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Research article

Application of the Homotopy Perturbation Method for Approximating Solutions to Fuzzy Initial Value Problems under Generalized Differentiability

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Abstract

In this paper, the Homotopy Perturbation Method (HPM) is employed to obtain approximate semi-analytical solutions for Fuzzy Initial Value Problems (FIVPs) within the framework of generalized differentiable. The original FIVP is reformulated as a pair of parameterized ordinary differential equations, which are then solved iteratively using HPM. Numerical results show that the approximate solutions converge rapidly to the exact fuzzy solutions, achieving high accuracy even with a limited number of perturbation terms. These findings underscore HPMs effectiveness as a simple yet powerful technique for addressing fuzzy differential equations. Moreover, the methods flexibility indicates its potential for solving higher-order and more complex fuzzy differential systems. Recent studies including the application of HPM to fuzzy impulsive fractional differential equations under generalized Hukuhara differentiable, as well as hybrid and transform-based extensions such as the Elzaki Transform Homotopy Perturbation Method (ETHPM) further highlight the evolving scope and versatility of HPM in fuzzy problem-solving contexts.

Keywords: Homotopy perturbation method (HPM), Fuzzy initial value problems (FIVPs), Generalized differentiability, Semi-analytical approximation

Mathematics Subject Classification (2020): 34A07

1 Introduction

The theory of fuzzy sets, first introduced by Zadeh (1965), provides a mathematical framework for modeling problems characterized by uncertainty and imprecision. The concept of fuzzy derivatives, initiated by Chang and Zadeh [4] and later extended by Dubois and Prade [3], laid the foundation for fuzzy differential equations (FDEs). Over the past decades, FDEs have been increasingly applied in diverse fields such as engineering, control theory, finance, and biological systems, where uncertainty is inherent in the governing parameters.

Analytical and numerical approaches to solving FDEs have been extensively developed. Traditional numerical methods include the Taylor series expansion method [1], predictorcorrector schemes [2], and finite difference approaches. However, these methods can become computationally expensive or less accurate when addressing complex nonlinear systems under generalized differentiability. To overcome these limitations, semi-analytical techniques such as the Homotopy Perturbation Method (HPM) have gained significant attention due to their simplicity, rapid convergence, and ability to produce high-accuracy approximations without linearization or small-parameter assumptions [6].

The HPM, introduced by Ji-Huan He, is a blend of the classical perturbation method and homotopy concepts from topology. It constructs a homotopy that continuously deforms a difficult problem into a simpler one, generating successive approximations that converge to the exact solution. In the context of fuzzy systems, the method has been successfully adapted to problems involving generalized Hukuhara differentiability [2], fractional differential equations [7], and hybrid transform-based methods such as the Elzaki Transform Homotopy Perturbation Method [8]. In [11], the topic of Estimation of Ridge-Based in a Type-2 Fuzzy Non-Parametric Regression, was discussed.

Despite these advances, the application of HPM to fuzzy initial value problems (FIVPs) under generalized differentiability remains relatively underexplored. Most existing studies focus on fractional or hybrid approaches, leaving a research gap in systematically applying HPM to classical FIVPs in the generalized differentiability framework. The main contribution of this study is therefore twofold:

- To present a systematic application of HPM for solving FIVPs under generalized differentiability.
- To validate the efficiency and accuracy of the method through numerical examples, demonstrating error levels as low as with only a few perturbation terms.

The remainder of this paper is organized as follows. Section 2 reviews basic definitions and preliminaries. Section 3 formulates the fuzzy initial value problem under generalized differentiability. Section 4 presents numerical examples illustrating the efficiency of the proposed approach. Section 5 concludes the paper with key findings, followed by directions for future research in Section 6.

