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Research article

Fiscal and Monetary Equilibrium: A Differential Game Between the Government and the Central Bank

Behzad Shahbaie[1], Alireza Bahiraie[1,2,*]

[1] Department of Mathematics, Semnan University, Semnan 35195–363, Iran

[2] Department of Mathematics, Washington University, St.Louis, MO, USA

* **Corresponding author(s):** Aalireza.bahiraie@semnan.ac.ir

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Abstract

This study employs a differential game-theoretic framework to analyze the strategic interaction between the government and the central bank in the context of public debt management. The core objective is to develop an analytical model that stabilizes government debt by optimizing three critical policy instruments: tax revenue, government spending, and the money supply. A major innovation of this research lies in disaggregating the fiscal deficit into two distinct control variablestaxation and spending, which allows for the derivation of optimal equilibrium trajectories for each. Numerical simulations, based on a 20-year dataset from the United States, reveal key differences in outcomes depending on the degree of independence or interdependence between fiscal instruments. When tax revenue and government spending are treated as independent from the monetary authority, the model converges to equilibrium values of 0.8992 for public debt, 0.0982 for the fiscal deficit, and 0.1288 for the monetary base. Conversely, when tax and spending decisions are jointly determined as components of the fiscal deficit, the corresponding equilibrium values shift slightly to 0.8975, 0.0852, and 0.1157, respectively. These findings suggest that enhanced coordination between the fiscal authoritys instruments and the central banks control over money creation can improve debt sustainability and overall macroeconomic stability. While the empirical focus is on the U.S. economy, the proposed framework offers a flexible foundation for evaluating fiscal-monetary dynamics in other institutional settings.

Keywords: Monetary and fiscal policies, Public debt, Cooperative games, Differential games.

Mathematics Subject Classification (2020): 91A12, 91A23, 91B64, 91B82

1 Introduction

Fiscal constraints dynamically establish a link between fiscal deficits, the monetary base, and the public debt structure. In many advanced economies, an increase in the monetary base typically accompanies the expansion of the fiscal deficit, stemming from the gap between government spending and revenues. These variables are often managed by two independent authorities, each pursuing distinct objectives and subject to different political pressures. Consequently, strategic interactions between these authorities play a crucial role in shaping fiscal and monetary policies. Researchers have increasingly focused on examining the evolving relationship between fiscal and monetary policies, particularly concerning their combined effects on economic stabilization. Government debt can be analyzed from various perspectives; for instance, Tabellini employed a second-order method in his analysis [18]. This paper investigates a dynamic differential game between fiscal and monetary authorities, interpreting and analyzing the system from multiple viewpoints. Furthermore, it discusses equilibrium concepts within the context of developments in money creation, fiscal deficits, and public debt.

Alessina and Tabellini [1], by introducing the private sector as a third player in the dynamic differential game of fiscal and monetary policies, demonstrated that mere commitment to monetary rules, without coordination between these two policies, is not only ineffective but also does not necessarily lead to an improvement in public welfare. Pettit [17], extended this analysis by examining both cooperative and non-cooperative behaviors and analyzing the interactions between the treasury and the central bank using differential equations within a dynamic framework. Levine and Pearlman [13], also studied this issue within the framework of a continuous-time econometric model. They emphasized the distinction between the fiscal incentives for coordinating fiscal policies within the European Union, on the one hand, and the credible low-inflation policy recommended by the European Central Bank, on the other.This study also considers the spillover effects of uncoordinated fiscal policy on monetary policy. Levine and Brociner [12] investigated the strategic interactions between the fiscal authorities of two entities operating within a shared fiscal framework in a monetary union. Their analysis focused on a model where the private sector is influenced by targeted microfinancing initiatives.

Utilizing a differential games model, Aarle et al. [19] analyzed the strategic interactions between monetary authorities responsible for revenue management and the oversight of initial fiscal deficits. These authorities are tasked with ensuring the alignment of government spending with available resources. Their study explored both cooperative and non-cooperative Nash open-loop equilibria and their implications for conservative versus more independent central banks. Addressing the issue of debt stabilization within a dynamic framework incorporating a moving horizon (which can approximate standard scenarios), Van Den Broek [20] found that the election of new policymakers often leads to a shorter planning horizon. This, in turn, can diminish debt stabilization efforts in the Nash equilibrium. Bartolome and Gioacchino [8], examined the strategic interactions among the policymaker, the central bank, and the government within a two-stage game framework. Their analysis revealed that the establishment of a robust institutional regime in the first stage can effectively ensure equilibrium in the subsequent second stage of the game. Furthermore, they demonstrated that when policymakers engage in communication prior to the game (potentially leading to multiple equilibria), coordination problems can be effectively addressed through the concept of correlated equilibrium, allowing for a more predictable and potentially welfare-enhancing outcome. Building upon earlier work, Engwerda et al. [9], solved and analyzed the nonlinear differential game arising in both cooperative and non-cooperative environments. Their innovative approach involved incorporating the endogenous risk premium imposed by financial markets directly into the well-established Tabellini model. This integration allowed for a more nuanced understanding of how market perceptions of risk influence the strategic choices of fiscal and monetary authorities.

Furthering this line of inquiry, Engwerda et al. [10] investigated the strategic conflict between the government and the central bank by employing the differential games method and the concept of Nash equilibrium within both cooperative and non-cooperative game settings. By explicitly considering the desired target levels for key macroeconomic variables such as government debt, budget deficits, the monetary base, and the quality of policymakers behavior in relation to these targets, and through the simulation of equilibrium models, their study yielded significant insights. They demonstrated that in cooperative game scenarios, the resulting equilibrium level of government debt is lower compared to non-cooperative settings, and the speed of convergence towards this equilibrium is notably faster. Additionally, their findings indicated that a lower rate of money issuance and smaller budget deficits are necessary to achieve long-term debt stabilization under cooperative policymaking. Anevlavis et al. [2] highlighted the crucial role of financial markets in assessing and pricing the risk associated with holding public debt, particularly for countries vulnerable to financial instability. Their work explored the implication that financial markets may demand a risk premium as a condition for maintaining such debt, thereby influencing the strategic choices available to fiscal and monetary authorities within a Nash equilibrium framework. Building upon this insight, the current study posits that the risk insurance premium function can be viewed as an inherent market-based disciplinary mechanism. This mechanism potentially shapes the behavior of policymakers by increasing the cost of unsustainable fiscal policies and incentivizing prudent debt management.

In a complementary vein, Barthélemy and Plantin [7] provided a comprehensive strategic analysis of the interplay between primary liabilities, encompassing the actions of both monetary and fiscal authorities. Utilizing the "chicken-wallace" game as their theoretical foundation, they analyzed the strategic interactions within the public sector regarding debt management. Their central finding was that the sustainability of public debt hinges on the inability of any single official to unilaterally commit to policies extending beyond their current mandate. This underscores the importance of establishing a consistent and credible long-term strategic framework for debt management. Furthermore, their analysis suggested a temporal dimension to effective coordination, arguing that the fiscal authority should ideally take the lead in policy action to ensure effective debt management, potentially setting the stage for subsequent monetary policy responses.

Providing empirical grounding to the theoretical considerations, Nguyen and Luong [16], investigated the determinants of public debt in a panel of 27 transition countries spanning the years 2000 to 2018. Employing an econometric model, they specifically examined the impact of fiscal policy levers and the quality of institutional frameworks on debt accumulation. Their findings revealed a significant positive correlation between weak corruption control management and the accumulation of public debt, indicating that deficiencies in governance exacerbate fiscal vulnerabilities. Conversely, they demonstrated that investments and financing directed towards improving institutional quality specifically enhancing government effectiveness, regulatory quality, and the rule of law had a debt-reducing effect. This empirical evidence strongly supports the notion that robust institutional structures are crucial for fostering fiscal sustainability and mitigating the risks associated with public debt in transitional economies. Li et al. [14] rigorously demonstrated the existence of an optimal solution within a Stackelberg game setting involving monetary and financial authorities. Their proof relied on nested observations within a linearquadraticGaussian (LQG) model tailored for Stackelberg games, employing established theorems to underpin their findings. Furthermore, they applied this framework to prove the existence of an optimal solution for the Stackelberg game of confrontation between fiscal and monetary policymakers, specifically aimed at stabilizing public debt. For a broader understanding of related theoretical and applied work in this domain, readers are directed to the contributions of Bahiraie et al. [3–6], Lashgari, Bahiraie, and Eshaghi [11], and Zoudbina, Shahbaie, and Bahiraie [21], which offer diverse perspectives on the strategic interactions of macroeconomic policymakers.

