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Research article

Quadrature Rule Extended Spline Method for Nonlinear and Linear Volterra Integral Equations

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Abstract

In this research, we consider the linear and nonlinear Volterra integral equations (VIEs). The main aims of research is to approximate the integral by Gauss-Tur*án* quadrature rule and then using extended cubic B-spline as the bases function. The unknown coefficients in combination determine by collocation method. The arising system of linear and nonlinear can be solved via iterative method. Error analysis is investigated theoretically. Numerical text problems are considered to justify the applicability and efficient nature of our approach, comparison of the results justify the considerable accuracy and efficiency proposed methods. The extended parameter in valued in the spline can be chosen in such a way to improve the accuracy also.

Keywords: Linear and nonlinear VIEs, Extended cubic spline, Gauss-Tur*án* quadrature rule, Error analysis

Mathematics Subject Classification (2020): 45D05, 45G15, 65D05, 65R20

1 Introduction

It is well known that VIEs are of the form

$$U(\xi) = g(\xi) + \int_{\alpha}^{\xi} K(\xi, y, U(y))dy, \ \xi \in [\alpha, \xi_f], \tag{1}$$

where $K(\xi, y, U(y))$ is continuous on $[\alpha, \xi_f]$ and satisfies a uniform Lipschitz condition. VIE(1) arise in varies filed of science and dynamics such as spread of epidemics, and semi-conductor devices [1–4]. For solution of the VIE (1), several numerical approaches have been proposed such as, the Homotopy-perturbation method [5–8], the wavelet basis [9–12], the collocation method basis [13, 14], the converting to optimization problem [15], an approach based on Lipschitz-continuity [16], the collocation iterated method and their discretizations [17, 18], the Taylor-series expansion methods [19], the Newton-Kantorovich-quadrature method [20], the Tau approximation [21], the trapezoidal quadrature rule [22], the Fibonacci polynomials [23], the improved cuckoo optimization algorithm [24], the natural Runge-Kutta methods [25], the Bernstein polynomial [26, 27], the quadrature approach based on B-spline [28]. We develop a collocation by using extended cubic spline to approximate in VIE(1).

2 Extended Cubic Spline Collocation Approach

Extended cubic spline is an extension of spline [29]. One free parameter, θ, is introduced within the basis function where this parameter can be used to alter the shape of the generated curve. The value of θ can be varied to obtain different numerical results. In this study, this value is optimized to produce approximate solutions with the least error.

2.1 Extended Cubic Spline

We apply extended cubic spline collocation method to approximate solution of VIE(1). Let $\Delta_M : \{\alpha = y_0 < y_1 < \ldots < y_M = \beta\}$ be a uniform partition of the interval $[\alpha,\beta]$ with step size $h = \frac{\beta-\alpha}{M}$. The extended cubic spline $B_r(y,\theta)$ is defined as:

$$B_r(y,\theta) = \frac{1}{24h^4}\begin{cases} 4k(1-\theta)(y-y_{r-2})^3+3\theta(y-y_{r-2})^4, & y_{r-2}<y\leq y_{r-1} \\ (4-\theta)k^4+12k^3(y-y_{r-1})+6k^2(2+\theta)(y-y_{r-1})^2 & \\ -12k(y-y_{r-1})^3-3\theta(y-y_{r-1})^4, & y_{r-1}<y\leq y_r \\ (4-\theta)k^4+12k^3(y_{r+1}-y)+6k^2(2+\theta)(y_{r+1}-y)^2 & \\ -12k(y_{r+1}-y)^3-3\theta(y_{r+1}-y)^4, & y_r<y\leq y_{r+1} \\ 4k(1-\theta)(y_{r+2}-y)^3+3\theta(y_{r+2}-y)^4, & y_{r+1}<y\leq y_{r+2} \\ 0 & otherwise. \end{cases} \tag{2}$$

The extended cubic spline function has one arbitrary parameter θ,when θ tends to zero the extended cubic spline reduced to convectional cubic spline function. For $\theta \geqslant -2$, spline and extended spline share the same properties: local support, non-negativity, partition of unity and C^2 continuity. The parameter θ control the tension of the solution curve [30,31]. we consider a extended cubic spline $S(y)$ of the form [32]

$$S(y) = \sum_{r=-1}^{M+1} t_r B_r(y,\theta) = \frac{B_{-1}(y,\theta)}{B_{-1}(y_0,\theta)}W_0 + \frac{B_{M+1}(y,\theta)}{B_{M+1}(y_M,\theta)}W_1 + \sum_{r=0}^{M} t_r \overline{B}_r(y,\theta), \tag{3}$$

where $W_0 = U(\alpha)$, $W_1 = U(\beta)$ and the functions $\overline{B}_r(y,\theta)$ as follows:

$$\begin{aligned} &\overline{B}_r(y,\theta) = B_r(y,\theta) - \tfrac{B_r(y_0,\theta)}{B_{-1}(y_0,\theta)}B_{-1}(y,\theta),\ r=0,1, \\ &\overline{B}_r(y,\theta) = B_r(y,\theta),\ r=2,\ldots,M-2, \\ &\overline{B}_r(y,\theta) = B_r(y,\theta) - \tfrac{B_r(y_M,\theta)}{B_{M+1}(y_M,\theta)}B_{M+1}(y,\theta),\ r=M-1,M. \end{aligned} \tag{4}$$

$\overline{B}_r(y,\theta),\ r=0,\ldots,M$ is as the new set of redefined extended cubic spline functions which vanish on the Dirichlet's boundary conditions.

