Analytical and Numerical Solutions for Nonlinear Equations
©Available online at https://ansne.du.ac.ir/
Online ISSN: 3060–785X
2024, Volume 9, Issue 2, pp. 176–183
Research article

Further Properties of Involutory and Idempotent Matrices

Mohammad Reza Ghaemi[1], Farzollah Mirzapour[1,*]

[1] Department of Mathematics and Computer Sciences, Faculty of Sciences, University of Zanjan, P.O. Box 45371–38791, Zanjan, Iran

* **Corresponding author(s):** f.mirza@znu.ac.ir

Received: 28/06/2025 **Revised:** 20/07/2025 **Accepted:** 23/08/2025 **Published:** 17/09/2025 doi 10.22128/ansne.2025.1036.1139

Abstract

In this paper, we investigate the real roots of a special class of square matrices, leveraging the properties of involutory and idempotent matrices. We focus on determining real roots for real orthogonal and symmetric matrices, demonstrating how involutory matrices facilitate this process. Our results show that a real involutory matrix of order n with a positive determinant always admits a real root. Furthermore, for real symmetric matrices, we establish that a real root exists if every negative eigenvalue appears with even multiplicity. We also explore the structure of idempotent matrices, presenting a general block form derived through similarity transformations. Specifically, we prove that for invertible submatrices A and D, along with arbitrary block matrices B and C, a constructed matrix P exhibits idempotency. An illustrative example is provided to clarify this construction, highlighting its application in generating idempotent and involutory matrices from simpler components. Additionally, we examine the root-approximability of orthogonal matrices, showing that certain sequences of matrices converge to the identity while their powers approximate the original matrix. This work extends existing results on matrix functions and diagonalization, offering practical insights into the analysis and computation of matrix roots. Our findings contribute to the broader understanding of matrix theory, with potential applications in numerical linear algebra and functional analysis.

Keywords: Matrix roots, Involutory matrices, Idempotent matrices, Symmetric matrices, Orthogonal matrices, Diagonalization

Mathematics Subject Classification (2020): 15A16, 47A60

1 Introduction and Preliminaries

Let $\mathscr{M}_n$ denote the C^*-algebra of all $n-$square matrices. We know that two matrices A and B are similar, if there exists an invertible matrix T such that $A = T^{-1}BT$, and A is diagonalizable if there exist $\lambda_1, \ldots, \lambda_n$ such that $A = T^{-1}\text{diag}(\lambda_1, \ldots, \lambda_n)T$ and unitarily diagonalizable if $T = U \in \mathscr{U}_n$, $A = U^*\text{diag}(\lambda_1, \ldots, \lambda_n)U$. Furthermore, matrix C is a root of A, if $A = C^2$, and it can also be said that A is root - approximable if there exists a sequence $\{C_k\}$ such that $C_k \longrightarrow I$ and $C_k^{2^k} = A$, for each $k = 0, 1, 2, \ldots$ [1, 2]. Matrix functions have been studied in [3–5]. In this paper, the matrix function $f(A) = \sqrt{A}$ for specific matrices is studied.
The square matrix A is said to be idempotent or a projection, if $A^2 = A$, and involutory if $A^2 = I$. In this article, we need the following propositions which are from [6–8].

Proposition 1. *Let A be an $n-$square complex matrix. Then*

1. *A is idempotent if and only if A is similar to a diagonal matrix of the form* $\mathrm{diag}(1,\ldots,1,0,\ldots,0)$.

2. *A is involutory if and only if A is similar to a diagonal matrix of the form* $\mathrm{diag}(1,\ldots,1,-1,\ldots,-1)$.

Proposition 2. *Let A and B be real square matrices of the same size. If P is a complex invertible matrix such that $P^{-1}AP = B$, then there exists a real invertible matrix Q such that $Q^{-1}AQ = B$.*

Proposition 3. *Let A and B be real square matrices of the same size. If $A = UBU^*$ for some unitary matrix U, then there exists a real orthogonal matrix Q such that $A = QBQ^T$.*

Proposition 4. *Every real orthogonal matrix is real orthogonally similar to a direct sum of real orthogonal matrices of order at most 2.*

2 Involution Matrices

In this section, we obtain some properties of the involutory matrices and by applying them we derive the real root of some special matrices. We start with matrices of order 2.

