_m^1\}, \{x_m^2\}$ are sequences in X, such that

$$\lim_{m\to+\infty} F(x_m^1, x_m^2) = \lim_{m\to+\infty} gx_m^1 = x^1,$$

$$\lim_{m\to+\infty} F(x_m^2, x_m^1) = \lim_{m\to+\infty} gx_m^2 = x^2,$$

for some $x^1, x^2 \in X$, then g and F are called *reciprocally continuous*, and if

$$\lim_{m\to+\infty} g(F(x_m^1, x_m^2)) = gx^1, or \lim_{m\to+\infty} F(gx_m^1, gx_m^2) = F(x^1, x^2),$$

$$\lim_{m\to+\infty} g(F(x_m^2, x_m^1)) = gx^2, or \lim_{m\to+\infty} F(gx_m^2, gx_m^1) = F(x^2, x^1),$$

such that

$$\lim_{m\to+\infty} F(x_m^1, x_m^2) = \lim_{m\to+\infty} gx_m^1 = x^1,$$

$$\lim_{m\to+\infty} F(x_m^2, x_m^1) = \lim_{m\to+\infty} gx_m^2 = x^2,$$

for some $x^1, x^2 \in X$, then g and $F$are called *w-reciprocally continuous.*

Definition 11 [12] Let $(X, G, \preceq)$ be a partially ordered G-metric space on X. We say that $(X, G, \preceq)$ is regular if the following conditions hold:

1. if a increasing sequence $x_m^1 \to x^1$, then $x_m^1 \preceq x^1$ for all m,
2. if a decreasing sequence $x_m^2 \to x^2$, then $x_m^2 \succeq x^2$ for all m.

Definition 12 [11] Let $\{a_n\}$ be a sequence of positive real numbers. A series $\sum_{n=1}^{+\infty} a_n$ is called an α-series, if there exist $0 < \alpha < 1$ and $n_\alpha \in \mathbf{N}$ such that $\sum_{i=1}^{k} a_i \leq \alpha k$ for each $k \geq n_\alpha$.

For example, we know that every convergent series is bounded hence every convergent series is an α-series. Moreover, we can show that for each α, $0 < \alpha < 1$, $\sum_{n=1}^{+\infty} \frac{1}{n}$ is an α-series. In other words, there exsist divergent series that is an α-series.

In this paper, for the sequence of mappings $T_m : X^2 \to X$ and $g : X \to X$, where (X, G) is a G-metric space, we consider existence and uniqueness of coupled common fixed point.

2 Main results

Inspired by the Definition 8 we have the following definition.

Definition 13 Let $(X, \preceq)$ be a poset and $T_m : X^2 \to X$, $m \in \mathbf{N}_0$, and $g : X \to X$ are given. We say that T_m has the *g-mixed monotone property* if for any $x^1, x'^1, x^2, x'^2 \in X$,

$$gx^1 \preceq gx'^1,\ gx'^2 \preceq gx^2 \ \textit{imply}\ T_m(x^1, x^2) \preceq T_{m+1}(x'^1, x'^2),$$
$$T_{m+1}(x'^2, x'^1) \preceq T_m(x^2, x^1).$$

Definition 14 Let $T_m : X^2 \to X$ and $g : X \longrightarrow X$ are given. We call $\{T_m\}_{m\in\mathbf{N}_0}$ and g are satisfied in (K) property if there exists $0 \le \beta_{m,m'}, \gamma_{m,m'} < 1$ for $m, m' \in \mathbf{N}_0$, such that

$$G(T_m(x^1, x^2), T_{m'}(u^1, u^2), T_{m'}(u^3, x^3)) \le \beta_{m,m'}[G(gx^1, T_m(x^1, x^2), T_m(x^1, x^2))$$
$$+G(gu^1, T_{m'}(u^1, u^2), T_{m'}(u^3, x^3))] + \gamma_{m,m'}(G(gx^1, gu^1, gu^3)) \qquad (1)$$

for $x^1, x^2, u^1, u^2 \in X$ with $gx^1 \preceq gu^1, gu^2 \preceq gx^2$ or $gx^1 \succeq gu^1, gu^2 \succeq gx^2$.