This paper extends the analytical solutions developed within the differential game framework introduced by Tabellini [18], aiming to capture the strategic interaction between fiscal and monetary authorities arising from government budget constraints. Unlike the original model, which relies on fiscal deficits as the main control variable, our framework defines government spending and tax revenue as two distinct control instruments. This reformulation allows for a more granular analysis of policy dynamics. We employ the model to characterize and interpret the equilibrium relationship between fiscal policy instruments and money creation, with the objective of stabilizing public debt. In the second part, we present a cooperative game-theoretic extension of the model, grounded in relevant theorems from cooperative game theory. This version of the model explores the equilibrium paths of monetary expansion, taxation, and public spending under both independent and coordinated fiscal behaviorwhether the government controls instruments autonomously or interacts strategically through a fiscal deficit framework. The third section introduces the models structural parameters and provides a numerical simulation of the equilibrium conditions. Finally, the resulting equilibrium trajectories are compared and interpreted within the context of a participatory (cooperative) policy game, highlighting the implications of fiscal-monetary coordination for debt stabilization.

2 A Differential Game on Government Debt Stabilization

To finance fiscal deficitseither through money creation or by accumulating additional debtmonetary policy is delegated to independent central banks, while decisions related to persistent and large-scale budget deficits are typically managed by decentralized treasury institutions. This division of responsibilities creates a structural separation between fiscal and monetary authorities. However, despite this institutional independence, monetary and fiscal policies remain closely interlinked due to the inherent limitations on government spending and borrowing capacity. The dynamic government budget constraint captures this interdependence by linking key variables over time: the primary fiscal deficit $f(t)$, seigniorage or base-money creation $m(t)$, interest payments on public debt $rd(t)$, and the accumulation of public debt $d(t)$, where the dot denotes a time derivative. This constraint formally illustrates how fiscal imbalances must ultimately be financed either through monetary expansion or increased borrowing, both of which involve trade-offs and require coordination between fiscal and monetary authorities

$$\dot{d}(t) = rd(t) + f(t) - m(t) \quad \text{with} \quad d(0) = d_0. \tag{1}$$

In this equation, $d(t)$, $f(t)$, and $m(t)$ are expressed as fractions of GDP. r represents the interest rate on unpaid government debt minus the growth rate of output and is assumed to be exogenous, so it is independent of the level of government debt.

When the total fiscal burdendefined as the sum of the primary deficit $f(t)$ and interest payments on outstanding debt $rd(t)$ exceeds the revenue generated through base-money creation $m(t)$, the government can defer necessary fiscal adjustments by accumulating additional debt. This dynamic highlights how the intertemporal government budget constraint embodies both short-term (intra-temporal) and long-term (inter-temporal) policy trade-offs between monetary and fiscal authorities. Consequently, decisions regarding monetary expansion, fiscal stance, and debt accumulation are inherently linked.

A key determinant of fiscal consolidation capacity is the initial stock of government debt $d(0)$, along with the real interest rate (adjusted for output growth). When initial debt levels and real interest rates are low, stabilizing public debt is more feasible. However, the challenge becomes more severe in high-debt and high-interest environments, where fiscal space is constrained.

Two broad strategies are typically considered for public debt stabilization: (1) reducing the initial fiscal deficit, and (2) increasing revenue from money creation. Yet, these approaches often lead to conflict when monetary and fiscal authorities pursue competing objectivessuch as price stability, debt sustainability, and espending priorities. To model this interaction, we adapt Tabellinis framework by establishing an explicit link between government spending, taxation, and monetary base dynamics. In this setting, fiscal authorities determine the levels of public spending and tax revenue, while the central bank independently sets the monetary base

$$\dot{d}(t) = rd(t) + G(t) - T(t) - m(t), \quad d(0) = d_0. \tag{2}$$

Here, $G(t)$ represents government spending and $T(t)$ represents tax revenue. We also take into account the existence of a budget deficit, $G(t) > T(t)$.

In the differential game between the government and the central bank, the government's goal is to increase tax revenues and reduce its expenses. By using the two control variables, government spending and tax revenue, instead of just a fiscal deficit, the government can identify an improved method for managing these variables to achieve maximum fiscal balance. Using two control variables helps the government optimize its fiscal strategy and minimize the likelihood of unexpected errors. Moreover, this approach enables the government to account for the interdependence between spending and tax revenue, adjusting both variables simultaneously.

We will not include cases where the no-Ponzi game condition is violated, i.e., unrestricted debt accumulation. This assumption of a finite steady-state level of debt ensures that the no-Ponzi game condition is maintained, preventing excessive debt accumulation and mandating that the discounted value of debt approaches zero as time approaches infinity. We assume that the players adopt strategies that converge to stable states, stabilizing government debt at a limited steady-state value and minimizing their respective loss functions. The set of admissible control plans in this study consists of locally square-integrable functions, denoted by:

$$U := \left\{ (G(t), T(t), m(t)) \in L_{2,\text{loc}} \mid \lim_{t\to\infty} d(t) = d^e, \lim_{t\to\infty} G(t) = G^e, \lim_{t\to\infty} T(t) = T^e, \lim_{t\to\infty} m(t) = m^e \right\}. \tag{3}$$

In this context, G^e, T^e, m^e, and d^e represent the long-run (steady-state) values of government spending, tax revenue, money growth, and government debt, respectively. There are two ways to incorporate $G(t)$ and $T(t)$ into the system's state equation:

1. The two control variables are used independently of each other in the system's state equation.
2. The two control variables are dependent on each other and are used as substitutes for the fiscal deficit (denoted by $f(t)$).

2.1 Introducing Two Independent Control Variables of Government Spending and Tax Revenue into the Model

We consider the equation for the state of public debt as follows:

The fiscal authority (or Treasury) has the following intertemporal loss function, which depends on the profiles of government spending, tax revenue, base money growth, and government debt:

$$L_f = \frac{1}{2}\int_0^{\infty} \left[(G(t)-\bar{G})^2 + (T(t)-\bar{T})^2 + \varphi(m(t)-\bar{m})^2 + \tau(d(t)-\bar{d})^2 \right] e^{-\rho t}\, dt. \tag{4}$$

We will eliminate instances of uncontrolled debt accumulation that violate the Ponzi game constraint. Thus, by assuming a fixed, limited level of debt, we ensure that the non-Ponzi game condition is maintained. This condition prevents excessive debt accumulation and requires that the discounted value of debt tends to zero as time approaches infinity, thereby ensuring the sustainability of the debt trajectory:

$$\lim_{t\to\infty} d(t)e^{-rt} = 0. \tag{5}$$

To minimize the intertemporal loss function, fiscal authorities manage the primary fiscal deficit within the constraints imposed by the dynamic government budget identity (Equation 2), the transversality condition (Equation 5), and the initial stock of public debt, $d(0)$. The target values for the monetary base ($\bar{m}$), government spending ($\bar{G}$), tax revenue ($\bar{T}$), and public debt ($\bar{d}$) represent the *bliss points* of the policy framework–benchmark values that reflect the institutional, political, and economic preferences shaping macroeconomic management.

These points serve as normative anchors, guiding the fiscal authority in evaluating deviations from its long-term objectives. The subjective discount rate ρ quantifies how heavily future costs are weighted relative to present ones, shaping the intertemporal trade-offs in policy design.

The parameters φ and τ capture the relative weight that the fiscal authority assigns to deviations in monetary policy and public debt, respectively. In the realm of non-interest public spending and taxation, the fiscal authority's objectives are summarized through the management of the primary balance. The policy targets for government spending and tax revenue, G and T, can thus be interpreted in two ways: either as the optimal level of non-interest spending given an exogenous tax path, or as the preferred tax-to-GDP ratio under an exogenously determined spending path.