Lemma 1. *The class of all real involutory matrices of order 2 is as follows:*

$$\left\{\begin{pmatrix} a & b \\ \frac{1-a^2}{b} & -a \end{pmatrix};\ a,b\in\mathbb{R},\ b\neq 0\right\}\cup\left\{\begin{pmatrix} \pm 1 & 0 \\ c & \mp 1 \end{pmatrix};\ c\in\mathbb{R}\right\}\cup\{\pm I_2\}.$$

Lemma 2. *The class of all real matrices A such that $A^2 = -I_2$ is as follows:*

$$\left\{\begin{pmatrix} a & -b \\ \frac{1+a^2}{b} & -a \end{pmatrix};\ a,b\in\mathbb{R},\ b\neq 0\right\}.$$

Herein after we refer to $\begin{pmatrix} a & -b \\ \frac{1+a^2}{b} & -a \end{pmatrix}$ as $\Psi(a,b)$.

Remark 1. *In the Lemma (2.1), if $|a|\leq 1$ and $b=\sqrt{1-a^2}$, then*

$$\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix};\quad \theta\in\mathbb{R}.$$

Remark 2. $\begin{pmatrix} \pm 1 & 0 \\ 0 & \pm i \end{pmatrix}$ *are the only roots of* $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Theorem 1. *Suppose A is a real involutory matrix of order n and $\det A > 0$, then A has a real root.*

Proof. Since A is a real involutory matrix, then by Propositions 1 and 2, there is an invertible real matrix B such that

$$A = B^{-1}\mathrm{diag}(1,\ldots,1,-1,\ldots,-1)B,$$

thus $\det A = 1$ or $\det A = -1$. By assumption $\det A = 1$, then the number of eigenvalues -1 is even. Therefore

$$\begin{aligned} C &= \mathrm{diag}(1,\ldots,1,-1,\ldots,-1) \\ &= I_k \oplus \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \oplus\cdots\oplus \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \end{aligned}$$

has many real roots, for instance, for arbitrary real numbers $a_1,\ldots,a_t$ and non-zero real numbers $b_1,\ldots,b_t$, if

$$D = \mathrm{diag}(\pm 1,\cdots,\pm 1)\oplus\Psi(a_1,b_1)\oplus\cdots\oplus\Psi(a_t,b_t),$$

then $A = B^{-1}D^2B = (B^{-1}DB)^2$. □

Theorem 2. *Let A be a real symmetric matrix of size n with the eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$. if every negative eigenvalue is repeated twice or an even number of times, then A has a real root.*

Proof. Given A is real symmetric, then by Proposition 3 there is a real orthogonal matrix Q such that $A = Q^T \operatorname{diag}(\lambda_1, \ldots, \lambda_n)Q$. If $\lambda_n \geq 0$, we have nothing to prove. Now if

$$\lambda_1 \geq \cdots \geq \lambda_k \geq 0 > \lambda_{k+1} \geq \cdots \geq \lambda_n,$$

then by assumption, each λ_j , $k+1 \leq j \leq n$ is repeated twice or even times, therefore

$$C = \operatorname{diag}(\lambda_1, \cdots, \lambda_k) \oplus \begin{pmatrix} \lambda_{k+1} & 0 \\ 0 & \lambda_{k+1} \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} \lambda_n & 0 \\ 0 & \lambda_n \end{pmatrix}.$$

Thus for all real numbers $a_{k+1}, \ldots, a_n$ and non-zero real numbers $b_{k+1}, \ldots, b_n$,

$$D = \operatorname{diag}(\pm\sqrt{\lambda_1}, \cdots, \pm\sqrt{\lambda_k}) \oplus \sqrt{-\lambda_{k+1}}\Psi(a_{k+1}, b_{k+1}) \oplus \cdots \oplus \sqrt{-\lambda_n}\Psi(a_n, b_n),$$

are real roots of C and $A = (Q^T DQ)^2$. □

In this case, note that this matrix has no symmetric root.