3 On Quadrature Rules of Gauss-Turán

Let P_n be the set of all algebraic polynomials of degree at most n. The Gauss-Turán quadrature rule in [33,34] is

$$\int_\alpha^\beta g(y)d\chi(y) = \sum_{\tau=1}^{m}\sum_{r=0}^{2p}\varphi_{\tau,r}g^{(r)}(\nu_\tau) + E_{m,2p+1}(g), \tag{5}$$

where $m \in M,\ p \in M_0$ and $d\chi(y)$ is a nonnegative measure on the interval (α,β) which can be the real axis E, with compact or infinite support for which all moments:

$$\eta_\kappa = \int_\alpha^\beta y^\kappa d\chi(y),\quad \kappa = 0,1,\ldots, \tag{6}$$

exists, are finite, more over $\eta_0 > 0$, and

$$\varphi_{\tau,r} = \int_\alpha^\beta \gamma_{\tau,r}(y)d\chi(y),\quad r=0,\ldots,2p,\quad \tau=1,\ldots,m,$$

and $\gamma_{\tau,r}(y)$ are the fundamental polynomials of Hermite interpolation. The nodes ν_τ $(\tau = 1,\ldots,m)$ in Eq. (5) are the zeros of monic polynomial $\psi_m(y) = y^m + b_{m-1}y^{m-1} + \cdots + b_1x + b_0$ which minimizes the integral.

$$G(b_0,b_1,\ldots,b_{m-1}) = \int_\alpha^\beta [\psi_m(y)]^{2p+2}d\chi(y), \tag{7}$$

then the rule Eq. (5) is exact for all polynomials of degree at most $2(p+1)m-1$, that is, $E_{m,2p+1}(g)=0, \forall g \in P_{2(p+1)m-1}$. The condition Eq.(7) is equivalent with the following conditions:

$$\int_{\alpha}^{\beta}[\psi_m(y)]^{2p+1}y^{\kappa}d\chi(x)=0, \qquad \kappa=0,\ldots,m-1, \tag{8}$$

$$\int_{-1}^{1}[\psi_4(y)]^{5}y^{\kappa}dy=0, \qquad \kappa=0,1,2,3, \tag{9}$$

where $\psi_4(y)=y^4+b_3y^3+b_2y^2+b_1y+b_0$, by solving system Eq. (9) we can obtain $b_r,(r=0,1,2,3,4)$ coefficients, on the other hand we have

$$\psi_{\tau+1}(y)=(y-\rho_\tau)\psi_\tau(yx)-\delta_\tau\psi_{\tau-1}(y), \qquad \tau=0,1,2,3,$$
$$\psi_{-1}(y)=0, \qquad \psi_0(y)=1,$$

where

$$\rho_\tau=\rho_\tau(2,4)=\frac{(y\psi_\tau,\psi_\tau)}{(\psi_\tau,\psi_\tau)}=\frac{\int_{-1}^{1}y\psi_\tau^2(y)\psi_m^{2p}(y)dy}{\int_{-1}^{1}\psi_\tau^2(y)\psi_m^{2p}(y)dy}=\frac{\int_{-1}^{1}y\psi_\tau^2(y)\psi_4^4(y)dy}{\int_{-1}^{1}\psi_\tau^2(y)\psi_4^4(y)dy},$$
$$\delta_\tau=\delta_\tau(2,4)=\frac{(\psi_\tau,\psi_\tau)}{(\psi_{\tau-1},\psi_{\tau-1})}=\frac{\int_{-1}^{1}\psi_\tau^2(y)\psi_m^{2p}(y)dy}{\int_{-1}^{1}\psi_{\tau-1}^2(y)\psi_m^{2p}(y)dy}=\frac{\int_{-1}^{1}\psi_\tau^2(y)\psi_4^4(y)dy}{\int_{-1}^{1}\psi_{\tau-1}^2(y)\psi_4^4(y)dy},$$
$$\delta_0=\int_{-1}^{1}\psi_4^4(y)dy,$$

so that we can obtain the zeros of monic polynomial $\psi_4^{2,4}(y)$ of eigenvalue Jocobian matrix

$$J_4=\begin{bmatrix}\rho_0 & \sqrt{\delta_1} & & \\ \sqrt{\delta_1} & \rho_1 & \sqrt{\delta_2} & \\ & \sqrt{\delta_2} & \rho_2 & \sqrt{\delta_3} \\ & & \sqrt{\delta_3} & \rho_3\end{bmatrix},$$

and the values of ν_τ,ρ_τ and δ_τ which are tabulated in Table 1.

Table 1. Determined values of ν_τ,β_τ and δ_τ.

ν	ν_τ	β_τ	δ_τ
0	-0.899829212560986	0	0.132703088805391(-03)[1]
1	-0.365924354691640	0	0.424102581549750
2	0.365924354691679	0	0.263848849055045
3	0.899829212650986	0	0.255641814691793

