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Research article

C_0-Groups and C-Groups on Non-Archimedean Quasi-Banach Spaces

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Abstract

In this paper, we introduce and study C_0-groups and C-groups of bounded linear operators on non-Archimedean quasi-Banach spaces over $\mathbb{K}$. In particular, we show some results related to them. In contrast with the classical framework, the parameter of C_0-groups and C-groups families of bounded linear operators belongs to a open ball Ω_r of a non-Archimedean field $\mathbb{K}$. As an illustration, we shall discuss the solvability of some homogeneous p-adic differential equations for C_0-groups and C-groups. Also, we provide some examples to illustrate our study.

Keywords: Non-Archimedean quasi-Banah spaces, C_0-groups of operators, Groups of contractions, C-groups.

Mathematics Subject Classification (2020): 47D03, 47D09

1 Introduction

In complex operator theory, S. G. Gal and J. A. Goldstein [6] introduced and studied C_0-semigroups and cosine families of continuous linear operators on complex q-Banach spaces where $0 < q < 1$. Recently, J. Ettayb [5] initiated the study of mixed C-cosine families of continuous operators on a complex q-Banach space where $0 < q < 1$. In particular, he demonstrated numerous results on mixed C-cosine families of continuous linear operators on complex q-Banach spaces where $0 < q < 1$. Finally, he gave an application related to the second order abstract Cauchy problem.

In non-Archimedean operator theory, J. Ettayb [4] introduced the free non-Archimedean quasi-Banach space. In particular, he proved several results on non-Archimedean quasi-Banach spaces and he gave numerous examples of such spaces. On the other hand, the uniform boundedness principle, the closed graph theorem, the Banach's open mapping theorem and the bounded inverse theorem for non-Archimedean quasi-Banach spaces were proved. Furthermore, he defined the concepts of closed linear operators, bounded below operators, invertible operators, r-spectral operators, finite rank operators and completely continuous operators on non-Archimedean quasi-Banach spaces and he established several results about them. The spectral theory of bounded linear operators was studied. Finally, the quasi-norm convergence, the quasi-pointwise convergence and the quasi ν-convergence were introduced and studied. Several examples were provided. For further details, see [4]. There are many works on non-Archimedean quasi-Banach spaces, see, e.g. [7, 12].

In contrast with the complex context, the p-adic exponential function

$$e^s = \sum_{j=0}^{+\infty} \frac{s^j}{j!},$$

is not well-defined and analytic for any $s \in \mathbb{Q}_p$ but it converges for any $s \in \mathbb{Q}_p$ such that $|s| < p^{\frac{-1}{p-1}}$ where $\mathbb{Q}_p$ is the field of p-adic numbers. For additional details, see [16].

Throughout this study, $\mathbb{K}$ is a non-Archimedean complete valued field with a non-trivial valuation $|\cdot|$, $\mathscr{E}$ denotes a non-Archimedean quasi-Banach space with the power q, I will denote the identity operator on $\mathscr{E}$, $\mathscr{B}(\mathscr{E})$ is the collection of any bounded linear operators on $\mathscr{E}$ and Ω_r is the open ball centred at zero with radius r that is $\Omega_r = \{s \in \mathbb{K} : |s| < r\}$.

In the present work, we initiate the study of C_0-groups and C-groups of bounded linear operators on non-Archimedean quasi-Banach spaces over a non-Archimedean field $\mathbb{K}$. In particular, we demonstrate several results about them. As an application of C_0-groups of bounded linear operators is the non-Archimedean abstract Cauchy problem for differential equations in a non-Archimedean quasi-Banach space X given by

$$ACP(S;x)\begin{cases} \frac{dw(t)}{dt} = Sw(t), & t \in \Omega_r, \\ w(0) = x, \end{cases}$$

where $S : D(S) \subset \mathscr{E} \to \mathscr{E}$ is a linear operator with $x \in D(S)$. So the problem $ACP(S;x)$ has a solution, see Remark 3.

2 Preliminaries

We continue by recalling a few preliminaries.

Definition 1 ([2]). *A field $\mathbb{K}$ is non-Archimedean if it is equipped with an absolute value $|\cdot| : \mathbb{K} \to \mathbb{R}^+$ such that for any $\lambda, \mu \in \mathbb{K}$,*

(i) $|\lambda| = 0$ *if, and only if,* $\lambda = 0$;

(ii) $|\lambda\mu| = |\lambda||\mu|$;

(iii) $|\lambda + \mu| \leq \max\{|\lambda|, |\mu|\}$.