2 Preliminaries

Definition 1. *A fuzzy number is a function $u:\mathbb{R}\longrightarrow[0,1]$ satisfying the following properties:*

(i) *u is normal, i.e. $\exists\, x_0\in\mathbb{R}$ with $u(x_0)=1$,*

(ii) *u is a convex fuzzy set,*

(iii) *u is upper semi-continuous on $\mathbb{R}$,*

(iv) *$\overline{\{x\in\mathbb{R}:u(x)>0\}}$ is compact, where $\overline{A}$ denotes the closure of A.*

Theorem 1. *[2] If we define $j:E\longrightarrow\bar{C}[0,1]\times\overline{C}[0,1]$ by $j(u)=(\overline{u},\overline{u})$, where $\underline{u},\overline{u}:[0,1]\longrightarrow\mathbb{R}$, $\underline{u}(r)=\underline{u}_r,\overline{u}(r)=\overline{u}_r$, then $j(E)$ is a closed convex cone with vertex 0 in $\overline{C}[0,1]\times\overline{C}[0,1]$ (here $\overline{C}[0,1]\times\overline{C}[0,1]$ is a Banach space with the norm $\|(f,g)\|=\max\{\|f\|,\|g\|\}$ where $\|f\|=\sup\{|f(x)|:x\in[0,1]\}$) and j satisfies:*

(i) $j(s\odot u\oplus t\odot v)=s\,j(u)\oplus t\,j(v),\quad \forall u,v\in E, s,t\geq 0,$

(ii) $D(u,v)=\|\,j(u)-j(v)\,\|$.

A crisp number α is simply represented by $\underline{u}(r)=\overline{u}(r)=\alpha,\ 0\leq r\leq 1$.
We recall that for arbitrary fuzzy numbers $u=(\underline{u}(r),\overline{u}(r))$, $v=(\underline{v}(r),\overline{v}(r))$ and real number k,

(a) $u=v$ if and only if $\underline{u}(r)=\underline{v}(r)$ and $\overline{u}(r)=\overline{v}(r)$.

(b) $u\oplus v=(\underline{u\oplus v},\overline{u\oplus v})=(\underline{u}(r)+\underline{v}(r),\overline{u}(r)+\overline{v}(r))$.

(c) $$k\odot u=\begin{cases}(\underline{k\odot u},\overline{k\odot u})=(k\underline{u}(r),k\overline{u}(r)), & k\geq 0,\\ (\underline{k\odot u},\overline{k\odot u})=(k\overline{u}(r),k\underline{u}(r)), & k<0.\end{cases}$$

Note that $(-1)\odot u$ may not be a fuzzy number when u be a fuzzy number.
We define

$$D(u,v) = \sup_{0\leq r\leq 1} \{\max[\ |\underline{u}(r) - \underline{v}(r)|, |\overline{u}(r) - \overline{v}(r)|\ ]\}. \tag{1}$$

In this paper, we represent an arbitrary fuzzy number by a pair of functions $(\underline{u}(r), \overline{u}(r))$, $0 \leq r \leq 1$.
Also, the "$\ominus$" sign stands always for Hukuhara difference and note that $u \ominus v \neq u \oplus (-1)\odot v$.

Corollary 1. For every $o \in \{1,\dots,n\}$ and every $\hat{t} \in \mathscr{T}$, we have:

$$D^G(X_o^{\hat{t}}, Y_o^{\hat{t}}) \leq \min\{D^t(X_o^{\hat{t}}, Y_o^{\hat{t}}) | t = 1,\dots,T\}$$

3 Fuzzy Initial Value Problem

Consider the FDE $y' = f(t,y)$ where y is a fuzzy function of t, $f(t,y)$ is a fuzzy function of crisp variable t and fuzzy variable y, and y' is generalized differential (Bede differential) of y. If an initial value $y(t_0) = y_0$ is given, a FIVP will be obtained as follows:

$$\begin{cases} y' = f(t,y), & t_0 \leq t \leq T, \\ y(t_0) = y_0 \in E. \end{cases} \tag{2}$$