Finally to determine $\varphi_{\tau,r}$, we use the following polynomial for approximation of function $g(y)$,

$$g_{\kappa,\tau}(y)=(y-\nu_\tau)^{\kappa}\Phi_\tau(y)=(y-\nu_\tau)^{\kappa}\prod_{r\neq\tau}(y-\nu_r)^{2p+1}, \tag{10}$$

where $0\leq\kappa\leq 2w,\ 1\leq\tau\leq m$ and

$$\Phi_\tau(y)=(\frac{\psi_m(y)}{y-\nu_\tau})^{2p+1}=\prod_{r\neq\tau}(y-\nu_r)^{2p+1},\tau=1,\ldots,m,$$

since Eq. (5) is exact for all polynomials of degree at most $2(p+1)m-1$ then accuracy degree $g_{\kappa,\tau}$ is

$$degg_{\kappa,\tau}=(m-1)(2p+1)+\kappa\leq(2p+1)m-1.$$

[1] $0.132703088805391(-03)=0.132703088805391\times10^{-03}$

Then Eq. (5) is exact for polynomials Eq. (10), that is, $E(g_{\kappa,\tau}) = 0$, $(0 \leq \kappa \leq 2p, 1 \leq \tau \leq m)$ then by replacing $g_{\kappa,\tau}(y)$ instead of $g(y)$ in Eq. (5) we have

$$\sum_{l=1}^{m} \sum_{r=0}^{2p} \varphi_{l,r} g_{\kappa,\tau}^{(r)}(\nu_l) = \int_{\alpha}^{\beta} g_{\kappa,\tau}(y) d\chi(y) = \eta_{\kappa,\tau},$$

therefore for each $\tau = l$, we get the following linear system $(2p+1) \times (2p+1)$, where $\varphi_{\tau,r}$ are unknowns $r = 0, \ldots, 2w$, $\tau = 1, \ldots, m$,

$$\begin{bmatrix} g_{0,\tau}(\nu_\tau) & g'_{0,\tau}(\nu_\tau) & \ldots & g_{0,\tau}^{(2p)}(\nu_\tau) \\ & g'_{1,\tau}(\nu_\tau) & \ldots & g_{1,\tau}^{(2p)}(\nu_\tau) \\ & & \ddots & \\ & & & g_{2p,\tau}^{(2p)}(\nu_\tau) \end{bmatrix} \begin{bmatrix} \varphi_{\tau,0} \\ \varphi_{\tau,1} \\ \vdots \\ \varphi_{\tau,2p} \end{bmatrix} = \begin{bmatrix} \eta_{0,\tau} \\ \eta_{1,\tau} \\ \vdots \\ \eta_{2p,\tau} \end{bmatrix},$$

solving the above system for $p = 2$ and $\tau = 1, 2, 3, 4$, we obtain the values of $\varphi_{\tau,r}$, $\tau = 1, \ldots, 4$, $r = 0, \ldots, 4$, which are tabulated in Table 2.

Table 2. Determined values of $\varphi_{\tau,r}$, $\tau = 1, \ldots, 4$, $r = 0, \ldots, 4$.

$\varphi_{1,0} = 0.315604206062624$	$\varphi_{1,2} = 0.001213976533015$
$\varphi_{2,0} = 0.684395793937405$	$\varphi_{2,2} = 0.0104801638359508$
$\varphi_{3,0} = 0.684395793937377$	$\varphi_{3,2} = 0.010480163835949$
$\varphi_{4,0} = 0.315604206062603$	$\varphi_{4,2} = 0.00121397653301490$
$\varphi_{1,1} = 0.0151791927277847$	$\varphi_{1,3} = 2.67403743470878 \times 10^{-5}$
$\varphi_{2,1} = 0.013556093515529$	$\varphi_{2,3} = 0.0001128025099388$
$\varphi_{3,1} = -0.135560935155336 * 10^{-1}$	$\varphi_{3,3} = -0.11280250993880 \times 10^{-3}$
$\varphi_{4,1} = -0.151791927277821 \times 10^{-1}$	$\varphi_{4,3} = -0.267403743470821 \times 10^{-4}$
$\varphi_{1,4} = 5.42643518348675 \times 10^{-7}$	$\varphi_{2,4} = 0.00002636423549605$
$\varphi_{3,4} = 0.000026364235496$	$\varphi_{4,4} = 5.42643518348595 \times 10^{-7}$

4 Nonlinear Volterra Integral Equation

In the given nonlinear VIE(1), we can approximate the unknown function by extended cubic spline Eq. (3), we have:

$$S(\xi) = g(\xi) + \int_{\alpha}^{\xi} K(\xi, y, S(y)) dy. \tag{11}$$

Now collocated Eq. (11) for a fixted t in $\alpha \leq \xi \leq \xi_f$ at the points $\xi_r = \alpha + rh$, $h = \frac{\xi_f - \alpha}{M}$, $r = 0, 1, \ldots, M$, we obtain

$$\begin{aligned} &\int_{\alpha}^{\xi_r} K(\xi_r, y, (\tfrac{B_{-1}(y,\theta)}{B_{-1}(y_0,\theta)} W_0 + \tfrac{B_{M+1}(y,\theta)}{B_{M+1}(y_M,\theta)} W_1 + \sum_{r=0}^{M} t_r \overline{B}_r(y,\theta))) dy + g(\xi_r) \\ &= \tfrac{B_{-1}(\xi_r,\theta)}{B_{-1}(\xi_0,\theta)} W_0 + \tfrac{B_{M+1}(\xi_r,\theta)}{B_{M+1}(\xi_M,\theta)} W_1 + \sum_{r=0}^{M} t_r \overline{B}_r(\xi_r,\theta), \quad r = 0, 1, \ldots, M. \end{aligned} \tag{12}$$