Definition 15 We call g and T_0 have a non-decreasing transcendence point in the first component and non-increasing transcendence point in the second component, which we call g and T_0 have *mixed coupled transcendence point*, if there exist $x_0^1, x_0^2 \in X$ such that

$$T_0(x_0^1, x_0^2) \succeq gx_0^1,\ T_0(x_0^2, x_0^1) \preceq gx_0^2. \qquad (2)$$

We begin with the following statement, which, in proof of the main theorem, considers the sequences that are made in the following way. Let $x_0^1, x_0^2 \in X$, such that condition (2) holds, since $T_0(X^2) \subseteq g(X)$, we can define $x_1^1, x_1^2 \in X$ such that $gx_1^1 = T_0(x_0^1, x_0^2)$, $gx_1^2 = T_0(x_0^2, x_0^1)$. Again since $T_0(X^2) \subseteq g(X)$, there exists $x_2^1, x_2^2 \in X$ such that $gx_2^1 = T_1(x_1^1, x_1^2)$, $gx_2^2 = T_1(x_1^2, x_1^1)$. Continuing this technique, for all $n \ge 0$, we get

$$gx_m^1 = T_{m-1}(x_{m-1}^1, x_{m-1}^2), \quad gx_m^2 = T_{m-1}(x_{m-1}^2, x_{m-1}^1). \qquad (3)$$

Now, using mathematical induction, we show that

$$gx_m^1 \preceq gx_{m+1}^1, \quad gx_m^2 \succeq gx_{m+1}^2, \qquad (4)$$

for all $n \ge 0$. To show this, since (2) holds in view of

$$gx_1^1 = T_0(x_0^1, x_0^2), \quad gx_1^2 = T_0(x_0^2, x_0^1),$$

we have $gx_0^1 \preceq gx_1^1$, $gx_0^2 \succeq gx_1^2$, that is, for $n = 0$, condition (4) holds. We assume that (4) holds for some $n > 0$. Now, by (3) and (4), we deduce that

$$gx_{m+1}^1 = T_m(x_m^1, x_m^2) \preceq T_{m+1}(x_{m+1}^1, x_{m+1}^2) = gx_{m+2}^1,$$
$$gx_{m+2}^2 = T_{m+1}(x_{m+1}^2, x_{m+1}^1) \preceq T_m(x_m^2, x_m^1) = gx_{m+1}^2.$$

Thus by mathematical induction, we have done. Therefore, we have

$$gx_0^1 \preceq \ldots \preceq gx_{m+1}^1 \preceq \ldots,$$
$$gx_0^2 \succeq \ldots \succeq gx_{m+1}^2 \succeq \ldots.$$

Due to the above considerations and [10] Definitions 9 and 10 are as follows.

Definition 16 Let (X, G) be a G-metric space, and let $g : X \to X$ and $T_m : X^2 \to X$ be given. g and $\{T_m\}_{m\in\mathbf{N}_0}$ are *compatible* if

$$\lim_{m\to+\infty} G(gT_m(x_m^1, x_m^2), T_m(gx_m^1, gx_m^2), T_m(gx_m^1, gx_m^2)) = 0,$$

$$\lim_{m\to+\infty} G(gT_m(x_m^2, x_m^1), T_m(gx_m^2, gx_m^1), T_m(gx_m^2, gx_m^1)) = 0,$$

whenever $\{x_m^1\}, \{x_m^2\}$ are sequences in X, such that

$$\lim_{m\to+\infty} T_m(x_m^1, x_m^2) = \lim_{m\to+\infty} gx_{m+1}^1 = gx^1,$$

$$\lim_{m\to+\infty} T_m(x_m^2, x_m^1) = \lim_{m\to+\infty} gx_{m+1}^2 = gx^2,$$

for some $x^1, x^2 \in X$.