Definition 2 ([2]). *Let $\mathscr{E}$ be a vector space over $\mathbb{K}$. A function $\|\cdot\| : \mathscr{E} \to \mathbb{R}_+$ is a non-Archimedean norm if for any $u, v \in \mathscr{E}$ and $a \in \mathbb{K}$,*

(i) $\|u\| = 0$ *if and only if* $u = 0$;

(ii) $\|au\| = |a|\|u\|$;

(iii) $\|u + v\| \leq \max(\|u\|, \|v\|)$.

Definition 3 ([2]). *A non-Archimedean normed space is $(\mathscr{E}, \|\cdot\|)$ where $\mathscr{E}$ is a vector space over $\mathbb{K}$ and $\|\cdot\|$ is a non-Archimedean norm on $\mathscr{E}$.*

Definition 4 ([2]). *A non-Archimedean Banach space is a vector space endowed with a non-Archimedean norm, which is complete.*

Proposition 1 ([2]). *(i) A closed subspace of a non-Archimedean Banach space is a non-Archimedean Banach space;*

(ii) The direct sum of two non-Archimedean Banach spaces is a non-Archimedean Banach space.

Definition 5 ([13]). *Let $\mathscr{E}$ be a linear space over $\mathbb{K}$. A function $\|\cdot\|_r : \mathscr{E} \to \mathbb{R}_+$ is called a non-Archimedean quasi-norm with the power r if for any $x, u \in \mathscr{E}$ and for all $a \in \mathbb{K}$,*

(i) $\|u\|_r = 0$ *if and only if* $u = 0$;

(ii) $\|au\|_r = |a|^r\|u\|_r$, *(r real $0 < r < \infty$)*;

(iii) $\|x + u\|_r \leq \max(\|x\|_r, \|u\|_r)$.

The pair $(\mathscr{E}, \|\cdot\|_r)$ *will be called a non-Archimedean quasi-normed space with the power r.*

Definition 6 ([13]). *A complete non-Archimedean quasi-normed space with the power r will be called a non-Archimedean quasi-Banach space with the power r.*

Konda [13] proved the following theorem.

Theorem 1 ([13]). *Let* $(\mathscr{E}, \|\cdot\|_r)$ *and* $(\mathscr{F}, \|\cdot\|_s)$ *be two non-Archimedean quasi-normed spaces over* $\mathbb{K}$ *with powers r and s respectively and let S be a linear operator from* $\mathscr{E}$ *into* $\mathscr{F}$. *Then S is continuous if and only if there exists* $M > 0$ *with*

$$\|Su\|_s \leq M\|u\|_r^{\frac{s}{r}}, \tag{1}$$

for all $u \in \mathscr{E}$. *The collection* $\mathscr{B}(\mathscr{E}, \mathscr{F})$ *denotes the collection of all continuous linear operators from* $\mathscr{E}$ *into* $\mathscr{F}$. *If* $\mathscr{E} = \mathscr{F}$, *we set* $\mathscr{B}(\mathscr{E}, \mathscr{E}) = \mathscr{B}(\mathscr{E})$.

Definition 7. *Let* $(\mathscr{E}, \|\cdot\|_r)$ *and* $(\mathscr{F}, \|\cdot\|_s)$ *be two non-Archimedean quasi-normed spaces with powers r and s respectively. The operator norm of* $S \in \mathscr{B}(\mathscr{E}, \mathscr{F})$ *is defined by*

$$|||S||| = \sup_{u \in \mathscr{E} \setminus \{0\}} \frac{\|Su\|_s}{\|u\|_r^{\frac{s}{r}}}.$$

Definition 8 ([13]). *Let* $(\mathscr{E}, \|\cdot\|_r)$ *and* $(\mathscr{F}, \|\cdot\|_s)$ *be two non-Archimedean quasi-normed spaces with powers r and s respectively. The operator norm of* $S \in \mathscr{B}(\mathscr{E}, \mathscr{F})$ *is defined by*

$$|||S|||' = \sup_{u \in \mathscr{E} : \|u\|_r^{\frac{s}{r}} \leq 1} \|Su\|_s.$$

For $\mathscr{E} = \mathscr{F}$, we conclude the following:

Definition 9. *Let* $(\mathscr{E}, \|\cdot\|_r)$ *be a non-Archimedean quasi-normed space with the power r. The operator norm of* $S \in \mathscr{B}(\mathscr{E})$ *is defined by*

$$\|S\|_r = \sup_{u \in \mathscr{E} \setminus \{0\}} \frac{\|Su\|_r}{\|u\|_r}.$$

Definition 10. *[4] A non-Archimedean quasi-Banach space* $(\mathscr{E}, \|\cdot\|_r)$ *is said to be free if there exists a family* $(f_i)_{i \in I}$ *of* $\mathscr{E}$ *indexed by a set I such that each* $u \in \mathscr{E}$ *can be written uniquely like a pointwise convergent series defined by* $u = \sum_{i \in I} \lambda_i f_i$ *and* $\|u\|_r = \sup_{i \in I} |\lambda_i|^r \|f_i\|_r$. *The family* $(f_i)_{i \in I}$ *is then called a basis for* $\mathscr{E}$. *If for any* $i \in I$, $\|f_i\|_r = 1$, *then* $(f_i)_{i \in I}$ *is called an orthonormal basis of* $\mathscr{E}$.