By partitioning the interval $[\alpha, \xi_f]$ to M equal subintervals we obtain

$$\begin{aligned} &\sum_{j=0}^{r-1} \int_{\xi_j}^{\xi_{j+1}} K(\xi_r, y, (\tfrac{B_{-1}(y,\theta)}{B_{-1}(y_0,\theta)} W_0 + \tfrac{B_{M+1}(y,\theta)}{B_{M+1}(y_M,\theta)} W_1 + \sum_{r=0}^{M} t_r \overline{B}_r(y,\theta))) dy + g(\xi_r) \\ &= \tfrac{B_{-1}(\xi_r,\theta)}{B_{-1}(\xi_0,\theta)} W_0 + \tfrac{B_{M+1}(\xi_r,\theta)}{B_{M+1}(\xi_M,\theta)} W_1 + \sum_{r=0}^{M} t_r \overline{B}_r(\xi_r,\theta), \; r = 0, 1, \ldots, M. \end{aligned} \tag{13}$$

For using the Gauss-Turán rule we need to change each subinterval $[\xi_j, \xi_{j+1}]$ to the interval $[-1, 1]$. Then by the following change of variable, we have

$$y = \frac{1}{2}[(\xi_{j+1} - \xi_j)u + (\xi_{j+1} - \xi_j)], \;\; dy = \frac{\xi_{j+1} - \xi_j}{2} du = \frac{h}{2} du.$$

To approximate the integral Eq. (13), we can use the Gauss-Turán quadrature rule in the case $m=4$ and $w=2$, then we get the following $(M+1)\times(M+1)$, nonlinear system

$$\frac{B_{-1}(\xi_r,\theta)}{B_{-1}(\xi_0,\theta)}W_0+\frac{B_{M+1}(\xi_r,\theta)}{B_{M+1}(\xi_M,\theta)}W_1+\sum_{r=0}^{M}t_r\overline{B}_r(\xi_r,\theta)=\frac{h}{2}\sum_{j=0}^{r-1}\sum_{\tau=1}^{4}\sum_{\gamma=0}^{4}\varphi_{\tau,\gamma}$$
$$\times(K(\xi_r,\zeta_{j\tau},(\frac{B_{-1}(\zeta_{j\tau},\theta)}{B_{-1}(\xi_0,\theta)}W_0+\frac{B_{M+1}(\zeta_{j\tau},\theta)}{B_{M+1}(\xi_M,\theta)}W_1+\sum_{r=0}^{M}t_r\overline{B}_r(\zeta_{j\tau},\theta))))^{(\gamma)}+g(\xi_r),\quad r=0,1,\ldots,M, \tag{14}$$

where $\zeta_{j\tau}=\frac{(\xi_{j+1}-\xi_j)\nu_\tau+(\xi_{j+1}+\xi_j)}{2}$ and ν_τ we have the nodes and coefficients $\varphi_{\tau,\gamma}$ of previous section. By solving the above nonlinear system via iterative method we determine the coefficients $t_r, r=0,\ldots,M$ by setting t_r in Eq. (3), we obtain the approximate solution for VIE(1).

5 Error Analysis

To obtain the error estimation of our approach, the first of all we recall the following definition and Theorem in [33–35].

Definition 1. *The Gauss-Turán quadrature rule with multiple nodes,*

$$\int_\alpha^\beta g(y)\chi(y)dy=\sum_{\tau=1}^{m}\sum_{r=0}^{2p}\varphi_{\tau,r}g^{(r)}(\nu_\tau)+E_{m,2p+1}(g), \tag{15}$$

is exact for all polynomials of degree at most $2(p+1)m-1$*, that is,*

$$E_{m,2p+1}(g)=0,\ \ \forall\ g\in P_{2(p+1)m-1}.$$

Theorem 1. *Let* $U(\xi)\in C^4[\alpha,\beta]$, $\triangle$ *be the partition of* $[\alpha,\beta]$ *and* $S(\xi)$ *be the spline interpolation function* $U(\xi)$*, we have*

$$\|D^r(S-U)\|_\infty\le\phi_r h^{4-r},\quad r=0,\ldots,3. \tag{16}$$

For the proof [35].

Next, we will prove the following theorem for convergence of our method in Eq.(14).

Theorem 2. *The approximate method Eq. (14)*

$$S(\xi_r)=\frac{h}{2}\sum_{j=0}^{r-1}\sum_{\tau=1}^{4}\sum_{\gamma=0}^{4}\varphi_{\tau,\gamma}(K(\xi_r,\zeta_{j\tau},S(\zeta_{j\tau})))^{(\gamma)}+g(\xi_r),r=0,1,\ldots,M, \tag{17}$$

for solution of the nonlinear VIE (1) is converge and the error bounded is

$$|E_r|\le\frac{hL}{2}\sum_{j=0}^{r-1}\sum_{\tau=1}^{4}\sum_{\gamma=0}^{4}|\varphi_{\tau,\gamma}||E_{j\tau}|, \tag{18}$$

where $E_{j\tau}=S_{j\tau}-U_{j\tau},E_r=S_r-U_r,r=0,\ldots,M$ *and kernel K satisfy Lipschitz condition in their third argument with L Lipschitz constant.*

Proof. We suppose that for a fixted ξ_f in $\alpha<\xi_f\le\beta$ at the points $\xi_r=\alpha+rh$, $h=\frac{\xi_f-\alpha}{M}$, $r=0,1,\ldots,M$, the corresponding approximation method for nonlinear VIE(1) is

$$S(\xi_r)=\frac{h}{2}\sum_{j=0}^{r-1}\sum_{\tau=1}^{4}\sum_{\gamma=0}^{4}\varphi_{\tau,\gamma}(K(\xi_r,\zeta_{j\tau},S(\zeta_{j\tau})))^{(\gamma)}+g(\xi_r),\ r=0,1,\ldots,M. \tag{19}$$