Definition 17 Let (X, G) be a G-metric space, and let $g : X \to X$ and $T_m : X^2 \to X$ be given. g and $\{T_m\}_{m\in\mathbf{N}_0}$ are called *w-reciprocally continuous* if

$$\lim_{m\to+\infty} g(T_m(x_m^1, x_m^2)) = g(x^1),$$
$$\lim_{m\to+\infty} g(T_m(x_m^2, x_m^1)) = g(x^2),$$

whenever $\{x_m^1\}, \{x_m^2\}$ are sequences in X, such that

$$\lim_{m\to+\infty} T_m(x_m^1, x_m^2) = \lim_{m\to+\infty} g(x_{m+1}^1) = x^1,$$

$$\lim_{m\to+\infty} T_m(x_m^2, x_m^1) = \lim_{m\to+\infty} g(x_{m+1}^2) = x^2,$$

for some $x^1, x^2 \in X$.

Definition 18 For $x^1, x^2 \in X$, we say that (x^1, x^2) is coupled comparable with (u^1, u^2) iff

$$x^1 \succeq u^1, x^2 \preceq u^2 \text{ or } x^1 \preceq u^1, x^2 \succeq u^2 \text{ or}$$

$$x^1 \succeq u^2, x^2 \preceq u^1 \text{ or } x^1 \preceq u^2, x^2 \succeq u^1.$$

If in the above definition replace (x^1, x^2) and (u^1, u^2) with (gx^1, gx^2) and (gu^1, gu^2), we call (x^1, x^2) is coupled comparable with (u^1, u^2) with respect to g.

At the following, we state the main result of this manuscript.

Theorem 1 *Let $(X, G, \preceq)$ be partially ordered G-metric on X such that (X, G) is a complete G-metric space, Let $g : X \to X$ and $T_m : X^2 \to X$ be a sequence of mappings, has g-mixed monotone property with $T_m(X^2) \subseteq g(X)$, g and $\{T_m\}_{m\in\mathbf{N}_0}$ are continuous, w-reciprocally continuous, and $g(X)$ is closed. Assume that the following holds:*

1. *$\{T_m\}_{m\in\mathbf{N}_0}$ and g are compatible,*
2. *there exists $(x_0^1, x_0^2) \in X$ such that condition (2) holds,*
3. *$\{T_m\}_{m\in\mathbf{N}_0}$ and g satisfying the condition (K), if $\sum_{m=1}^{+\infty}\left(\frac{\beta_{m,m+1}+\gamma_{m,m+1}}{1-\beta_{m,m+1}}\right)$ is an α-series, then $\{T_m\}_{m\in\mathbf{N}_0}$ and g have a coupled coincidence point.*
4. *If $\{T_m\}_{m\in\mathbf{N}_0}$ and g, have coupled coincidence points comparable with respect to g, then there exists $(x^1, x^2) \in X$ such that*

$$x^1 = gx^1 = T_m(x^1, x^2), \quad x^2 = gx^2 = T_m(x^2, x^1),$$

for $m \in \mathbf{N}$, that is, $\{T_m\}_{m\in\mathbf{N}_0}$ and g have a unique coupled common fixed point.

Proof For any $x_0^1, x_0^2 \in X$, we can consider the sequences $\{x_r^1\}, \{x_r^2\}$ constructed above, that is,

$$gx_r^1 = T_{r-1}(x_{r-1}^1, x_{r-1}^2), \quad gx_r^2 = T_r(x_{r-1}^2, x_{r-1}^1).$$

By (1), we get

$$\begin{aligned} G(gx_1^1, gx_2^1, gx_2^1) &= G(T_0(x_0^1, x_0^2), T_1(x_1^1, x_1^2), T_1(x_1^1, x_1^2)) \\ &\leq \beta_{1,2}[G(gx_0^1, T_0(x_0^1, x_0^2), T_0(x_0^1, x_0^2)) \\ &+G(gx_1^1, T_1(x_1^1, x_1^2), T_1(x_1^1, x_1^2))] + \gamma_{1,2}G(gx_0^1, gx_1^1, gx_1^1) \\ &= \beta_{1,2}[G(gx_0^1, gx_1^1, gx_1^1) + G(gx_1^1, gx_2^1, gx_2^1)] \\ &+\gamma_{1,2}G(gx_0^1, gx_1^1, gx_1^1). \end{aligned}$$

It follows that

$$G(gx_1^1, gx_2^1, gx_2^1) \leq \left(\frac{\beta_{1,2}+\gamma_{1,2}}{1-\beta_{1,2}}\right) G(gx_0^1, gx_1^1, gx_1^1). \tag{5}$$