Since higher public debt levels inevitably require higher future taxes, the model embeds debt directly into the loss function. An increase in debt necessitates fiscal adjustments that account for fluctuations in real interest rates and output. In the absence of Ricardian equivalence, elevated debt burdens can crowd out private investment and exacerbate intergenerational inequality, as future taxpayers bear the cost of present-day fiscal imbalances.

On the monetary side, the central bank determines the optimal growth path of the monetary base to minimize its own loss function, ensuring consistency between fiscal and monetary objectives within a dynamic macroeconomic policy environment.

Monetary authorities set the growth of base money to minimize the following loss function:

$$L_m = \frac{1}{2}\int_0^\infty \left[(m(t)-\bar{m})^2+\eta(T(t)-\bar{T})^2+\eta(G(t)-\bar{G})^2+\theta(d(t)-\bar{d})^2\right]e^{-\rho t}\,dt. \tag{6}$$

The parameter θ represents the central banks degree of conservatism. When $\theta = 0$ (implying $1/\theta \to \infty$), the central bank places an extreme emphasis on price stability and behaves in an ultra-conservative manner.

The gap between $(rd+G-T)$ and $\bar{m}$, which is assumed to be positive, is a key factor in the accumulation of government debt. It reflects the tension between the government's desired level of financing $(rd+G-T)$ and the central banks desired level of monetary accommodation $\bar{m}$. A larger gap intensifies the conflict between monetary and fiscal policy goals. Additionally, a high initial debt stock $d(0)$ or a low target for government debt further amplifies this tension.

Another important determinant of debt accumulation is the difference between the rate of time preference (ρ) and the net interest rate (r). When $\rho \geq r$ and public debt is not included in the policymakers' objective functions (i.e., $\theta = \tau = 0$), policymakers are more likely to prioritize short-term fiscal benefits over long-term debt sustainability.

In such a scenario, the perceived benefits of increasing government debtsuch as financing public spending without immediate taxationmay outweigh the long-term economic costs associated with debt accumulation. This imbalance can lead to a continuous and unchecked rise in public debt, as there is little to no institutional incentive to constrain borrowing.

Over time, excessive public debt may cause macroeconomic instability, forcing future policymakers to implement strict fiscal measuressuch as raising taxes or cutting government spendingto restore fiscal sustainability. Furthermore, persistent debt accumulation can undermine investor confidence, increase borrowing costs, and deepen intergenerational wealth inequality, particularly in the absence of Ricardian equivalence.

The weights assigned by fiscal and monetary authorities to debt stabilization, denoted by τ and θ, play a crucial role in determining how the stabilization burden is distributed. When θ is high and τ is low, the central bank may resolve the fiscal-monetary tension by creating money rather than requiring small primary fiscal surpluses. This reflects a dominant fiscal authority and a relatively weak central bank. Conversely, when both θ and τ are low, neither authority is willing to take significant action to stabilize public debt. In such cases, if policymakers are relatively impatient (i.e., $\rho \geq r$), the burden of adjustment shifts to future generations, leading to further debt accumulation.

2.1.1 Solving the Differential Game

Policy coordination and information structure are two key factors that determine how monetary and fiscal authorities interact dynamically. If macroeconomic policies are coordinated, the stabilization of government debt can yield beneficial externalities for both parties. The resulting cooperative equilibrium is Pareto-effective and represents the desired outcome. In contrast, noncooperative equilibria tend to be less effective. In the cooperative game, the goals of the monetary and fiscal authorities are weighted by $1-\omega$ and ω, respectively, reflecting the results of a previous negotiation process. These weights are assumed to be constant. Two important aspects of the information structure must be taken into account when analyzing the strategic interactions between the authorities.

First, it is essential to distinguish between equilibrium settings where players can credibly commit to future actions and those where they cannot. When commitment is possible, open-loop equilibria emerge. In these scenarios, an optimal time path for policy variables is established based on the assumption that the other player's policies remain fixed. Conversely, closed-loop equilibria arise when commitment is not feasible, and each player adjusts their strategy in response to the others actions at every stage.

Second, we need to distinguish between noncooperative equilibria where only one player leads, and those where both players act simultaneously. A Stackelberg equilibrium occurs when one player leads, while a Nash equilibrium arises when no player takes the leadership role.

From a group rationality perspective, the cooperative game can be reformulated as the following simultaneous minimization problem:

$$L^c(\omega) = \omega L_f + (1-\omega)L_m. \tag{7}$$

Using the loss functions defined earlier in Equations (4) and (6), we can rewrite $L^c(\omega)$ as:

$$\begin{aligned} L^c(\omega) =& \frac{1}{2}\int_0^\infty \Big\{(\omega+(1-\omega)\eta)(G(t)-\bar{G})^2 + (\omega+(1-\omega)\eta)(T(t)-\bar{T})^2 \\ &+(\omega\varphi+(1-\omega))(m(t)-\bar{m})^2 + (\omega\tau+(1-\omega)\theta)(d(t)-\bar{d})^2\Big\} e^{-\rho t}\,dt. \end{aligned} \tag{8}$$

The equilibrium is found by minimizing the present-value Hamiltonian:

$$\begin{aligned} H(t) =& \frac{1}{2}\Big[(\omega+(1-\omega)\eta)(G(t)-\bar{G})^2 + (\omega+(1-\omega)\eta)(T(t)-\bar{T})^2 \\ &+(\omega\varphi+(1-\omega))(m(t)-\bar{m})^2 + (\omega\tau+(1-\omega)\theta)(d(t)-\bar{d})^2\Big] e^{-\rho t} \\ &+\lambda(t)\left[rd(t)+G(t)-T(t)-m(t)\right], \end{aligned} \tag{9}$$

where $\lambda(t)$ is the co-state variable associated with the dynamic government budget constraint. It represents the marginal cost of public funds as perceived by the coalition of policymakers. Assuming $e^{\rho t}\lambda(t) = \mu(t)$, the first-order conditions of this dynamic optimization problem yield the following equations for the control variables:

$$\frac{\partial H}{\partial G} = 0 \quad \Rightarrow \quad (\omega+(1-\omega)\eta)(G(t)-\bar{G})+\mu(t)=0 \quad \Rightarrow \quad G(t) = \frac{-\mu(t)}{\omega+(1-\omega)\eta}+\bar{G}, \tag{10}$$

$$\frac{\partial H}{\partial T} = 0 \quad \Rightarrow \quad (\omega+(1-\omega)\eta)(T(t)-\bar{T})+\mu(t)=0 \quad \Rightarrow \quad T(t) = \frac{\mu(t)}{\omega+(1-\omega)\eta}+\bar{T}, \tag{11}$$

$$\frac{\partial H}{\partial m} = 0 \quad \Rightarrow \quad (\omega\varphi+(1-\omega))(m(t)-\bar{m})-\mu(t)=0 \quad \Rightarrow \quad m(t) = \frac{\mu(t)}{\omega\varphi+(1-\omega)}+\bar{m}. \tag{12}$$

According to (10) and (11)

$$G(t)-T(t) = \frac{-2\mu(t)}{\omega+(1-\omega)\eta}+\bar{G}-\bar{T}. \tag{13}$$

Which indicates a government financial deficit.

As illustrated in equations (10), (11), and (13), the equilibrium equation for government spending $G(t)$ is structurally independent of the equation for government tax revenue $T(t)$, and vice versa. According to the principle of bilateral determination in economic systems, each variable is influenced by distinct sets of determinants. Nevertheless, the outcomes derived from these equations must ultimately be internally consistent within the broader macroeconomic framework.

In essence, the equilibrium conditions for government spending and tax revenue represent interdependent components of the economic system. Although determined separately, they interact dynamically with each other and with other macroeconomic variables, jointly contributing to the overall formation of economic outcomes.