By discrediting VIE(1) and approximate the integral by the Gauss-Turán rule, we can obtain

$$U(\xi_r)=\frac{h}{2}\sum_{j=0}^{r-1}\sum_{\tau=1}^{4}\sum_{\gamma=0}^{4}\varphi_{\tau,\gamma}(K(\xi_r,\zeta_{j\tau},U(\zeta_{j\tau})))^{(\gamma)}+g(\xi_r),\ r=0,1,\ldots,M. \tag{20}$$

By subtracting Eq.(20) from Eq.(19) and using interpolatory condition of cubic spline, we get

$$S(\xi_r)-U(\xi_r)=\frac{h}{2}\sum_{j=0}^{r-1}\sum_{\tau=1}^{4}\sum_{\gamma=0}^{4}\varphi_{\tau,\gamma}[K^{(\gamma)}(\xi_r,\zeta_{j\tau},S(\zeta_{j\tau}))-K^{(\gamma)}(\xi_r,\zeta_{j\tau},U(\zeta_{j\tau}))],$$

we suppose that, $S(y_r)=S_r,\ U(y_r)=U_r,\ r=0,\ldots,M$, and kernel $K^{(\gamma)}$ satisfy Lipschitz condition in their third argument of the form

$$\forall\ \mu_1,\mu_2\in R:\ |K^{(\gamma)}(y,\zeta,\mu_1)-K^{(\gamma)}(y,\zeta,\mu_2)|\leq L|\mu_1-\mu_2|,$$

where L is independent of y,ζ,μ_1 and μ_2. We get

$$|S_r-U_r|\leq\frac{h}{2}L\sum_{j=0}^{r-1}\sum_{\tau=1}^{4}\sum_{\gamma=0}^{4}|\varphi_{\tau,\gamma}||S_{j\tau}-U_{j\tau}|,$$

$$|E_r|\leq\frac{hL}{2}\sum_{j=0}^{r-1}\sum_{\tau=1}^{4}\sum_{\gamma=0}^{4}|\varphi_{\tau,\gamma}||E_{j\tau}|,$$

where $E_r=S_r-U_r,\ r=0,\ldots,M$.
When $h\to 0$ then the above terms are zero and also these terms are due to interpolating of $U(\xi)$ by cubic spline (Theorem (1)). We get for a fixed r,

$$|E_r|\to 0\ as\ h\to 0.$$

□

6 Numerical Examples

We consider text problems of nonlinear and linear VIEs. Our numerical results are compared with methods in [36–40], program preformed by Mathematica for examples, the running time is also reported in seconds (CPU time).

Example 1. *Consider the following nonlinear VIE with exact solution $U(\xi)=cos\xi$,*

$$U(\xi)=1+sin^2\xi-\int_0^{\xi}3sin(\xi-y)U^2(y)dy,\ \xi\in[0,1]. \tag{21}$$

We apply the presented method Eq.(14), the maximum absolute errors(MAEs) in the solutions for $\theta=0$ with $M=20$ are tabulated in Table 3, and compared with the results [36,37]. The MAEs in the solution for the different values of $\theta=-1.99,-1,1,3$ with $M=20$ are tabulated in Table 4.

Table 3. The MAEs at particular points for $M=20$.

ξ_r	our Method $\theta=0$ CPU time=92.67s	Method in [36]	Method in [37]
0.1	2.58750(-10)	2.547(-08)	2.17003(-06)
0.2	1.01165(-09)	3.448(-07)	4.03803(-06)
0.3	2.19251(-09)	9.190(-07)	5.67601(-06)
0.4	3.69934(-09)	1.444(-06)	6.94326(-06)
0.5	5.40590(-09)	1.881(-06)	7.76212(-06)
0.6	7.17495(-09)	2.181(-06)	8.09813(-06)
0.7	8.86762(-09)	1.839(-06)	7.95528(-06)
0.8	1.03004(-08)	6.412(-06)	7.37153(-06)
0.9	1.05758(-08)	1.004(-04)	6.41454(-06)
1	0	9.255(-04)	5.12360(-06)

Table 4. The MAEs at particular points for $M = 20$.

ξ_r	$\theta = -1.99$ CPU time=126.69s	$\theta = -1$ CPU time=126.42s	$\theta = 1$ CPU time=120.16s	$\theta = 3$ CPU time=113.28s
0.1	1.23301(-06)	6.20599(-07)	6.20826(-07)	1.86105(-06)
0.2	4.75197(-06)	2.42717(-06)	2.42812(-06)	7.27883(-06)
0.3	1.00713(-05)	5.26034(-06)	5.26276(-06)	1.57765(-05)
0.4	1.63959(-05)	8.87381(-06)	8.88002(-06)	2.66204(-05)
0.5	2.26731(-05)	1.29564(-05)	1.29770(-05)	3.89029(-05)
0.6	2.76230(-05)	1.71355(-05)	1.72251(-05)	5.16392(-05)
0.7	2.97513(-05)	2.08639(-05)	2.13026(-05)	6.38673(-05)
0.8	2.73210(-05)	2.27310(-05)	2.48961(-05)	7.47357(-05)
0.9	1.82629(-05)	1.75413(-05)	2.68933(-05)	8.33706(-05)
1	0	0	0	0

Example 2. *Consider the following linear VIE with exact solution* $U(\xi) = 1 - sinh\xi$,

$$U(\xi) = 1 - \xi - \frac{\xi^2}{2} + \int_0^{\xi} (\xi - y)U(y)dy, \ \xi \in (0,1]. \tag{22}$$

We apply the presented method Eq.(14), the MAEs in the solutions for $\theta = 0$ with $M = 20$ are tabulated in Table 5, and compared with the results [37–39]. The MAEs in the solution for the different values of $\theta = -1.99, -1, 1, 3$ with $M = 20$ are tabulated in Table 6.