Also, we get

$$\begin{aligned} G(gx_2^1, gx_3^1, gx_3^1) &= G(T_1(x_1^1, x_1^2), T_2(x_2^1, x_2^2), T_2(x_2^1, x_2^2)) \\ &\leq (\frac{\beta_{2,3}+\gamma_{2,3}}{1-\beta_{2,3}})G(gx_1^1, gx_2^1, gx_2^1) \\ &\leq (\frac{\beta_{2,3}+\gamma_{2,3}}{1-\beta_{2,3}})(\frac{\beta_{1,2}+\gamma_{1,2}}{1-\beta_{1,2}})G(gx_0^1, gx_1^1, gx_1^1). \end{aligned}$$

Similarly, we obtain

$$G(gx_r^1, gx_{r+1}^1, gx_{r+1}^1) \leq \prod_{m=1}^{r} \left(\frac{\beta_{m,m+1} + \gamma_{m,m+1}}{1 - \beta_{m,m+1}} \right) G(gx_0^1, gx_1^1, gx_1^1). \tag{6}$$

Using the same method as above, we can also show that

$$G(gx_r^2, gx_{r+1}^2, gx_{r+1}^2) \leq \prod_{m=1}^{r} \left(\frac{\beta_{m,m+1} + \gamma_{m,m+1}}{1 - \beta_{m,m+1}} \right) G(gx_0^2, gx_1^2, gx_1^2). \tag{7}$$

Adding (6) and (7), we get

$$\begin{aligned}
\delta_r &:= G(gx_r^1, gx_{r+1}^1, gx_{r+1}^1) + G(gx_r^2, gx_{r+1}^2, gx_{r+1}^2) \\
&= \sum_{i=1}^{2} G(gx_r^i, gx_{r+1}^i, gx_{r+1}^i) \\
&\leq \prod_{m=1}^{r} \left(\frac{\beta_{m,m+1} + \gamma_{m,m+1}}{1 - \beta_{m,m+1}} \right) \sum_{i=1}^{2} G(gx_0^i, gx_1^i, gx_1^i) \\
&= \prod_{m=1}^{r} \left(\frac{\beta_{m,m+1} + \gamma_{m,m+1}}{1 - \beta_{m,m+1}} \right) \delta_0.
\end{aligned}$$

Moreover, by repeated use of (G5) and for $p > 0$, we have

$$\begin{aligned}
\sum_{i=1}^{2} G(gx_r^i, gx_{r+p}^i, gx_{r+p}^i) &\leq \sum_{i=1}^{2} G(gx_r^i, gx_{r+1}^i, gx_{r+1}^i) \\
&+ \sum_{i=1}^{2} G(gx_{r+1}^i, gx_{r+2}^i, gx_{r+2}^i) \\
&+ \dots \\
&+ \sum_{i=1}^{2} G(gx_{r+p-1}^i, gx_{r+p}^i, gx_{r+p}^i) \\
&\leq \prod_{m=1}^{r} \left(\frac{\beta_{m,m+1} + \gamma_{m,m+1}}{1 - \beta_{m,m+1}} \right) \delta_0 \\
&+ \prod_{m=1}^{r+1} \left(\frac{\beta_{m,m+1} + \gamma_{m,m+1}}{1 - \beta_{m,m+1}} \right) \delta_0 \\
&+ \dots \\
&+ \prod_{m=1}^{r+p-1} \left(\frac{\beta_{m,m+1} + \gamma_{m,m+1}}{1 - \beta_{m,m+1}} \right) \delta_0
\end{aligned}$$

$$= \sum_{k=0}^{p-1} \prod_{m=1}^{r+k} \left(\frac{\beta_{m,m+1} + \gamma_{m,m+1}}{1 - \beta_{m,m+1}} \right) \delta_0$$

$$= \sum_{k=r}^{r+p-1} \prod_{m=1}^{k} \left(\frac{\beta_{m,m+1} + \gamma_{m,m+1}}{1 - \beta_{m,m+1}} \right) \delta_0.$$