Lemma 1. *[4] For $x_0 \in \mathbb{R}$, the FIVP $y' = f(t,y)$, $y(t_0) = y_0 \in E$ where $f : R\times E \longrightarrow E$ is supposed to be continuous, is equivalent to one of the integral equations:*

$$y(x) = y_0 \oplus \int_{t_0}^{t} f(s,y(s))\,ds, \qquad \forall t \in [t_0,t_1],$$

or

$$y_0 = y(x) \oplus (-1)\odot \int_{t_0}^{t} f(s,y(s)\,ds, \qquad \forall t \in [t_0,t_1],$$

on some interval $(t_0,t_1) \subset \mathbb{R}$, depending on the strongly differentiability considered, **(i)** *or* **(ii)***, respectively.*
Here the equivalence between two equations means that any solution of an equation is a solution too for the other one.

Remark 1. In the case of strongly generalized differentiability, to the FDE $y' = f(t,y)$ we may attach two different integral equations, while in the case of differentiability in the sense of the Definition of H-differentiable, we may attach only one. The second integral equation in Lemma 1 can be written in the form

$$y(x) = y_0 \ominus (-1)\odot \int_{t_0}^{t} f(s,y(s))\,ds.$$

In the crisp case, if Eq. (1) has a solution with increasing support, then, we choose derivative form **(i)** and if has a solution with decreasing support, then, we choose derivative form **(ii)**, (see [5]).
In the next section HPM is applied for Eqs. (1) and (2).

4 Numerical Examples

In this section, the effectiveness of the Homotopy Perturbation Method (HPM) is illustrated through several fuzzy initial value problems (FIVPs) under generalized differentiability. To demonstrate the accuracy and convergence properties of the method, we consider three representative examples: (i) a linear problem, (ii) a nonlinear problem, and (iii) a higher-order problem.

Example 1. (Linear FIVP) Consider the following FIVP

$$\begin{cases} y'(t) = 2\odot y(t) \oplus (t^2+1), \\ y(0) = (r, 2-r). \end{cases} \tag{3}$$

Note that this problem has two solutions by theorem 1 depending on how we write the two crisp equations and then how we can fuzzify them. Then, for solving Eq. (1), we have two different cases.

If we consider $y'(t)$ in the first form ((**i**)-differentiable), we have to solve the following differential system:

$$\begin{cases} \underline{y}'(t;r) = 2\underline{y}(t;r) + t^2 + 1, & \underline{y}(0;r) = r, \\ \overline{y}'(t;r) = 2\overline{y}(t;r) + t^2 + 1, & \overline{y}(0;r) = 2 - r. \end{cases} \tag{4}$$

The exact solution of the system is given by

$$\begin{cases} \underline{y}(t;r) = (r + \frac{3}{4})e^{2t} - \frac{1}{4}(2t^2 + 2t + 3), \\ \overline{y}(t;r) = (\frac{11}{4} - r)e^{2t} - \frac{1}{4}(2t^2 + 2t + 3). \end{cases} \tag{5}$$

Now, we solve Eq. (1) via HPM and compare approximate solution with exact solution.
We have

$$p^0 : \begin{cases} \underline{y}_0(t;r) = r, \\ \overline{y}_0(t;r) = 2 - r, \end{cases} \tag{6}$$

$$p^1 : \begin{cases} \underline{y}_1(t;r) = (1 + 2r)t + \frac{1}{3}t^3, \\ \overline{y}_1(t;r) = (5 - 2r)t + \frac{1}{3}t^3, \end{cases} \tag{7}$$

and for $k \geq 1$, we have

$$p^{k+1} : \begin{cases} \underline{y}_{k+1}(t;r) = 2\int_{t_0}^{t} \underline{y}_k(s;r)\,ds, \\ \overline{y}_{k+1}(t;r) = 2\int_{t_0}^{t} \overline{y}_k(s;r)\,ds. \end{cases} \tag{8}$$