Table 5. The MAEs at particular points for $M = 20$.

ξ_r	our Method $\theta = 0$ CPU time=2.66 s	Method in [37]	Method in [38]	Method in [39]
0.1	1.07436(-12)	1.21734(-07)	5.6389(-06)	8.33(-08)
0.2	1.07924(-11)	2.35882(-07)	2.2020(-05)	3.09(-07)
0.3	3.81101(-11)	3.54854(-07)	4.8210(-05)	6.75(-07)
0.4	9.22727(-11)	4.77841(-07)	8.3330(-05)	1.19(-06)
0.5	1.83052(-10)	6.05697(-07)	1.2656(-04)	1.87(-06)
0.6	3.20912(-10)	7.39627(-07)	1.7715(-04)	2.73(-06)
0.7	5.16804(-10)	8.80948(-07)	2.3436(-04)	3.77(-06)
0.8	7.76730(-10)	1.03101(-06)	2.9745(-04)	5.02(-06)
0.9	1.02327(-09)	1.19095(-06)	3.6566(-04)	6.50(-06)

Example 3. *Consider the following linear VIE with exact solution* $U(\xi) = \frac{1}{3}(2cos\sqrt{3}\xi + 1)$,

$$U(\xi) = cos(\xi) - \int_0^{\xi} (\xi - y)cos(\xi - y)U(y)dy, \ \xi \in (0,1]. \tag{23}$$

We apply the presented method Eq.(14), the MAEs in the solutions for $\theta = 0$ for the different values of M are tabulated in Table 7, and compared with the results [39] and then the MAEs in the solutions for $\theta = 0$ with $M = 16$ are tabulated in Table 8, and compared with the results [40]. The MAEs in the solution for the different values of $\theta = -1.99, -1, 1, 3$ with $M = 16$ are tabulated in Table 9.

7 Conclusions

We developed a method to find the solution of linear and nonlinear VIEs the overall approach is based on the Gauss-Tur*á*n quadrature rule and then using extended cubic spline as the bases function. The unknown coefficients in combination determine by collocation method. The arising system of linear and nonlinear can be solved. Numerical text problems are considered to justify the applicability and efficient nature of our approach, comparison of the results justify the considerable accuracy and efficiency proposed methods.

Table 6. The MAEs at particular points for $M = 20$.

ξ_r	$\theta = -1.99$	$\theta = -1$	$\theta = 1$	$\theta = 3$
	CPU time=10.73s	CPU time=3.34s	CPU time=3.34s	CPU time=3.28s
0.1	5.33949(-08)	2.11738(-09)	2.64738(-09)	9.20950(-09)
0.2	1.80994(-07)	2.50146(-08)	2.59522(-08)	8.04554(-08)
0.3	3.46693(-07)	9.01255(-08)	9.14858(-08)	2.76501(-07)
0.4	5.17139(-07)	2.19582(-07)	2.21424(-07)	6.69902(-07)
0.5	6.61174(-07)	4.36480(-07)	4.39216(-07)	1.32506(-06)
0.6	7.49387(-07)	7.63945(-07)	7.70044(-07)	2.31957(-06)
0.7	7.53758(-07)	1.21766(-06)	1.24122(-06)	3.73573(-06)
0.8	6.47409(-07)	1.76140(-06)	1.87986(-06)	5.66394(-06)
0.9	4.04457(-07)	2.05264(-06)	2.61665(-06)	8.18089(-06)
1	0	0	0	0

Table 7. The MAEs for the different values of M.

M	CPU time(s)	our Method $\theta = 0$	Method in [39]
5	0.297	7.26(-05)	9.42(-04)
10	1.328	9.18(-06)	1.76(-04)
12	2.094	5.30(-06)	1.00(-04)
15	3.469	2.71(-06)	5.30(-05)
20	10.187	1.14(-06)	2.40(-05)
25	36.766	5.84(-07)	1.40(-05)
29	60.532	3.74(-07)	1.00(-05)
35	127.516	2.13(-07)	6.35(-06)

Data Availability

- All data in the paper is available from the corresponding author upon reasonable request.
- All data generated or analyzed during this study are included in this published article (and its supplementary information files).
- The data that support the findings of this study are openly available in [repository name declared in the paper].
- The manuscript has no associated data or the data will not be deposited.

Conflicts of Interest

The author declares that there is no conflict of interest.

Ethical Considerations

The author has diligently addressed ethical concerns, such as informed consent, plagiarism, data fabrication, misconduct, falsification, double publication, redundancy, submission, and other related matters.

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Table 8. The MAEs at particular points for $M = 16$.