If

$$R(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \text{ and } E = R(\frac{\pi}{2}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{1}$$

For the roots of orthogonal matrices, we need the following lemmas:

Lemma 3. *Let A and B be two matrices as follows:*

$$A = \begin{pmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{pmatrix}, \qquad B = \begin{pmatrix} \cos\beta & \sin\beta \\ \sin\beta & -\cos\beta \end{pmatrix}.$$

Then we have

$$(A \oplus B)^{1/2} = T_{\alpha\beta}(I_2 \oplus E)T_{\alpha\beta}^{-1},$$

where

$$T_{\alpha\beta} = \begin{pmatrix} \cos\frac{\alpha}{2} & 0 & -\sin\frac{\alpha}{2} & 0 \\ \sin\frac{\alpha}{2} & 0 & \cos\frac{\alpha}{2} & 0 \\ 0 & \cos\frac{\beta}{2} & 0 & -\sin\frac{\beta}{2} \\ 0 & \sin\frac{\beta}{2} & 0 & \cos\frac{\beta}{2} \end{pmatrix}.$$

Proof. Suppose that

$$S_{\alpha\beta} = \begin{pmatrix} \cos\frac{\alpha}{2} & -\sin\frac{\alpha}{2} \\ \sin\frac{\alpha}{2} & \cos\frac{\alpha}{2} \end{pmatrix} \oplus \begin{pmatrix} \cos\frac{\beta}{2} & -\sin\frac{\beta}{2} \\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{pmatrix}.$$

Then we have

$$\begin{aligned} A \oplus B &= S_{\alpha\beta} \left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \oplus \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right) S_{\alpha\beta}^{-1} \\ &= S_{\alpha\beta} I_{2,3} \left(I_2 \oplus -I_2\right) I_{2,3} S_{\alpha\beta}^{-1} = T_{\alpha\beta} \left(I_2 \oplus -I_2\right) T_{\alpha\beta}^{-1} \end{aligned}$$

where

$$I_{2,3} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \text{ and } T_{\alpha\beta} = S_{\alpha\beta} I_{2,3}. \tag{2}$$

This gives us

$$(A \oplus B)^{1/2} = T_{\alpha\beta}(I_2 \oplus E)T_{\alpha\beta}^{-1}.$$

□

Let

$$A = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \qquad B = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix},$$

then

$$A \odot B = \begin{pmatrix} a_1 & 0 & b_1 & 0 \\ 0 & a_2 & 0 & b_2 \\ c_1 & 0 & d_1 & 0 \\ 0 & c_2 & 0 & d_2 \end{pmatrix},$$

we see that $A \odot B = I_{2,3}(A \oplus B)I_{2,3}$ where $I_{2,3}$ is in (2). Let A and B be in Lemma 3, then $A \odot B$ has a real root, in general let

$$A_i = \begin{pmatrix} \cos\theta_i & \sin\theta_i \\ \sin\theta_i & -\cos\theta_i \end{pmatrix}, \quad i = 1, 2, \ldots, 2n, \tag{3}$$

we have $A_1 \odot A_2 \odot \cdots \odot A_{2n} = U(A_1 \oplus A_2 \oplus \cdots \oplus A_{2n})U^T$, where $U = (e_1, e_3, \ldots, e_{4n-1}, e_2, e_4, \ldots, e_{4n})^T$ [8]. Therefore we have the following corollary:

Corollary 1. *Let $A_i, i = 1, 2, \ldots, 2n$ be in (3), then $A_1 \odot A_2 \odot \cdots \odot A_{2n}$ has a real root.*

Lemma 4. *Let A be the matrix in Lemma 3. Then*

$$(-1 \oplus A)^{1/2} = V_\alpha(1 \oplus E)V_\alpha^{-1},$$

where

$$V_\alpha = \begin{pmatrix} 0 & 1 & 0 \\ \cos\frac{\alpha}{2} & 0 & -\sin\frac{\alpha}{2} \\ \sin\frac{\alpha}{2} & 0 & \cos\frac{\alpha}{2} \end{pmatrix}.$$