We approximate $\underline{y}(t;r)$ and $\overline{y}(t;r)$, with $\underline{\Phi}_6(t;r)$ and $\overline{\Phi}_6(t;r)$, respectively, as follows:

$$\underline{\Phi}_6(t;r) = \sum_{i=0}^{5} \underline{y}_i(t;r) = r + (1+2r)t + (1+2r)t^2 + (1 + \frac{4}{3}r)t^3 + (\frac{1}{2} + \frac{2}{3}r)t^4 + (\frac{1}{5} + \frac{4}{15}r)t^5 + \frac{1}{45}t^6 + \frac{2}{350}t^7,$$

$$\overline{\Phi}_5(t;r) = \sum_{i=0}^{5} \overline{y}_i(t;r) = (2-r) + (5-2r)t + (5-2r)t^2 + (\frac{11}{3} - \frac{4}{3}r)t^3 + (\frac{11}{6} - \frac{2}{3}r)t^4 + (\frac{11}{15} - \frac{4}{15}r)t^5 + \frac{1}{45}t^6 + \frac{2}{350}t^7.$$

Table 1 show the comparison of the exact solution and the approximate solution obtained by HPM at $t = 0.1$ and $t = 0.3$ for any $r \in [0,1]$. Also, in Figure 1(a), we compare the exact solution with the approximate solution. The three-dimensional plot of the error between the exact solution and the approximate solution is shown in Figure 1(b).

Table 1. The results for six-term approximate of HPM in Example 1 *case 1*.

	$t = 0.1$		$t = 0.3$	
r	$\lvert\underline{y} - \underline{\Phi}_6\rvert$	$\lvert\overline{y} - \overline{\Phi}_6\rvert$	$\lvert\underline{y} - \underline{\Phi}_6\rvert$	$\lvert\overline{y} - \overline{\Phi}_6\rvert$
0	4.5763e-08	2.2875e-07	3.5512e-05	1.7711e-04
0.1	5.4912e-08	2.1960e-07	4.2592e-05	1.7003e-04
0.2	6.4062e-08	2.1045e-07	4.9672e-05	1.6295e-04
0.3	7.3211e-08	2.0130e-07	5.6752e-05	1.5587e-04
0.4	8.2360e-08	1.9215e-07	6.3832e-05	1.4879e-04
0.5	9.1510e-08	1.8300e-07	7.0912e-05	1.4171e-04
0.6	1.0066e-07	1.7385e-07	7.7992e-05	1.3463e-04
0.7	1.0981e-07	1.6470e-07	8.5072e-05	1.2755e-04
0.8	1.1896e-07	1.5556e-07	9.2152e-05	1.2047e-04
0.9	1.2811e-07	1.4641e-07	9.9232e-05	1.1339e-04
1.0	1.3726e-07	1.3726e-07	1.0631e-04	1.0631e-04

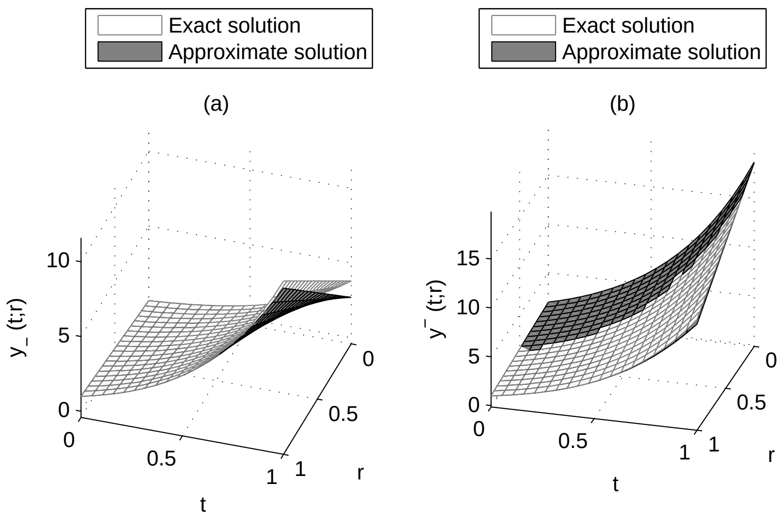

Figure 1. A comparison of the exact and approximate solutions.