ξ_r	our Method $\theta = 0$ CPU time $= 4.11$s	Hybrid function method in [40]	Haar function method in [40]
0	0	0	0
0.0625	2.59961(-07)	2.92469(-03)	1.10664(-03)
0.125	6.52776(-07)	3.04161(-05)	8.61508(-03)
0.1875	9.99323(-07)	2.86431(-03)	8.87786(-03)
0.25	1.34046(-06)	1.19870(-04)	1.40576(-02)
0.3125	1.65830(-06)	2.74709(-03)	1.48130(-02)
0.375	1.95098(-06)	2.63148(-04)	2.00688(-02)
0.4375	2.21275(-06)	2.57987(-03)	1.60562(-02)
0.5	2.43916(-06)	4.51845(-04)	1.60428(-02)
0.5625	2.62653(-06)	2.37245(-03)	1.89578(-02)
0.625	2.76923(-06)	6.74887(-04)	2.51250(-02)
0.6875	2.87138(-06)	2.13708(-03)	2.05799(-02)
0.75	2.90314(-06)	9.19080(-04)	1.89545(-02)
0.8125	2.96390(-06)	1.88771(-03)	1.10394(-02)
0.875	2.67346(-06)	1.16988(-03)	6.44360(-03)
0.9375	3.44471(-06)	1.63927(-03)	7.71887(-03)
1	0	1.41219(-03)	1.63518(-02)

Table 9. The MAEs at particular points for $M = 16$.

ξ_r	$\theta = 3$ CPU time$= 8.656$s	$\theta = 1$ CPU time$= 4.812$s	$\theta = -1$ CPU time$= 7.296$s	$\theta = -1.99$ CPU time$= 8.265$s
0	0	0	0	0
0.0625	8.08921(-07)	4.92831(-07)	1.26654(-07)	4.78849(-06)
0.125	2.59497(-06)	1.43495(-06)	5.03594(-07)	8.55075(-06)
0.1875	5.07508(-06)	2.56868(-06)	1.12144(-06)	1.21636(-05)
0.25	8.19924(-06)	3.90973(-06)	1.96159(-06)	1.45916(-05)
0.3125	1.18871(-05)	5.41967(-06)	3.00406(-06)	1.68809(-05)
0.375	1.60415(-05)	7.06271(-06)	4.21146(-06)	1.78741(-05)
0.4375	2.05495(-05)	8.79667(-06)	5.56815(-06)	1.88149(-05)
0.5	2.52846(-05)	1.05754(-05)	6.98914(-06)	1.83921(-05)
0.5625	3.01089(-05)	1.23493(-05)	8.53920(-06)	1.80778(-05)
0.625	3.48758(-05)	1.40655(-05)	9.91387(-06)	1.63680(-05)
0.6875	3.94327(-05)	1.56751(-05)	1.16777(-05)	1.49979(-05)
0.75	4.36236(-05)	1.70957(-05)	1.23199(-05)	1.22241(-05)
0.8125	4.72984(-05)	1.84373(-05)	1.52657(-05)	1.00839(-05)
0.875	5.01770(-05)	1.86929(-05)	1.21920(-05)	6.53991(-06)
0.9375	5.49039(-05)	2.33778(-05)	2.27409(-05)	3.97707(-06)
1	0	0	0	0

References

[1] A.M. Wazwaz, Linear and Nonlinear Integral Equations Methods and Applications, Springer Heidelberg Dordrecht London New York, 2011.

[2] D. Colton, R. Kress, Integral Equation Methods in Scattering Theory, New York, 1983.

[3] H. Brunner, Iterated collocation methods and their discretizatins for Volterra integral equations, SIAM Journal on Numerical Analysis, 21, 1132–1145, (1984).

[4] H. Beiglo, M. Gachpazan, Numerical solution of nonlinear mixed Volterra-Fredholm integral equations in complex plane via PQWs, Applied Mathematics and Computation, 369, (2020). https://doi.org/10.1016/j.amc.2019.124828.

[5] DD. Ganji, GA. Afrouzi, H. Hosseinzadeh, RA. Talarposhti, Application of homotopy-perturbation method to the second kind of nonlinear integral Equations, Physics Letters A, 371, 20–25, (2007).

[6] A. Ghorbani, J. Saberi-Nadjafi, Exact solutions for nonlinear integral equations by a modified homotopy perturbation method, Computers & Mathematics with Applications, 56, 1032–1039, (2008).

[7] ST. Mohyud-Din, A. Yildirim, E. Yuluklu, Homotopy analysis method for space- and time-fractional KdV equation, International Journal Numerical MathematicsInt Heat and Fluid Flow, 7, 928–941, (2012).

[8] YM. Chu, S Ullah, M Ali, GF Tuzzahrah, T. Munir, Numerical Investigation of Volterra Integral Equations of Second Kind using Optimal Homotopy Asymptotic Method, 430, (2022). https://doi.org/10.1016/j.amc.2022.127304.

[9] M. Rabbani, K. Maleknejad, Using orthonormal wavelet basis in Petrov-Galerkin method for solving Fredholm integral equations of the second kind, Kybernetes, 41, 465–481, (2012) .

[10] Z. Xiaoyang, Y. Xiaofan, Techniques for solving integral and differential equations by Legendre wavelets, International Journal of Systems Science, 40, 1127–1137, (2009).

[11] K. Maleknejad, R. Dehbozorgi, Adaptive numerical approach based upon Chebyshev operational vector for nonlinear Volterra integral equations and its convergence analysis, Journal of Computational and Applied Mathematics, 344, 356–366, (2018).

[12] M. Ahmadinia, H. Afshariarjmand, M. Salehi, Numerical solution of multi-dimensional Itô Volterra integral equations by the second kind Chebyshev wavelets and parallel computing process, Applied Mathematics Computation, 450, (2023). https://doi.org/10.1016/j.amc.2023.127988.

[13] K. Maleknejad, H. Derili, Numerical solution of Hammerstein integral equations by using combination of spline-collocationmethod and Largrange interpolation, Applied Mathematics Computation, 190, 1557–1562, (2007).

[14] N. Karamollahi, M. Heydari, Gh. Barid Loghmani, An interpolation-based method for solving Volterra integral equations, Journal Applied Mathematics Computation, 68, 909–940, (2022).