Let α and n_α as in Definition 12, then, for $r \geq n_\alpha$ and using that the non-negative numbers geometric mean is less than or equal to the arithmetic mean, it follows that

$$\sum_{i=1}^{2} G(gx_r^i, gx_{r+p}^i, gx_{r+p}^i) \leq \sum_{k=r}^{r+p-1} \left[\frac{1}{k} \sum_{m=1}^{k} \left(\frac{\beta_{m,m+1} + \gamma_{m,m+1}}{1 - \beta_{m,m+1}} \right) \right]^k \delta_0$$

$$\leq \left(\sum_{k=r}^{r+p-1} \alpha^k \right) \delta_0$$

$$\leq \frac{\alpha^r}{1 - \alpha} \delta_0.$$

Now, letting $r \to +\infty$, we conclude that

$$\lim_{r \to \infty} \sum_{i=1}^{2} G(gx_r^i, gx_{r+p}^i, gx_{r+p}^i) = 0,$$

which implies that

$$\lim_{r \to \infty} G(gx_r^i, gx_{r+p}^i, gx_{r+p}^i) = 0,$$

Thus $\{gx_r^1\}, \{gx_r^2\}$ are Cauchy sequences in X. Since $g(X)$ is closed in a complete G-metric space, there exist $(x^1, x^2) \in X$, with $\lim_{r \to +\infty}\{gx_r^1\} = g(x^1) := x^1, \lim_{r \to +\infty}\{gx_r^2\} = g(x^2) := x^2$. By construction we have

$$\lim_{r \to \infty} g(x_{r+1}^1) = \lim_{r \to \infty} T_r(x_r^1, x_r^2) = x^1,$$

$$\lim_{r \to \infty} g(x_{r+1}^2) = \lim_{r \to \infty} T_r(x_r^2, x_r^1) = x^2.$$

Now, by w-reciprocally continuous and the compatibility of $\{T_m\}_{m \in \mathbf{N}_0}$ and g, we have

$$\lim_{r \to +\infty} T_r(g(x_r^1), g(x_r^2)) = g(x^1),$$

$$\lim_{r \to +\infty} T_r(g(x_r^2), g(x_r^1)) = g(x^2).$$

Now suppose that $\{T_m\}_{m \in \mathbf{N}}$ is continuous. Using triangle inequality we get

$$G(T_m(x^1, x^2), T_r(gx_r^1, gx_r^2), T_r(gx_r^1, gx_r^2)) \leq G[T_m(x^1, x^2), gT_r(x_r^1, x_r^2),$$
$$gT_r(x_r^1, x_r^2)] + G[gT_r(x_r^1, x_r^2), T_r(gx_r^1, gx_r^2), T_r(gx_r^1, gx_r^2)]$$

and

$$G(T_m(x^2, x^1), T_r(gx_r^2, gx_r^1), T_r(gx_r^2, gx_r^1)) \leq G[T_m(x^2, x^1), gT_r(x_r^2, x_r^1),$$

$$gT_r(x_r^2,x_r^1)] + G[gT_r(x_r^2,x_r^1), T_r(gx_r^2,gx_r^1), T_r(gx_r^2,gx_r^1)].$$

Now using continuity of $\{T_m\}_{m\in\mathbf{N}}$ and g, and taking the limit as $r\to\infty$ we get

$$G(T_m(x^1,x^2), gx^1, gx^1) = 0, \qquad G(T_m(x^2,x^1), gx^2, gx^2) = 0$$

i.e. $T_m(x^1,x^2) = gx^1$, and $T_m(x^2,x^1) = gx^2$. Thus, (x^1,x^2) is a coupled coincidence point of $\{T_m\}_{m\in\mathbf{N}}$ and g. Then the set of coupled coincidences is non-empty.
Now, we show that if (x^1,x^2) and (u^3,x^3) are coupled coincidence points, that is, if $gx^1 = T_m(x^1,x^2), gx^2 = T_m(x^2,x^1), gu^3 = T_m(u^3,x^3)$ and $gx^3 = T_m(x^3,u^3)$, then $gx^1 = gu^3$ and $gx^2 = gx^3$. Since the set of coupled coincidence points is comparable, applying condition (1), we get