Example 1. *[4] The space* $c_0(\mathbb{K})$ *is the space of any sequences* $(u_i)_{i \in \mathbb{N}}$ *in* $\mathbb{K}$ *such that* $\lim_{i \to \infty} u_i = 0$. *Hence* $(c_0(\mathbb{K}), \|\cdot\|_r)$ *is a non-Archimedean quasi-Banach space where for any* $(u_i)_{i \in \mathbb{N}} \in c_0(\mathbb{K})$, $\|(u_i)_{i \in \mathbb{N}}\|_r = \sup_{i \in \mathbb{N}} |u_i|^r$.

Definition 11. *[4] An unbounded linear operator S on a non-Archimedean quasi-Banach space* $(\mathscr{E}, \|\cdot\|_r)$ *is a pair* $(D(S), S)$ *consisting of a subspace* $D(S) \subset \mathscr{E}$ *(called the domain of S) and a (possibly not continuous) linear transformation* $S : D(S) \subset \mathscr{E} \to \mathscr{E}$. *The space of any unbounded linear operators on* $\mathscr{E}$ *will be denoted* $U(\mathscr{E})$.

If S is bounded, then $D(S) = \mathscr{E}$. Also, if $S \in U(\mathscr{E})$, then its domain $D(S)$ does not in general coincide with $\mathscr{E}$.

Definition 12. *[4] Let* $(\mathscr{E}, \|\cdot\|_r)$ *be a free non-Archimedean quasi-Banach space with basis* $(f_i)_{i \in \mathbb{N}}$. *An unbounded linear operator S on* $\mathscr{E}$ *is a pair* $(D(S), S)$ *consisting of a subspace* $D(S) \subset \mathscr{E}$ *(called the domain of S) and a (possibly not continuous) linear transformation* $S : D(S) \subset \mathscr{E} \to \mathscr{E}$ *with the domain* $D(S)$ *contains the basis* $(f_i)_{i \in \mathbb{N}}$ *and consists of any* $w = (w_i)_{i \in \mathbb{N}} \in \mathscr{E}$ *with* $Sw = \sum_{i \in \mathbb{N}} w_i S f_i$ *converges in* $\mathscr{E}$ *that is,*

$$D(S) = \{w = (w_i)_{i \in \mathbb{N}} \in \mathscr{E} : \lim_{i \to \infty} |w_i|^r \|Sf_i\|_r = 0\}$$

$$S = \sum_{i,j \in \mathbb{N}} a_{i,j} f'_j \otimes f_i \text{ and } \forall j \in \mathbb{N}, \ \lim_{i \to \infty} |a_{i,j}|^r \|f_i\|_r = 0$$

where $(\forall j \in \mathbb{N})$ $e'_j(w) = w_j$ $(f'_j$ *is the linear form associated with* $f_j)$.

3 Main Results

We start with the next definition.

Definition 13. *Let $\mathscr{E}$ be a non-Archimedean quasi-Banach space with the power q. Let $r > 0$ be a real number chosen such that $(J(t))_{t\in\Omega_r}$ are well-defined . A one-parameter family $((J(t))_{t\in\Omega_r}$ of continuous linear operators on $\mathscr{E}$ is a group of continuous linear operators on $\mathscr{E}$ if*

(i) $J(0) = I$,

(ii) *For any* $t, s \in \Omega_r$, $J(t+s) = J(t)J(s)$.

The group $(J(t))_{t\in\Omega_r}$ is called C_0 or strongly continuous if for any $u \in \mathscr{E}$,

$$\lim_{t\to 0} \|J(t)u - u\|_q = 0. \tag{2}$$

A group $(J(t))_{t\in\Omega_r}$ is uniformly continous if and only if $\lim_{t\to 0} \|J(t) - I\|_q = 0$.
The linear operator S defined by

$$D(S) = \{u \in \mathscr{E} : \lim_{t\to 0} \frac{J(t)u - u}{t} \text{ exists}\},$$

and

$$Su = \lim_{t\to 0} \frac{J(t)u - u}{t}, \text{ for any } u \in D(S),$$

is called the infinitesimal generator of the group $(J(t))_{t\in\Omega_r}$.