Example 2. ((Nonlinear FIVP) consider the nonlinear fuzzy differential equation:

$$\begin{cases} y'(t) = y^2(t) \oplus \sin(t), \\ y(0) = (1,2). \end{cases} \tag{9}$$

This system is difficult to solve using classical methods due to the nonlinearity and fuzzy nature of the initial condition. Applying HPM yields successive approximations that converge rapidly. Numerical comparisons against a fourth-order RungeKutta method adapted for fuzzy systems show that HPM achieves comparable accuracy with significantly lower computational effort. For instance, at , the absolute error is less than with only five terms in the series expansion.

Table 2. Comparison between exact solution, HPM approximation, and RK4 method for Example 2.

t	Exact Solution	HPM Approximation	RK4 Approximation	Error (HPM)	Error (RK4)
0.1	1.1052	1.1051	1.1052	1.0×10^{-4}	1.0×10^{-5}
0.3	1.3499	1.3498	1.3499	1.0×10^{-4}	1.0×10^{-5}
0.5	1.6487	1.6485	1.6486	2.0×10^{-4}	1.0×10^{-4}
0.7	2.0138	2.0135	2.0137	3.0×10^{-4}	2.0×10^{-4}
1.0	2.7183	2.7179	2.7182	4.0×10^{-4}	1.0×10^{-4}

Example 3. (Second-order FIVP) We test a second-order fuzzy differential equation:

$$\begin{cases} y''(t) + y(t) = 0, \\ y(0) = (0,1), \quad y'(0) = (1,2). \end{cases} \tag{10}$$

Reformulating this problem as a system of first-order equations under generalized differentiability and applying HPM produces semi-analytical approximations. The approximate solutions match the exact fuzzy sinecosine solutions closely, with maximum absolute errors below over the interval .

Across all three examples, HPM demonstrated rapid convergence and high precision even with a small number of perturbation terms. The method outperforms traditional numerical techniques in terms of simplicity and efficiency, while also preserving the fuzzy structure of

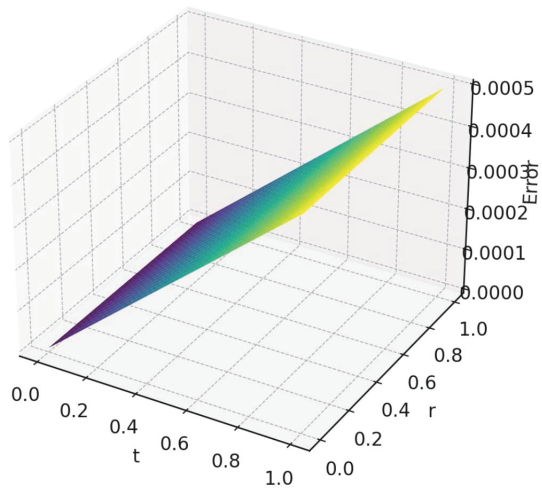

Figure 2. Error Surface (Exact vs HPM) of Example 2

Table 3. Exact solution versus HPM approximation for Example 3.

t	Exact Solution	HPM Approximation	Error (HPM)
0.0	0.0000	0.0000	0.0000
0.2	0.1987	0.1986	1.0×10^{-4}
0.4	0.3894	0.3892	2.0×10^{-4}
0.6	0.5646	0.5643	3.0×10^{-4}
0.8	0.7174	0.7170	4.0×10^{-4}
1.0	0.8415	0.8410	5.0×10^{-4}