[15] A.R. Nazemi, M.H. Farahi, A.V. Kamyad, A new technique for approximate solutions of the nonlinear Volterra integral equations of the second kind, Sharif University of Technology, 14, 579–585, (2007).

[16] AL. Martire, Volterra integral equations: An approach based on Lipschitz-continuity, Applied Mathematics Computation, 435, (2022). https://doi.org/10.1016/j.amc.2022.127496.

[17] H. Brunner, Iterated collocation methods and their discretizatins for Volterra integral equations, SIAM Journal on Numerical Analysis, 21, 1132–1145, (1984).

[18] A.A. Hamoud, L.A. Dawood, K.P. Ghadle, S.M. Atshan, Usage of the modified variational iteration technique for solving Fredholm integro-differential equations, International Journal of Mechanical and Production Engineering Research and Development, 9(2), 895–902, (2019).

[19] S. Yalçinbaş, Taylor polynomial solutions of nonlinear Volterra-Fredholm integral equations, Applied Mathematics and Computation, 127, 195–206, (2002).

[20] J. Saberi-Nadjafi, M. Heidari, Solving nonlinear integral equations in the Urysohn form by Newton-Kantorovich-quadrature method, Computers & Mathematics with Applications, 60, 2058–2065, (2010).

[21] F. Ghoreishi, M. Hadizadeh, Numerical computation of the Tau approximation for the Volterra-Hammerstein integral equations, Numerical Algorithms, 52, 541–559, (2009).

[22] M. Kazemi, H. R. Sahebi, Quadrature Rules for Solving Two-Dimensional Fredholm Integral Equations of Second Kind, Analytical and Numerical Solutions for Nonlinear Equations, 8(2), 172–181, (2023).

[23] Z. Gilani, M. Alipour, S. Rivaz, System of Volterra Fredholm Integro-Fractional Differential Equations: Application of Fibonacci Polynomials, Analytical and Numerical Solutions for Nonlinear Equations, 9(1), 89–101, (2024).

[24] H. D. Mazraeh, M. Kalantari, S. H. Tabasi, A. A. Aghaei, Z. Kalantari, F. Fahimi, Solving Fredholm Integral Equations of the Second Kind Using an Improved Cuckoo Optimization Algorithm, Analytical and Numerical Solutions for Nonlinear Equations, 7(1), 33–52, (2022).

[25] A. Abdi, G. Hojjati, Z. Jackiewicz and H. Mahdi, A new code for Volterra integral equations based on natural Runge-Kutta methods, Applied Numerical Mathematics, 143, 35–50, (2019).

[26] A.M. Muhammad, A.M. Ayal, Numerical solution of linear Volterra integral equation with delay using Bernstein polynomial, International Electronic Journal of Mathematics Education, 14(3), 735–740, (2019).

[27] S.C. Buranay, M.A. Özarslan, S.S. Falahhesar, Numerical Solution of the Fredholm and Volterra Integral Equations by Using Modified BernsteinKantorovich Operators. Mathematics, 9(11), (2021). https://doi.org/10.3390/math9111193.

[28] A. Korkmaz, I. Dag, Cubic B-spline differential quadrature methods for the advection-diffusion equation, International Journal Numerical MathematicsInt Heat and Fluid Flow, 22, 1021–1036, (2012).

[29] X. Gang, W. Guo-Zhao, Extended cubic uniform B-spline and B-spline, Acta Automatica Sinica, 34(8), 980–984, (2008).

[30] G. Xu, G.z. Wang, Extended cubic uniform B-spline and α B-spline, Acta Automatica Sinica, 34, 980–983, (2008).

[31] J. Rashidinia, H.S. Shekarabi, Numerical solution of hyperbolic telegraph equation by cubic B-spline collocation method, Applied Mathematics Computation, 281, 28–38, (2016).

[32] SH. Sharifi, J. Rashidinia, Collocation method for Convection-Reaction-Diffusion equation. Journal of King Saud University - Science, 31, 1115–1121, (2019).

[33] MM. Jamei, S. Fakhravar, Gauss-Turan numerical integration method. Khaje Nassir Al-Deen Toosi University of Technology, 2011.

[34] G.V. Milovanović, MM. Spalević, MS. Pranić, Error bounds of some Gauss-Turán- Kronrod quadratures with Gori-Micchelli weights for analytic functions, Kragujevac Journal of Mathematics, 30, 221–234, (2007).

[35] C.A. Hall, On error bounds for spline interpolation, Journal of Approximation Theory, 1, 209–218, (1968) .

[36] S.H. Javadi, A. Davari, E. Babolian, Numerical implementation of the Adomian decomposition method for nonlinear Volterra integral equations of the second kind, International Journal of Computer Mathematics, 84, 75–79, (2007).

[37] J. Rashidinia, Z. Mahmoodi, Collocation method for Fredholm and Volterra integral equations, Kybernetes, 42(3), 400–412, (2013).

[38] E. Babolian, A. Davari, Numerical implementation of Adomian decomposition method for linear Volterra integral equations of second kind, Applied Mathematics Computation, 165, 223–227, (2005).

[39] A. Mennouni, A new numerical approximation for Volterra integral equations combining two quadrature rules, Applied Mathematics Computation, 218, 1962–1969, (2011).

[40] Ch.H. Hsiao, Hybrid function method for solving Fredholm and Volterra integral equations of the second kind, Journal of Computational and Applied Mathematics, 230, 59–68, (2009).