Proof. If

$$S_\alpha = \begin{pmatrix} \cos\frac{\alpha}{2} & -\sin\frac{\alpha}{2} \\ \sin\frac{\alpha}{2} & \cos\frac{\alpha}{2} \end{pmatrix},$$

then

$$\begin{aligned}
(-1) \oplus A &= (-1) \oplus S_\alpha \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} S_\alpha^{-1} \\
&= (1 \oplus S_\alpha)\left((-1) \oplus \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right)(1 \oplus S_\alpha^{-1}) \\
&= (1 \oplus S_\alpha)\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}(1 \oplus S_\alpha)^{-1} \\
&= V_\alpha(1 \oplus -I_2)V_\alpha^{-1},
\end{aligned}$$

we have $((-1) \oplus A)^{1/2} = V_\alpha(1 \oplus E)V_\alpha^{-1}$. □

According to proposition 4, for any real orthogonal matrix A there exist $\alpha_1,\dots,\alpha_s$ and $\beta_1\dots,\beta_t$ such that A is similar to matrix

$$\begin{aligned} C &= P^T A P \\ &= I_k \oplus -I_l \oplus R(\alpha_1) \oplus R(\alpha_2) \oplus \cdots \oplus R(\alpha_s) \oplus \begin{pmatrix} \cos\beta_1 & \sin\beta_1 \\ \sin\beta_1 & -\cos\beta_1 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} \cos\beta_t & \sin\beta_t \\ \sin\beta_t & -\cos\beta_t \end{pmatrix}, \end{aligned} \tag{4}$$

where s and t are non-negative integer numbers. In the following theorem, we will show the roots of orthogonal matrices.

Theorem 3. *Suppose A is a real orthogonal matrix with $\det(A) > 0$, then $A^{1/2}$ is real.*

Proof. Let C be in the (4) where l and t have the same parity (i.e., are both even or both odd), according to the last lemmas we have two cases:

(i) if l and t are even, then we have

$$\begin{aligned} C^{1/2} = I_k \oplus R(\frac{\pi}{2})\operatorname{diag}(\overbrace{I_2,\dots,I_2}^{l/2}) \oplus R(\frac{\alpha_1}{2}) \oplus R(\frac{\alpha_2}{2}) \oplus \cdots \oplus R(\frac{\alpha_s}{2}) \\ \oplus T_{\beta_1\beta_2}(I_2 \oplus E)\,T^{-1}_{\beta_1\beta_2} \oplus \cdots \oplus T_{\beta_{t-1}\beta_t}(I_2 \oplus E)\,T^{-1}_{\beta_{t-1}\beta_t}, \end{aligned}$$

(ii) If l and t are odd numbers, then

$$\begin{aligned} C^{1/2} = I_k \oplus R(\frac{\pi}{2})&\operatorname{diag}(\overbrace{I_2,\dots,I_2}^{\frac{l-1}{2}}) \\ &\oplus R(\frac{\alpha_1}{2}) \oplus R(\frac{\alpha_2}{2}) \oplus \cdots \oplus R(\frac{\alpha_s}{2}) \oplus V_{\beta_1}(1 \oplus E)V^{-1}_{\beta_1} \\ &\oplus T_{\beta_2\beta_3}(I_2 \oplus E)\,T^{-1}_{\beta_2\beta_3} \oplus \cdots \oplus T_{\beta_{t-1}\beta_t}(I_2 \oplus E)\,T^{-1}_{\beta_{t-1}\beta_t}. \end{aligned}$$

Both cases yield that $A^{1/2} = (PCP^T)^{1/2} = PC^{1/2}P^T$, where $A^{1/2}$ is real. □

Of course, there are many solutions to the real roots of orthogonal matrices, as stated in the remark below.

Remark 3. *For an even number l*

$$\begin{aligned} D = \operatorname{diag}(\pm 1,\dots,\pm 1) \oplus R(\frac{\pi}{2})\operatorname{diag}(\overbrace{\pm I_2,\dots,\pm I_2}^{l/2}) \\ \oplus \pm R(\frac{\alpha_1}{2}) \oplus \pm R(\frac{\alpha_2}{2}) \oplus \cdots \oplus \pm R(\frac{\alpha_s}{2}) \oplus \\ T_{\beta_1\beta_2}(\pm I_2 \oplus \Psi(a_1,b_1))\,T^{-1}_{\beta_1\beta_2} \oplus \cdots \oplus T_{\beta_{t-1}\beta_t}(\pm I_2 \oplus \Psi(a_t,b_t))\,T^{-1}_{\beta_{t-1}\beta_t}, \end{aligned}$$