According to the Theorem (Euler-Lagrange):

$$\dot{\lambda}(t) = \rho\lambda(t) - \frac{\partial H}{\partial d} \longrightarrow \dot{\lambda}(t) = [(\omega\tau+(1-\omega)\theta)(d(t)-\bar{d})e^{-\rho t}+r\lambda(t)]. \tag{14}$$

By multiplying the sides of the above equation in $e^{\rho t}$ and putting $e^{\rho t}\lambda(t) = \mu(t)$, the following first-order differential equation is obtained:

$$\dot{\mu}(t) = (\rho-r)\mu(t) - (\omega\tau+(1-\omega)\theta)(d(t)-\bar{d}). \tag{15}$$

The following first-order differential equation is obtained according to the Euler-Lagrange theorem:

$$\frac{\partial H(t)}{\partial \mu(t)} = \dot{d}(t) \longrightarrow \dot{d}(t) = rd(t) + G(t) - T(t) - m(t). \tag{16}$$

With respect to relations (10), (11), and (12):

$$\dot{d}(t) = rd(t) + \frac{-2\mu(t)}{\omega + (1-\omega)\eta} + \frac{-\mu(t)}{\omega\varphi + (1-\omega)} + \bar{G} - \bar{T} - \bar{m}. \tag{17}$$

Theorem 1. *If $(G^*(.), T^*(.), m^*(.)) \in U$ is a set of open-loop Nash strategies for 2,4 and 6, there exist a trajectory for debt $d^*(.)$ and an associated costate variable $\mu^*(.)$ that satisfy the set of nonlinear differential equations:*

$$\dot{d}(t) = rd(t) + \frac{-2\mu(t)}{\omega + (1-\omega)\eta} + \frac{-\mu(t)}{\omega\varphi + (1-\omega)} + \bar{G} - \bar{T} - \bar{m}, \tag{18}$$

$$\dot{\mu}(t) = (\rho - r)\mu(t) - (\omega\tau + (1-\omega)\theta)(d(t) - \bar{d}), \tag{19}$$

with $\dot{d}(0) = d_0$ and where both $\lim_{t\to\infty} d^*(t) = d^e$ and $\lim_{t\to\infty} \mu^*(t) = \mu^e$ exist.

Now, $d^*(t)$ and $\mu^*(t)$, the system of differential equations formed by equations (18) and (19), must be solved. The matrix form of the above system is as follows:

$$\begin{bmatrix} \dot{d}(t) \\ \dot{\mu}(t) \end{bmatrix} = \begin{pmatrix} r & -\frac{2}{k_2} - \frac{1}{k_3} \\ -k_5 & \rho - r \end{pmatrix} \begin{bmatrix} d(t) \\ \mu(t) \end{bmatrix} + \begin{bmatrix} k_4 \\ k_5\bar{d} \end{bmatrix}$$

where $k_2 = \omega + (1-\omega)\eta$, $k_3 = \omega\varphi + (1-\omega)$, $k_4 = \bar{G} - \bar{T} - \bar{m}$, and $k_5 = \omega\tau + (1-\omega)\theta$.

We use the following theorem to solve the system of differential equations (Mahmoudinia et al., [15]).

Theorem 2. *Assume that the matrix A is in the form $A = SDS^{-1}$, with eigenvalues d_1 and d_2, where $D = \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix}$, and $d_1 < 0$ while $\sigma(d_2) \subset C^+$. Then the solution to the differential equation system:*

$$\begin{pmatrix} \dot{d}(t) \\ \dot{\mu}(t) \end{pmatrix} = A \begin{pmatrix} d(t) \\ \mu(t) \end{pmatrix} + \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}; \qquad \begin{pmatrix} d(0) \\ \mu(0) \end{pmatrix} = \begin{pmatrix} d_0 \\ \mu_0 \end{pmatrix},$$

is given by:

$$d(t) = \left(d_0 + [I \quad 0]A^{-1} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \right) e^{d_1 t} - [I \quad 0]A^{-1} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix},$$

and

$$\mu(t) = \alpha e^{d_1 t} [0 \quad I] S \begin{pmatrix} 1 \\ 0 \end{pmatrix} - [0 \quad I]A^{-1} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix},$$

where μ_0, α, and s_{11} are calculated using the following relations:

$$\mu_0 = \alpha[0 \quad I] \left(S \begin{pmatrix} 1 \\ 0 \end{pmatrix} - A^{-1} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \right),$$

$$\alpha = \frac{1}{s_{11}} \left(d_0 + [I \quad 0]A^{-1} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \right),$$

$$s_{11} = [I \quad 0] S \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Now, using Theorem 2, we can obtain the equilibrium levels of government expenses, tax revenue, and the monetary base as follows:

$$d^e(t) = \left(d_0 + \frac{(\rho - r)k_2k_3k_4 + (2k_3 + k_2)(k_5\bar{d})}{\beta_1} \right) e^{d_1 t} - \frac{(\rho - r)k_2k_3k_4 + (2k_3 + k_2)(k_5\bar{d})}{\beta_1}, \tag{20}$$

$$\mu^e(t) = \alpha e^{d_1 t} - \left(\frac{k_2k_3k_4k_5}{\beta_1} + \frac{rk_2k_3k_5\bar{d}}{\beta_1}\right), \tag{21}$$

where

$$d_1 = \frac{\rho k_2k_3 - \sqrt{\beta_2}}{2k_2k_3}, \quad \alpha = -\frac{\rho k_2k_3 - \sqrt{\beta_2} - 2rk_2k_3}{4k_3+2k_2}\left(d_0 + \frac{(\rho-r)k_2k_3k_4 + (2k_3+k_2)(k_5\bar{d})}{\beta_1}\right),$$
$$\beta_1 = rk_2k_3\rho - r^2k_2k_3 - k_5k_2 - 2k_5k_3,$$
$$\beta_2 = \rho^2k_2^2k_3^2 - 4\rho rk_2^2k_3^2 + 4r^2k_2^2k_3^2 + 4k_2^2k_3k_5 + 8k_2k_3^2k_5.$$

Now, according to relations (10)–(13), we have the following equilibrium expressions for government expenditure, tax revenue, and monetary base:

$$G^e(t) = -\frac{\alpha e^{d_1 t} - \left(\frac{k_2k_3k_4k_5}{\beta_1} + \frac{rk_2k_3k_5\bar{d}}{\beta_1}\right)}{\omega + (1-\omega)\eta} + \bar{G}, \tag{22}$$

$$T^e(t) = \frac{\alpha e^{d_1 t} - \left(\frac{k_2k_3k_4k_5}{\beta_1} + \frac{rk_2k_3k_5\bar{d}}{\beta_1}\right)}{\omega + (1-\omega)\eta} + \bar{T}, \tag{23}$$

$$m^e(t) = \frac{\alpha e^{d_1 t} - \left(\frac{k_2k_3k_4k_5}{\beta_1} + \frac{rk_2k_3k_5\bar{d}}{\beta_1}\right)}{\omega\varphi + (1-\omega)} + \bar{m}, \tag{24}$$

$$G^e(t) - T^e(t) = \frac{-2\left(\alpha e^{d_1 t} - \left(\frac{k_2k_3k_4k_5}{\beta_1} + \frac{rk_2k_3k_5\bar{d}}{\beta_1}\right)\right)}{\omega + (1-\omega)\eta} + \bar{G} - \bar{T}. \tag{25}$$

Here, we define the following:

$$\psi_d = -\frac{(\rho-r)k_2k_3k_4 + (2k_3+k_2)(k_5\bar{d})}{\beta_1},$$
$$\psi_G = \frac{\frac{k_2k_3k_4k_5}{\beta_1} + \frac{rk_2k_3k_5\bar{d}}{\beta_1}}{\omega + (1-\omega)\eta} + \bar{G},$$
$$\psi_T = -\left(\frac{\frac{k_2k_3k_4k_5}{\beta_1} + \frac{rk_2k_3k_5\bar{d}}{\beta_1}}{\omega + (1-\omega)\eta}\right) + \bar{T},$$
$$\psi_m = -\left(\frac{\frac{k_2k_3k_4k_5}{\beta_1} + \frac{rk_2k_3k_5\bar{d}}{\beta_1}}{\omega\varphi + (1-\omega)}\right) + \bar{m},$$
$$\psi_{(G-T)} = 2\left(\frac{\frac{k_2k_3k_4k_5}{\beta_1} + \frac{rk_2k_3k_5\bar{d}}{\beta_1}}{\omega + (1-\omega)\eta}\right) + \bar{G} - \bar{T}.$$

Here, ψ_d denotes the steady state of public debt, ψ_G the steady state of government spending, ψ_T the steady state of tax revenue, ψ_m the steady state of the monetary base, and $\psi_{(G-T)}$ the steady state of the fiscal deficit. In addition, d_1 represents the speed of convergence toward the steady state.