$$G(gx^1,gu^3,gu^3) = G(T_m(x^1,x^2), T_{m'}(u^3,x^3), T_{m'}(u^3,x^3))$$

$$\le \beta_{m,m'}[G(gx^1, T_m(x^1,x^2), T_m(x^1,x^2))$$

$$+G(gu^3, T_{m'}(u^3,x^3), T_{m'}(u^3,x^3))]$$

$$+\gamma_{m,m'}G(gx^1,gu^3,gu^3)$$

and so as $\gamma_{m,m'} < 1$, it follows that $G(gx^1,gu^3,gu^3) = 0$, that is, $gx^1 = gu^3$. Similarly, it can be proved that $gx^2 = gx^3$. Hence, $\{T_m\}_{m\in\mathbf{N}_0}$ and g have a unique coupled point of coincidence (gx^1,gx^1), since two compatible mappings, commute at their coincidence points. Thus, clearly, $\{T_m\}_{m\in\mathbf{N}_0}$ and g have a unique coupled common fixed point whenever $\{T_m\}_{m\in\mathbf{N}_0}$ and g are w-compatible.

As a result of Theorem 1, if g is the identity mapping, then we state the following corollary.

Corollary 1 *Let $(X,G,\preceq)$ be poset G-metric space on X such that (X,G) is a complete G-metric space. Let $\{T_m\}_{m\in\mathbf{N}_0}$ be a sequence of mappings from X^2 into X, which $\{T_m\}_{m\in\mathbf{N}_0}$ and $Id : X \to X$ satisfying the (K) property. Also, T_0 and Id have mixed coincidence point. If $\sum_{m=1}^{+\infty}\left(\frac{\beta_{m,m+1}+\gamma_{m,m+1}}{1-\beta_{m,m+1}}\right)$ is an α-series and X is regular, then $\{T_m\}_{m\in\mathbf{N}_0}$ has a coupled fixed point, that is, there exists $(x^1,x^2)\in X^2$ such that*

$$x^1 = T_m(x^1,x^2), x^2 = T_m(x^2,x^1), \text{ for } m\in\mathbf{N}_0.$$

Theorem 2 *Let $(X,G,\preceq)$ be partially ordered G-metric space on X such that $(X,G,\preceq)$ is regular and let g and $\{T_m\}_{m\in\mathbf{N}_0}$ be as in the preceding theorem and $lim_{r\to+\infty}\sup\beta_{r,m} < 1$. Therefore, the conditions $(1)-(3)$ in theorem 1 hold.*

Proof According to Theorem 1, sequences $\{gx_r^1\}$ and $\{gx_r^2\}$ are Cauchy sequences in the complete G-metric space $(g(X), G)$. Since $\{gx_r^1\}$ and $\{gx_r^2\}$ are non-decreasing and non-increasing respectively, using the regularity of $(X, G, \preceq)$, we have $gx_r^1 \preceq x^1, x^2 \preceq gx_r^2$ for all $r \geq 0$. Then by (1), we obtain

$$G(T_r(gx_r^1, gx_r^2), T_m(x^1, x^2), T_m(x^1, x^2))$$

$$\leq \beta_{r,m}[G(ggx_r, T_r(gx_r^1, gx_r^2), T_r(gx_r^1, gx_r^2))$$

$$+G(gx^1, T_m(x^1, x^2), T_m(x^1, x^2))] + \gamma_{r,m}(G(ggx_r, gx^1, gx^1)).$$

Taking the limit as $r \to +\infty$, we obtain $T_m(x^1, x^2) = gx^1$ as $\beta_{r,m} < 1$. Similarly, it can be proved that $gx^2 = T_m(x^2, x^1)$. Thus, (x^1, x^2) is a coupled coincidence point of $\{T_m\}_{m \in \mathbf{N}}$ and g.