and for an odd number l

$$\begin{aligned} D = \operatorname{diag}(\pm 1,\dots,\pm 1) \oplus R(\frac{\pi}{2})\operatorname{diag}(\overbrace{\pm I_2,\dots,\pm I_2}^{\frac{l-1}{2}}) \\ \oplus \pm R(\frac{\alpha_1}{2}) \oplus \pm R(\frac{\alpha_2}{2}) \oplus \cdots \oplus \pm R(\frac{\alpha_s}{2}) \oplus V_{\beta_1}(\exp(\pm 1 \oplus \Psi(a_0,b_0))V^{-1}_{\beta_1} \\ \oplus T_{\beta_2\beta_3}(\pm I_2 \oplus \Psi(a_1,b_1))\,T^{-1}_{\beta_2\beta_3} \oplus \cdots \oplus T_{\beta_{t-1}\beta_t}(\pm I_2 \oplus \Psi(a_t,b_t))\,T^{-1}_{\beta_{t-1}\beta_t}, \end{aligned}$$

where arbitrary real numbers $a_1,\dots,a_t$ and non-zero real numbers $b_1,\dots,b_t$. Then $A = (PDP^T)^2$.

Remark 4. *If*

$$\begin{aligned} D_k = I_r \oplus \begin{pmatrix} \cos\frac{\pi}{2^k} & \sin\frac{\pi}{2^k} \\ -\sin\frac{\pi}{2^k} & \cos\frac{\pi}{2^k} \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} \cos\frac{\pi}{2^k} & \sin\frac{\pi}{2^k} \\ -\sin\frac{\pi}{2^k} & \cos\frac{\pi}{2^k} \end{pmatrix} \\ \oplus \begin{pmatrix} \cos\frac{\alpha_1}{2^k} & \sin\frac{\alpha_1}{2^k} \\ -\sin\frac{\alpha_1}{2^k} & \cos\frac{\alpha_1}{2^k} \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} \cos\frac{\alpha_s}{2^k} & \sin\frac{\alpha_s}{2^k} \\ -\sin\frac{\alpha_s}{2^k} & \cos\frac{\alpha_s}{2^k} \end{pmatrix}, \end{aligned}$$

then $D_k \longrightarrow I$ and $(PD_kP^T)^{2^k} = A$, i.e. A is root-approximable.

Remark 5. *With the above given if*

$$P^TAP = I_r \oplus -I_l \oplus \begin{pmatrix} \cos\beta_1 & \sin\beta_1 \\ \sin\beta_1 & -\cos\beta_1 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} \cos\beta_t & \sin\beta_t \\ \sin\beta_t & -\cos\beta_t \end{pmatrix},$$

then A is an involutory matrix.

3 Idempotent Matrices

By Proposition 1, if P is an idempotent matrix, then it is similar to $\begin{pmatrix} I & O \\ O & O \end{pmatrix}$ where I is an identity matrix, i.e. there are matrices A,B,C and D such that A and D are square, A and I are of the same size, then $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is invertible [9, 10] and

$$P = M \begin{pmatrix} I & O \\ O & O \end{pmatrix} M^{-1}.$$

If $M^{-1} = \begin{pmatrix} X & Y \\ U & V \end{pmatrix}$ and A is invertible, then we have

$$X = A^{-1} + A^{-1}B(D-CA^{-1}B)^{-1}CA^{-1}, \quad Y = -A^{-1}B(D-CA^{-1}B)^{-1},$$
$$U = -(D-CA^{-1}B)^{-1}CA^{-1}, \quad V = (D-CA^{-1}B)^{-1},$$

$$P = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} I & O \\ O & O \end{pmatrix}\begin{pmatrix} X & Y \\ U & V \end{pmatrix} = \begin{pmatrix} AX & AY \\ CX & CY \end{pmatrix}$$
$$= \begin{pmatrix} I + B(D-CA^{-1}B)^{-1}CA^{-1} & -B(D-CA^{-1}B)^{-1} \\ CA^{-1} + CA^{-1}B(D-CA^{-1}B)^{-1}CA^{-1} & -CA^{-1}B(D-CA^{-1}B)^{-1} \end{pmatrix},$$

and if D is invertible, we have

$$A^{-1} + A^{-1}B(D-CA^{-1}B)^{-1}CA^{-1} = (A-BD^{-1}C)^{-1}.$$

Consider $S = (D-CA^{-1}B)^{-1}$ and $T = (A-BD^{-1}C)^{-1}$. We get

$$P = \begin{pmatrix} AT & -BS \\ CT & -CA^{-1}BS \end{pmatrix} = \begin{pmatrix} A & O \\ C & O \end{pmatrix}\begin{pmatrix} T & O \\ O & O \end{pmatrix} - \begin{pmatrix} O & B \\ O & CA^{-1}B \end{pmatrix}\begin{pmatrix} O & O \\ O & S \end{pmatrix}.$$