2.2. Control Variable Dependency

The two control variables are interdependent and used in place of the function f. Based on the conditions outlined in subsection 2.1, we consider a scenario in which government spending (G) and tax revenue (T) are algebraically dependent. The state equation, the intertemporal loss function of the fiscal policymaker, and the intertemporal loss function of the monetary policymaker are specified as follows:

$$\dot{d} = rd(t) + (G(t) - T(t)) - m(t). \tag{26}$$

The fiscal authority (or Treasury) has the following intertemporal loss function:

$$L_{G,T} = \frac{1}{2}\int_0^\infty \left\{\left((G(t)-\bar{G})-(T(t)-\bar{T})\right)^2 + \varphi(m(t)-\bar{m})^2 + \tau(d(t)-\bar{d})^2\right\} e^{-\delta t}\,\mathrm{d}t,$$
$$L_{G,T} = \frac{1}{2}\int_0^\infty \left\{(G(t)-\bar{G})^2 + (T(t)-\bar{T})^2 - 2(G(t)-\bar{G})(T(t)-\bar{T}) + \varphi(m(t)-\bar{m})^2 + \tau(d(t)-\bar{d})^2\right\} e^{-\delta t}\,\mathrm{d}t. \tag{27}$$

As in subsection 2.1, the parameters φ and τ represent the relative weight that the fiscal policy assigns to the issuance of money by the central bank and public debt, respectively.

Given the goal of both fiscal and monetary policymakers to stabilize public debt, the "bliss point" for tax revenue and government spending for the fiscal policymaker is as follows: $G(t) > \bar{G}$, $T(t) < \bar{T}$. Therefore, the term $(G(t)-\bar{G})(T(t)-\bar{T})$ in the loss function yields a negative number, which does not pose a problem in the minimization objective of the function.

Monetary authorities set the growth of base money so as to minimize the following loss function:

$$L_m = \frac{1}{2}\int_0^\infty \left\{(m(t)-\bar{m})^2 + \eta\left((G(t)-\bar{G})-(T(t)-\bar{T})\right)^2 + \theta(d(t)-\bar{d})^2\right\} e^{-\delta t}\,\mathrm{d}t,$$
$$L_m = \frac{1}{2}\int_0^\infty \left\{(m(t)-\bar{m})^2 + \eta(G(t)-\bar{G})^2 + \eta(T(t)-\bar{T})^2 - 2\eta(G(t)-\bar{G})(T(t)-\bar{T}) + \theta(d(t)-\bar{d})^2\right\} e^{-\delta t}\,\mathrm{d}t.$$

The cooperative game problem can be reformulated as the following simultaneous minimization problem:

$$L^c(\omega) = \omega L_{G,T} + (1-\omega)L_m.$$

According to (27) and (28), we can express $L^c(\omega)$ as:

$$\begin{aligned} L^c(\omega) =& \frac{1}{2}\int_0^\infty \Big\{(\omega+(1-\omega)\eta)(G(t)-\bar{G})^2 + (\omega+(1-\omega)\eta)(T(t)-\bar{T})^2 - 2(\omega+(1-\omega)\eta)(G(t)-\bar{G})(T(t)-\bar{T}) \\ &+ (\omega\varphi+(1-\omega))(m(t)-\bar{m})^2 + (\omega\tau+(1-\omega)\theta)(d(t)-\bar{d})^2\Big\} e^{-\rho t}\,\mathrm{d}t. \end{aligned} \tag{28}$$

The equilibrium is found by minimizing the following present-value Hamiltonian:

$$\begin{aligned} H_2 =& \frac{1}{2}\Big[(\omega+(1-\omega)\eta)(G(t)-\bar{G})^2 + (\omega+(1-\omega)\eta)(T(t)-\bar{T})^2 - 2(\omega+(1-\omega)\eta)(G(t)-\bar{G})(T(t)-\bar{T}) \\ &+ (\omega\varphi+(1-\omega))(m(t)-\bar{m})^2 + (\omega\tau+(1-\omega)\theta)(d(t)-\bar{d})^2\Big] e^{-\rho t} + \lambda(t)\left[rd(t) + (G(t)-T(t)) - m(t)\right]. \end{aligned}$$

Assuming $e^{\rho t}\lambda(t) = \mu(t)$, the first-order conditions of this dynamic optimization problem are:

$$\frac{\partial H_2}{\partial G} = 0 \longrightarrow (\omega+(1-\omega)\eta)(G(t)-\bar{G}) - (\omega+(1-\omega)\eta)(T(t)-\bar{T}) + \mu(t) = 0 \longrightarrow G(t) = \frac{-\mu(t)}{\omega+(1-\omega)\eta} + T(t) + \bar{G} - \bar{T}, \tag{29}$$

$$\frac{\partial H_2}{\partial T} = 0 \longrightarrow (\omega+(1-\omega)\eta)(T(t)-\bar{T}) - (\omega+(1-\omega)\eta)(G(t)-\bar{G}) - \mu(t) = 0 \longrightarrow T(t) = \frac{\mu(t)}{\omega+(1-\omega)\eta} + \bar{T} + G(t) - \bar{G}, \tag{30}$$

$$\frac{\partial H_2}{\partial m} = 0 \longrightarrow (\omega\varphi+(1-\omega))(m(t)-\bar{m}) - \mu(t) = 0 \longrightarrow m(t) = \frac{\mu(t)}{\omega\varphi+(1-\omega)} + \bar{m}. \tag{31}$$

According to relationships (30) and (31):

$$G(t) - T(t) = \frac{-\mu(t)}{\omega+(1-\omega)\eta} + \bar{G} - \bar{T}. \tag{32}$$

According to the Euler-Lagrange theorem and $e^{\rho t}\lambda(t) = \mu(t)$, we have:

$$\begin{aligned} \dot{\lambda}(t) = \rho\lambda(t) - \frac{\partial H}{\partial d} &\longrightarrow \dot{\lambda}(t) = [(\omega\tau+(1-\omega)\theta)(d(t)-\bar{d})e^{-\rho t} + r\lambda(t)] \\ &\longrightarrow \dot{\mu}(t) = (\rho - r)\mu(t) - (\omega\tau+(1-\omega)\theta)(d(t)-\bar{d}). \end{aligned} \tag{33}$$

The following first-order differential equation is obtained according to the theorem (Euler-Lagrange):

$$\frac{\partial H(t)}{\partial \mu(t)} = \dot{d}(t) \longrightarrow \; \dot{d}(t) = rd(t) + (G(t)-T(t)) - m(t). \tag{34}$$

With regard to relationships (32) and (33):

$$\dot{d}(t) = rd(t) + \frac{-\mu(t)}{\omega+(1-\omega)\eta} + \frac{-\mu(t)}{\omega\varphi+(1-\omega)} + \bar{G} - \bar{T} - \bar{m}. \tag{35}$$

Theorem 3. *If $(G^*(.), T^*(.), m^*(.)) \in U$ is a set of open-loop Nash strategies for equations (25), (26), and (27), there exists a trajectory for debt $d^*(.)$ and an associated co-state variable $\mu^*(.)$ that satisfy the set of nonlinear differential equations:*

$$\dot{d}(t) = rd(t) + \frac{-\mu(t)}{\omega + (1-\omega)\eta} + \frac{-\mu(t)}{\omega\varphi + (1-\omega)} + \bar{G} - \bar{T} - \bar{m},$$

$$\dot{\mu}(t) = (\rho - r)\mu(t) - (\omega\tau + (1-\omega)\theta)(d(t) - \bar{d}),$$

with $\dot{d}(0) = d_0$, and where both $\lim_{t\to\infty} d^*(t) = d^e$ and $\lim_{t\to\infty} \mu^*(t) = \mu^e$ exist.