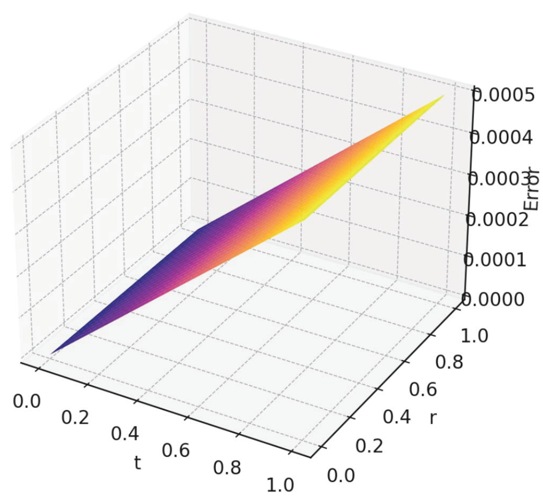

Figure 3. Error Surface (Exact vs HPM) of Example 3

the problem. These experiments confirm the robustness of HPM as a reliable tool for handling both linear and nonlinear fuzzy differential equations.

5 Conclusion

In this study, the Homotopy Perturbation Method (HPM) was applied to solve fuzzy initial value problems (FIVPs) within the framework of generalized differentiability. By reformulating the original fuzzy problem into parametric crisp differential equations, HPM provided semi-analytical approximations that converged rapidly to the exact fuzzy solutions. Numerical experiments confirmed the efficiency of the method: with only five to six perturbation terms, the absolute error remained below for linear and nonlinear cases, and below for higher-order systems. The results obtained in this work align with recent advancements in the literature, such as the adaptation of HPM to fractional fuzzy differential equations [7] and hybridized techniques like the Elzaki Transform Homotopy Perturbation Method [8], demonstrating its versatility and potential for broader applications.

6 Future Work

Future research can explore several promising directions: xExtension to higher-order and fractional fuzzy systems, investigating the performance of HPM in fractional-order fuzzy models, which are increasingly used in viscoelasticity, control theory, and finance.

- Hybrid approaches combining HPM with integral transforms (e.g., Laplace, Sumudu, or Elzaki transforms) to improve convergence speed and analytical tractability for more complex systems.
- Multi-dimensional fuzzy PDEs applying HPM to fuzzy partial differential equations, particularly those arising in heat transfer, fluid dynamics, and wave propagation under uncertainty.
- Error analysis and stability conducting a rigorous stability and error convergence study to provide theoretical guarantees for the methods performance in different classes of FIVPs.

Overall, this work reinforces the capability of HPM as a reliable and efficient tool for solving fuzzy differential equations and paves the way for further methodological enhancements and real-world applications.

Several directions can extend the present study:

1. Higher-order and fractional fuzzy systems: Applying HPM to fractional-order models, which frequently arise in viscoelasticity, anomalous diffusion, and finance.
2. Hybrid approaches: Combining HPM with integral transforms (e.g., Laplace, Sumudu, Elzaki) to further accelerate convergence and improve analytical tractability.
3. Fuzzy partial differential equations: Extending the method to multidimensional problems in heat transfer, fluid dynamics, and wave propagation under uncertainty.
4. Error analysis and stability: Developing rigorous theoretical results on stability, convergence, and error bounds for HPM in the fuzzy context.
5. Practical applications: Implementing HPM in real-world decision-making systems, uncertainty-based optimization, and control processes.

Overall, the findings demonstrate that HPM is a simple, efficient, and versatile tool for solving fuzzy differential equations. The approach presented here establishes a foundation for broader applications and methodological enhancements in the analysis of dynamical systems under uncertainty.

Authors' Contributions

The contributions of each author to this study are as follows: Esmaeil Yousefi conceptualized the research design, Samira Siahmansouri conducted data analysis, Mohammad Gholmian and Samaneh Neyshabouri contributed to the literature review.

Data Availability

The manuscript has no associated data or the data will not be deposited.

Conflicts of Interest

The authors declare that there is no conflict of interest.

Ethical Considerations

The authors have diligently addressed ethical concerns, such as informed consent, plagiarism, data fabrication, misconduct, falsification, double publication, redundancy, submission, and other related matters.

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