Thus we have proved the following theorem:

Theorem 4. *Let A and D be two invertible matrices of orders n and m respectively, while B and C are two matrices of orders $n \times m$ and $m \times n$ respectively. Then*

$$P = \begin{pmatrix} A(A-BD^{-1}C)^{-1} & -B(D-CA^{-1}B)^{-1} \\ C(A-BD^{-1}C)^{-1} & -CA^{-1}B(D-CA^{-1}B)^{-1} \end{pmatrix}$$

is an idempotent.

Example 1. *Let a,b,c and d be real numbers, with $bc \neq ad \neq 0$, and*

$$A = aI_n, B = b\begin{pmatrix} I_m \\ O \end{pmatrix}, C = c\begin{pmatrix} I_m & O \end{pmatrix}, D = dI_m, \quad (n \geq m),$$

$$M = \begin{pmatrix} aI_n & b\begin{pmatrix} I_m \\ O \end{pmatrix} \\ c\begin{pmatrix} I_m & O \end{pmatrix} & dI_m \end{pmatrix},$$

$$S = \frac{a}{ad-bc} I_m, \quad T = \begin{pmatrix} \frac{d}{ad-bc} I_m & O \\ O & \frac{1}{a} I_{n-m} \end{pmatrix},$$

therefore

$$P = \frac{1}{ad-bc} \begin{pmatrix} \begin{pmatrix} adI_m & O \\ O & (ad-bc)I_{n-m} \end{pmatrix} & \begin{pmatrix} -abI_m \\ O \end{pmatrix} \\ \begin{pmatrix} cdI_m & O \end{pmatrix} & -bcI_m \end{pmatrix}$$

is idempotent and

$$T = 2P - I = \frac{2}{ad-bc} \begin{pmatrix} \begin{pmatrix} \frac{ad+bc}{2} I_m & O \\ O & \frac{ad-bc}{2} I_{n-m} \end{pmatrix} & \begin{pmatrix} -abI_m \\ O \end{pmatrix} \\ \begin{pmatrix} cdI_m & O \end{pmatrix} & -\frac{ad+bc}{2} I_m \end{pmatrix}$$

is an involutory matrix.

Authors' Contributions

All authors have the same contribution.

Data Availability

The manuscript has no associated data or the data will not be deposited.

Conflicts of Interest

The authors declare that there is no conflict of interest.

Ethical Considerations

The authors have diligently addressed ethical concerns, such as informed consent, plagiarism, data fabrication, misconduct, falsification, double publication, redundancy, submission, and other related matters.

Funding

This research did not receive any grant from funding agencies in the public, commercial, or nonprofit sectors.

References

[1] A. Chademan and F. Mirzapour, Midconvex functions in locally compact groups, Proc. Amer. Math. Soc., 127(10), 2961–2968 (1999).

[2] F. Mirzapour, On Semi-E-Convex and Quasi-Semi-E-Convex Functions, Hacettepe Journal of Mathematics and Statistics, 41(6), 841–845 (2012).

[3] R. Bhatia, Matrix Analysis, Springer-Verlag, Inc., New York, (1973).

[4] R. A. Horn and C. R. Johnson, Topics in Matrix Analysis, Cambridge University Press, (1991).

[5] N. J. Higham, Functions of Matrices, theory and Computation, SIAM, first edition, (2008).

[6] N. J. Higham, Computing real square roots of a real matrix, Linear Algebra and its Applications, 88, 405–430, (1987).

[7] F. Zhang, Matrix Theory, Springer-Verlag, (2011).

[8] S. Friedland and P. J. Freitas, Revisiting the Siegel upper half plane I Linear Algebra and its Applications 376, 27–39, (2004).

[9] R. C. Thompson, The characteristic polynomial of a principal subpencil of a Hermitian matrix pencil, Linear Algebra Appl., 14, 136–177, (1976).

[10] R. C. Thompson, Pencils of complex and real symmetric and skew symmetric matrices, Linear Algebra Appl., 147, 323–371, (1991).