The matrix form of the above system is as follows:

$$\begin{bmatrix} \dot{d}(t) \\ \dot{\mu}(t) \end{bmatrix} = \begin{pmatrix} r & -\frac{1}{k_2} - \frac{1}{k_3} \\ -k_5 & \rho - r \end{pmatrix} \cdot \begin{bmatrix} d(t) \\ \mu(t) \end{bmatrix} + \begin{bmatrix} k_4 \\ k_5 \bar{d} \end{bmatrix},$$

where

$$k_2 = \omega + (1-\omega)\eta, \quad k_3 = \omega\varphi + (1-\omega), \quad k_4 = \bar{G} - \bar{T} - \bar{m}, \quad k_5 = \omega\tau + (1-\omega)\theta.$$

Using Theorem 2, we can obtain the equilibrium levels for government spending, tax revenue, and the monetary base:

$$d^e(t) = \left(d_0 + \frac{(\rho - r)k_2k_3k_4 + (k_3 + k_2)(k_5\bar{d})}{\gamma_1}\right) e^{d_1 t} - \frac{(\rho - r)k_2k_3k_4 + (k_3 + k_2)(k_5\bar{d})}{\gamma_1}, \tag{36}$$

$$\mu^e(t) = \alpha e^{d_1 t} - \left(\frac{k_2k_3k_4k_5 + rk_2k_3k_5\bar{d}}{\gamma_1}\right), \tag{37}$$

where

$$d_1 = \frac{\rho k_2 k_3 - \sqrt{\gamma_2}}{2k_2k_3}, \quad \alpha = -\frac{\rho k_2 k_3 - \sqrt{\gamma_2} - 2rk_2k_3}{2k_3 + 2k_2}\left(d_0 + \frac{(\rho - r)k_2k_3k_4 + (k_3 + k_2)(k_5\bar{d})}{\gamma_1}\right),$$

$$\gamma_1 = \rho r k_2 k_3 - r^2 k_2 k_3 - k_5 k_2 - k_5 k_3,$$

$$\gamma_2 = \rho^2 k_2^2 k_3^2 - 4\rho r k_2^2 k_3^2 + 4r^2 k_2^2 k_3^2 + 4k_2^2 k_3 k_5 + 4k_2 k_3^2 k_5.$$

According to relations (32) and (33):

$$G^e(t) - T^e(t) = -\frac{\alpha e^{d_1 t} - \left(\frac{k_2k_3k_4k_5 + rk_2k_3k_5\bar{d}}{\gamma_1}\right)}{k_2} + \bar{G} - \bar{T}, \tag{38}$$

$$m^e(t) = \frac{\alpha e^{d_1 t} - \left(\frac{k_2k_3k_4k_5 + rk_2k_3k_5\bar{d}}{\gamma_1}\right)}{k_3} + \bar{m}. \tag{39}$$

As expected, since the two variables of government spending and tax revenues have been introduced as a fiscal deficit in the differential game, equations (30) and (31) show that these two equations are linearly dependent. Therefore, the parametric equations of each cannot be calculated separately. That here:

$$\phi_d = -\frac{(\rho - r)k_2k_3k_4 + (k_3 + k_2)(k_5\bar{d})}{\gamma_1}, \tag{40}$$

$$\phi_{(G-T)} = \frac{\frac{k_2k_3k_4k_5 + rk_2k_3k_5\bar{d}}{\gamma_1}}{k_2} + \bar{G} - \bar{T}, \qquad \phi_m = -\frac{\frac{k_2k_3k_4k_5 + rk_2k_3k_5\bar{d}}{\gamma_1}}{k_3} + \bar{m}, \tag{41}$$

ϕ_d represents the stable state of public debt, $\phi_{(G-T)}$ represents the stable state of financial deficit, and ϕ_m represents the state of the monetary base. Also, d_1 is the convergence speed toward the steady state.

Table 1. Parameters

$\bar{G}$	$\bar{T}$	$\bar{m}$	$\bar{d}$	d_0	φ	τ	η	θ	r	ρ	ω
0.149	0.103	0.155	0.9	1.219	0.05	0.05	0.05	0.05	0.034	0.04	0.5

3 Numerical Simulation and Analysis of the Results

In this section, we apply numerical simulation to the model presented in the previous sections. Using real data from the United States, we estimate the equilibrium trajectories based on the models described in subsections 2.1 and 2.2. We assume that the government and the central bank assign equal weights to public debt, tax revenue, government spending, and the monetary base (similar to the values used in Engwerda [10]). Target levels are based on their 20-year average rates from 2003 to 2022, derived from US Federal Reserve Bank data. Additionally, the initial public debt is assumed to be much higher than the target debt level.

As stated in this study, three variablesgovernment spending, tax revenue, and the monetary baseare considered within a cooperative game between the government and the central bank to determine equilibrium paths for stabilizing public debt.

It is assumed that both the government and the central bank assign equal weights in their respective loss functions, meaning they have equal bargaining power; thus, ω is set to 1/2. The results obtained from the numerical simulation of the cooperative gamedefined according to Theorems 1 and 2 in subsection 2.1, where tax revenue and government spending are included separately and independently in loss functions (4), (6), and then in (7)show that the steady-state value of public debt is 0.8993, and the convergence rate toward equilibrium is 0.5246.

In the steady state, government spending, tax revenue, and the monetary base stabilize at 0.1751, 0.0768, and 0.1289, respectively, while the fiscal deficit (the difference between government spending and tax revenue) stabilizes at 0.0983. These dynamics are illustrated in Figures 1 through 5, where the time axis is measured in years and equilibrium values are shown for each control variable.

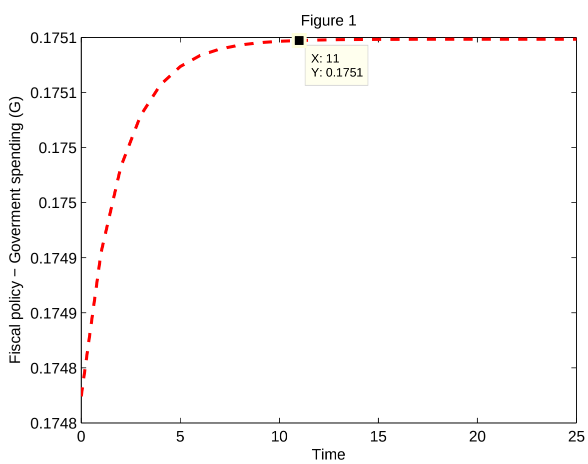

Figure 1. government spending plot.

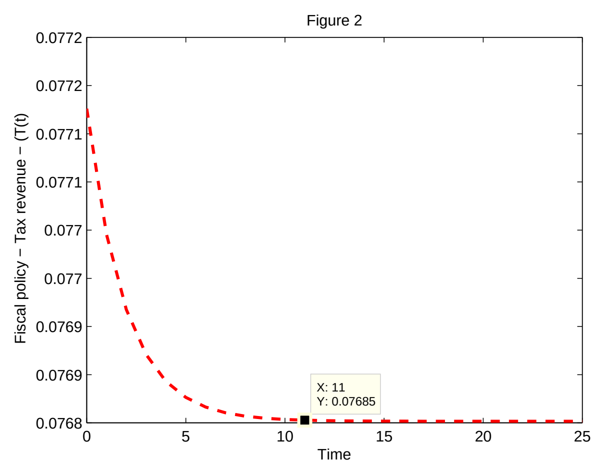

Figure 2. tax revenue plot.

We now consider the steady state and analyze how changes in the assigned weight to fiscal and monetary policymakers (i.e., changes in the value of ω) affect public debt. As illustrated in Figure 6, the minimum level of public debt occurs when $\omega = 0.6$. In this case, public debt reaches 0.8981, while the government surplus, tax revenue, and base money are 0.1717, 0.0802, and 0.1221, respectively. The fiscal deficit is stabilized at 0.0915.