Example 1 Let $X = [0, 1]$ and

$$G(x^1, x^2, x^3) = \max\{|x^1 - x^2|, |x^2 - x^3|, |x^3 - x^1|\}.$$

It is clear that (X, G) is a complete G-metric space. Also define $\beta_{m,m'} = \dfrac{1}{2^{2m+1}}, \gamma_{m,m'} = \frac{1}{2^m}$ for all $m, m' = 1, 2, \ldots$. Consider the mapping $T_m : X^2 \to X$ and $g : X \to X$ with

$$T_m(x^1, x^2) = \frac{x^1 + x^2}{2^m}, \; g(x^1) = 6x^1$$

for all $x^1, x^2 \in X$, $m = 1, 2, \ldots$.

$$G(T_m(x^1, x^2), T_{m'}(u^1, u^2), T_{m'}(u^3, x^3)) = |\frac{x^1 + x^2}{2^m} - \frac{u^3 + x^3}{2^{m'}}|$$

and

$$G(gx^1, T_m(x^1, x^2), T_m(x^1, x^2)) + G(gx^2, T_{m'}(u^1, u^2), T_{m'}(u^3, x^3))$$

$$= |6x^1 - \frac{x^1 + x^2}{2^m}| + |6x^2 - \frac{u^3 + x^3}{2^{m'}}|,$$

$$G(gx^1, gx^2, gu^3) = 6|x^1 - u^3|$$

Then by mathematical induction condition (1) is satisfied for all $x^1, x^2, u^1, u^2 \in X$ with $gx^1 \preceq gu^1, gu^2 \preceq gx^2$ or $gx^1 \succeq gu^1, gu^2 \succeq gx^2$ Moreover, the series

$$\sum_{m=1}^{+\infty} \left(\frac{\beta_{m,m+1} + \gamma_{m,m+1}}{1 - \beta_{m,m+1}} \right) = \sum_{m=1}^{+\infty} \frac{2^{m+1} + 1}{2^{2m+1} - 1}$$

is an α-series with $\alpha = \frac{1}{2}$. Then $(0, 0)$ is a unique coupled fixed point for T_m and g.

References

1. H. Aydi, B. Damjanovic, B. Samet, W. Shatanawi, Coupled fixed point theorems for nonlinear contractions in partially ordered G-metric spaces, Math. Comput. Modell., 54, 2443–2450 (2011).
2. T.G. Bhaskar, V. Lakshmikantham, Fixed point theorems in partially ordered metric spaces and applications, Nonlinear Anal., 65, 1379–1393 (2006).
3. B. S. Choudhary, A. Kundu, A coupled coincidence point result in partially ordered metric spaces for compatible mappings, Nonlinear Anal., 73, 2524–2531 (2010).
4. S. Hadi Bonab, R. Abazari, A. Bagheri Vakilabad, Partially ordered cone metric spaces and coupled fixed point theorems via α-series, MACA, 1, 50–61 (2019).
5. S. Hadi Bonab, R. Abazari, A. Bagheri Vakilabad, H. Hosseinzadeh, Generalized metric spaces endowed with vector-valued metrics and matrix equations by tripled fixed point theorems, J. Inequal. Appl., 2020, 1–16 (2020).
6. H. Hosseinzadeh, Some Fixed Point Theorems in Generalized Metric Spaces Endowed with Vector-valued Metrics and Application in Linear and Nonlinear Matrix Equations, Sahand Commun. Math. Anal., 17, 37–53 (2020).
7. V. Lakshmikantham, L. Ciric, Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces, Nonlinear Anal., 70, 4341–4349 (2009).
8. Z. Mostafa, A new structure for generalized metric spaces with applications to fixed point theory, Ph.D. thesis, The University of Newcastle, Callaghan, Australia, (2005).
9. Z. Mostafa, B. Sims, A new approach to generalized metric spaces, J. Nonlinear Convex Anal., 7, 289–297 (2006).
10. RK. Vats, K. Tas, V. Sihag and A. Kumar, Triple fixed point theorems via α-series in partially ordered metric spaces, J. Inequal. Appl., 1–12 (2014).
11. V. Sihag, C. Vetro, RK. Vats, A fixed point theorem in G-metric spaces via α-series, Quaest. Math., 37, 1–6 (2014).
12. S.H. Rasouli, M.H. Malekshah, Coupled fixed point results for mappings without mixed monotone property in partially ordered G-metric spaces, J. Egyptian Math. Soc., 22, 471–475 (2014).