Since the government controls the two independent policy instrumentsgovernment spending and tax revenueassigning a higher weight to the government allows it to better stabilize public debt by adjusting these tools. This approach enables the government to avert potential economic crises and implement policies that boost investor confidence and reduce public debt. Furthermore, the central bank may adjust its monetary policies to align more closely with the governments fiscal policies, thereby improving policy coordination.

In subsection 2.2 of the differential game, it is assumed that two control variablesgovernment spending and tax revenueare

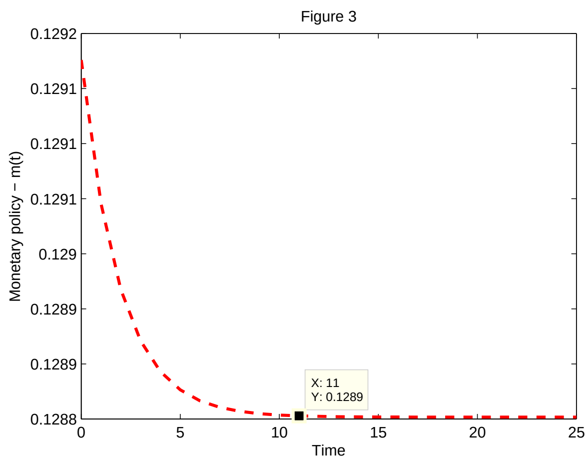

Figure 3. monetary base plot.

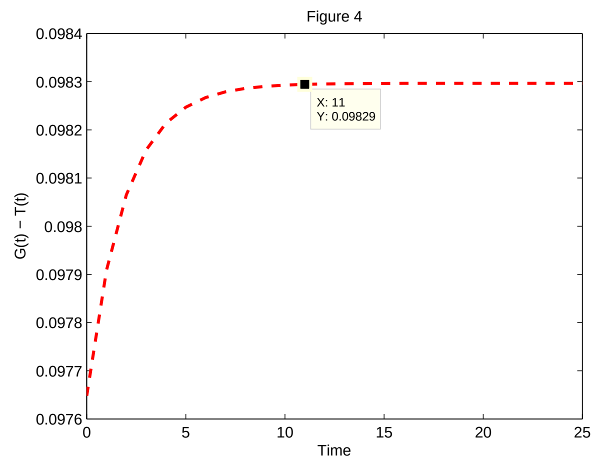

Figure 4. fiscal deficit plot.

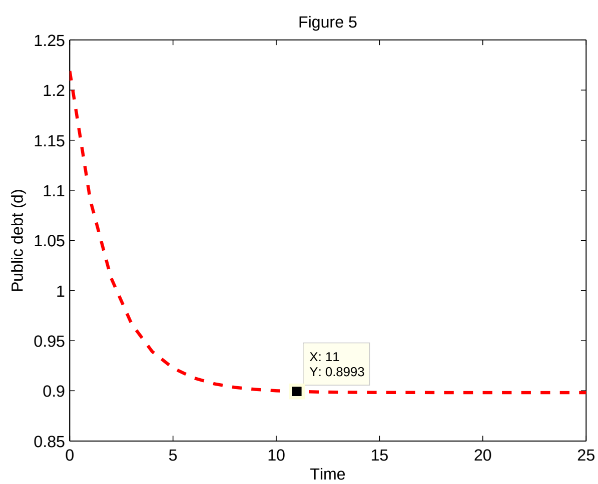

Figure 5. public debt plot.

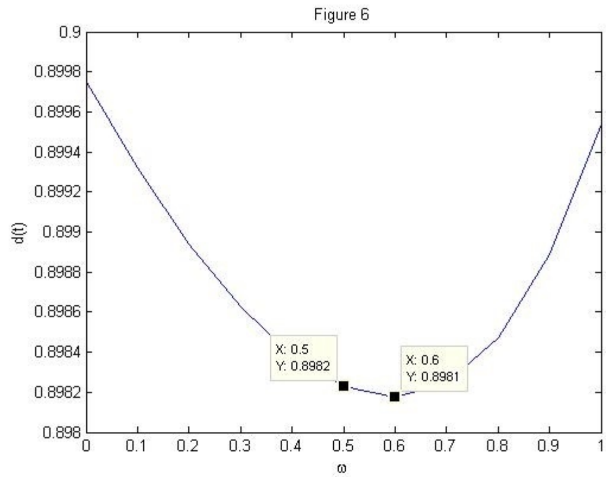

Figure 6. Public debt in steady state across different policy weights plot.

interdependent as a fraction of the financial deficit in the loss functions (27) and (28). Subsequently, in the cooperative game, these variables are included in the loss function (29).

The results of the numerical simulation of the cooperative game, defined according to Theorems 2 and 3, indicate that the steady-state value of public debt is 0.8979, and the convergence rate toward equilibrium is 0.4181. Additionally, in the steady state, the base money value is fixed at 0.1157, and the government fiscal deficit (the difference between government spending and tax revenue) stabilizes at a level of 0.0852 (see Figures 7–9).

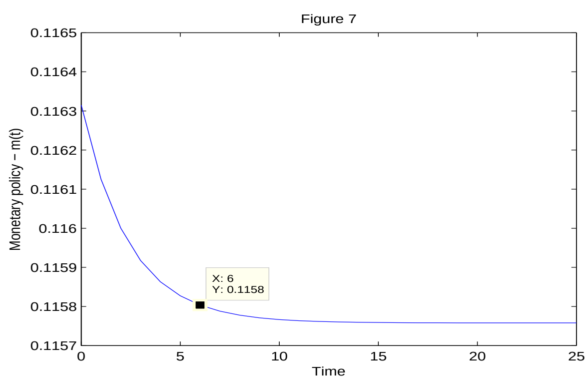

Figure 7. monetary base plot.

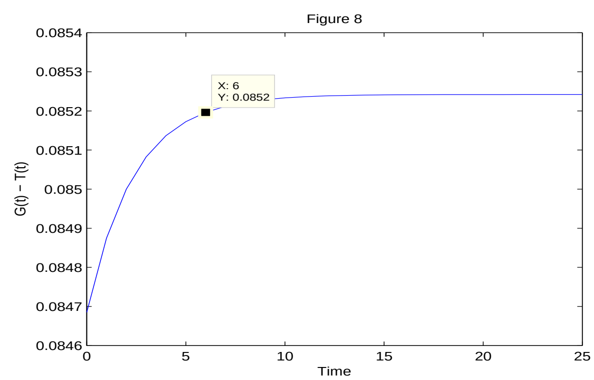

Figure 8. fiscal deficit plot.

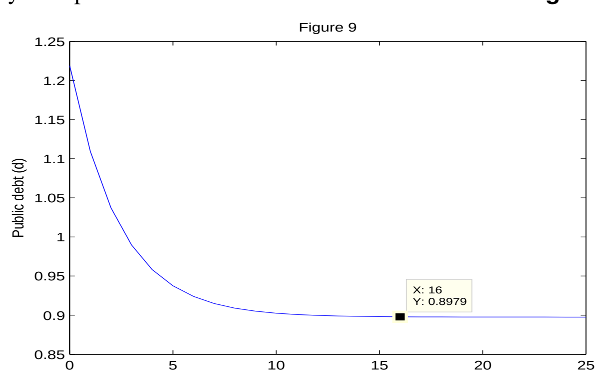

Figure 9. Public debt plot.

As shown in Figure 10, the variation in the weight assigned to the two policymaker positions (fiscal and monetary) results in the minimum stable public debt occurring precisely at $\omega = \frac{1}{2}$.

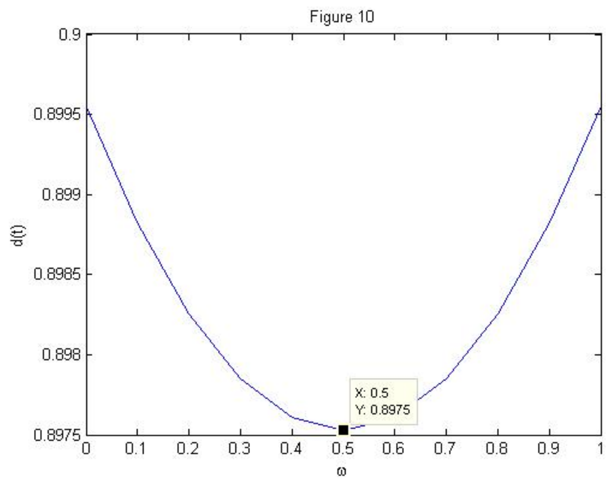

Figure 10. The Public debt in steady state across different policy weights plot.

As previously mentioned, government expenditure and tax revenue variables interact independently with the central bank as part of the government's control mechanisms. This paper examines the interactions between two decision-makersone in monetary policy (the central bank) and the other in fiscal policy (the government)in public debt management within a differential game model. The goal of this modeling approach is to stabilize government debt by balancing three variables: tax revenue, government spending, and the money supply.

The results of the numerical simulation of the cooperative game show that, when the two control variablestax revenue and government spendinginteract independently with the central bank (as government control variables in the government equation), the stable value of public debt is 0.8992. Furthermore, the levels of fiscal deficit and monetary base are fixed at 0.0982 and 0.1288, respectively. On the other hand, when these two control variablestax revenue and government expenditureare included in the government equation in the form of the fiscal deficit, the stable values of public debt, fiscal deficit, and monetary base are fixed at 0.8975, 0.0852, and 0.1157, respectively.

The numerical simulation results for the three game typescooperative, non-cooperative, and Stackelbergbased on the two approaches, are summarized in the tables below.

comparison of Dependent and Independent Control Structures under Non-Cooperative, Cooperative, and Stackelberg Games

1) The two control variables are dependent on each other and are used instead of the function f:

Table 2. Simulation Results with Dependent Fiscal and Monetary Control Variables under Game-Theoretic Regimes

Control Variable	Non-Cooperative	Cooperative	Stackelberg
Fiscal Deficit	0.08528	0.08524	0.04967
Monetary Base	0.011575	0.1158	0.08012
Public Debt	0.89528	0.8979	0.89550
Convergence Rate	0.29653	0.41815	0.42360

1) Entering two independent control variables of government spending and tax revenue in the model independently of each other:

4 Summary and Conclusion

The findings of this research indicate that in the interaction between the government and the central bank, when the government establishes a dependency between the two control variablesgovernment spending and tax revenuesin the form of a fiscal deficit, the level of public debt

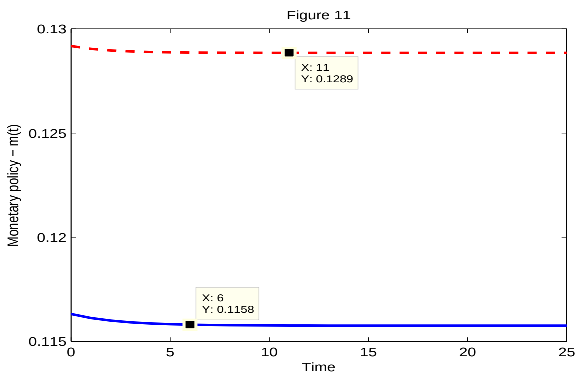

Figure 11. monetary base plot.

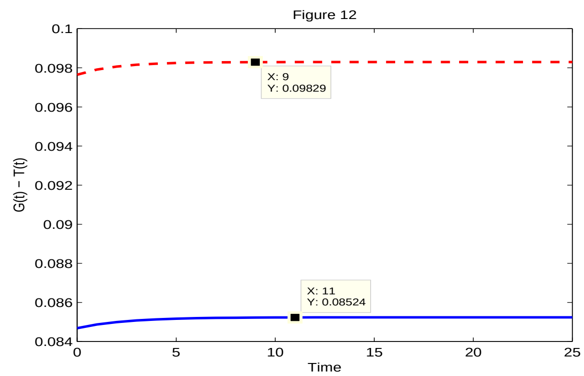

Figure 12. fiscal deficit plot.

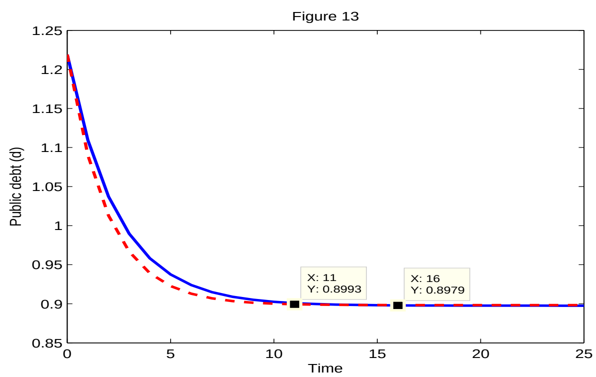

Figure 13. Public debt plot.

Table 3. Simulation Results with Independent Government Spending and Tax Control Variables under Game-Theoretic Regimes.

Control Variable	Non-Cooperative	Cooperative	Stackelberg
Fiscal Deficit	0.09833	0.09829	0.03897
Monetary Base	0.12883	0.012885	0.02847
Public Debt	0.89685	0.89922	0.89570
Convergence Rate	0.37855	0.51620	0.49738

is lower compared to when tax revenues and government spending are independently controlled. In this case, a smaller fiscal deficit emerges, and the need for money issuance by the central bank is also reduced.

Since government debt is considered an essential economic variable, stabilizing it is crucial for maintaining overall economic stability. In this context, a balance among the three variablestax revenue, government spending, and money supplyplays a significant role. If tax revenue falls short of government spending, the government must issue new debt to access the financial resources necessary to cover its spending. This increase in debt tends to raise interest rates and, through expanded money supply, can lead to inflation.

Moreover, an increase in government spending without a corresponding rise in tax revenue contributes to growing public debt and ultimately undermines economic stability. Therefore, to stabilize public debt, government spending should be aligned with the capacity of tax revenue. Excessive money supply can fuel inflation, thereby reducing the national currencys value and the publics purchasing power. Hence, money supply must be carefully managed to support debt stabilization and minimize its adverse economic effects.

Stabilizing government debt requires coordinated policymaking across tax revenue, government spending, and money supply. Maintaining a balance among these three elements is essential to ensure fiscal sustainability. The numerical simulations conducted in this study reveal that when the government engages in a cooperative game with the central bankusing interdependent control variables, namely government spending and tax revenue expressed as financial fractions, the resulting level of public debt is lower than in scenarios where the two variables operate independently. Additionally, this cooperative framework reduces the fiscal burden on the government and lessens the central banks need to issue money (see Figures 11–13).

When the government collaborates with the central bank to manage fiscal deficits through these interdependent variables, it demonstrates improved coordination in fiscal policy. Essentially, this reflects an effort to optimize the balance between revenue and spending. By reducing the necessity for money issuance by the central bank, the government mitigates associated economic costs and contributes to a lower public debt level.

Furthermore, when the government is incorporated into the state equation (2) with two independent control variablesgovernment spending and tax revenuewhile the central bank manages the monetary base, the lowest sustainable level of public debt occurs when the government holds greater bargaining power than the central bank (refer to Figure 6, where $\omega = 0.6$). Simulation results show that in such a scenario, compared to one in which both policymakers have equal bargaining power, government spending decreases, the monetary base contracts, and tax revenue rises. These shifts collectively contribute to a reduction in the fiscal deficit.

Finally, based on the simulation results presented in Tables 2 and 3, the Stackelberg game structure, where the government acts as the leader and the central bank as the follower, yields the most favorable outcomes. In this setting, the equilibrium levels of public debt, fiscal deficit, and monetary base are minimized.

Authors' Contributions

All authors have the same contribution.

Data Availability

The manuscript has no associated data or the data will not be deposited.

Conflicts of Interest

The authors declare that there is no conflict of interest.

Ethical Considerations

The authors have diligently addressed ethical concerns, such as informed consent, plagiarism, data fabrication, misconduct, falsification, double publication, redundancy, submission, and other related matters.

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