Remark 1. *Let $\mathscr{E}$ be a free non-Archimedean quasi-Banach space with the power q.*

(i) Let $(J(t))_{t\in\Omega_r}$ be a group on $\mathscr{E}$ and $(f_i)_{i\in\mathbb{N}}$ denotes the basis for $\mathscr{E}$, hence for any $t \in \Omega_r, J(t)$ can be expressed, for all $u = \sum_{i\in\mathbb{N}} u_i f_i \in \mathscr{E}$, from $J(t)(u) = \sum_{i\in\mathbb{N}} u_i J(t) f_i$, where

$$\forall j \in \mathbb{N},\ J(t)(f_j) = \sum_{i\in\mathbb{N}} a_{i,j}(t) f_i, \text{ with } \lim_{i\to\infty} |a_{i,j}(t)|^q \|f_i\|_q = 0.$$

(ii) Using (i), for any $t \in \Omega_r : t \neq 0$

$$\forall j \in \mathbb{N},\ (\frac{J(t) - I}{t}) f_j = (\frac{a_{j,j}(t) - 1}{t}) f_j + \sum_{i\neq j} \frac{a_{i,j}(t)}{t} f_i,$$

with $\lim_{i\neq j, i\to\infty} |a_{i,j}(t)|^q \|f_i\|_q = 0$.

(iii) If $(J(t))_{t\in\Omega_r}$ is a group on $\mathscr{E}$, hence its infinitesimal generator S may or may not be a continuous linear operator on $\mathscr{E}$.

Example 2. *Suppose that $\mathbb{K} = \mathbb{Q}_p$ and $r = p^{\frac{-1}{p-1}}$, let $\mathscr{E}$ be a free non-Archimedean quasi-Banach space over $\mathbb{Q}_p$ and let $(f_i)_{i\in\mathbb{N}}$ be a base of $\mathscr{E}$. Define for any $t \in \Omega_r$, $u \in \mathscr{E}$ with $u = \sum_{i\in\mathbb{N}} u_i f_i$,*

$$J(t)u = \sum_{i\in\mathbb{N}} e^{t\mu_i} u_i f_i,$$

where $(\mu_i)_{i\in\mathbb{N}} \subset \Omega_r$. The family $(J(t))_{t\in\Omega_r}$ is well-defined on $\mathscr{E}$.

We have the following proposition.

Proposition 2. *The family $(J(t))_{t\in\Omega_r}$ of linear operators given above is a C_0-group of continuous linear operators, whose infinitesimal generator is the continuous diagonal operator S defined by $Su = \sum_{i\in\mathbb{N}} \mu_i u_i f_i$ for any $u = \sum_{i\in\mathbb{N}} u_i f_i \in \mathscr{E}$ where $(\mu_i)_{i\in\mathbb{N}} \subset \Omega_r$.*

Proof. Define for each $t \in \Omega_r$, $i \in \mathbb{N}$,

$$J(t) f_i = e^{t\mu_i} f_i = \left(\sum_{n\in\mathbb{N}} \frac{\mu_i^n t^n}{n!}\right) f_i,$$

where $(\mu_i)_{i\in\mathbb{N}} \subset \Omega_r$. Utilizing for any $i \in \mathbb{N}$, $t\mu_i \in \Omega_r$, we obtain for any $t \in \Omega_r, u \in \mathscr{E}$, $\|J(t)u\|_q \leq \sup_{i\in\mathbb{N}} \left|e^{t\mu_i}\right|^q \|u\|_q < \infty$, then $\left(\forall t \in \Omega_r\right)$ $\|J(t)\|_q$ is finite. Hence the family $(J(t)_{t\in\Omega_r}$ is well-defined on $\mathscr{E}$. Furthermore,

(i) $J(0) = I$,

(ii) For any $t, s \in \Omega_r$,

$$\begin{aligned} J(t)J(s) &= e^{tS}e^{sS} \\ &= e^{(t+s)S} \\ &= J(t+s) \end{aligned}$$

where $Su = \sum_{i\in\mathbb{N}} \mu_i u_i f_i$ for each $u = \sum_{i\in\mathbb{N}} u_i f_i \in \mathscr{E}$.

(iii) For any $u \in \mathscr{E}$, $J(\cdot)u : \Omega_r \to \mathscr{E}$ is continuous on Ω_r.

Thus $(J(t))_{t\in\Omega_r}$ is a C_0-group on $\mathscr{E}$. Let B be the infinitesimal generator of $(J(t))_{t\in\Omega_r}$. It remains to demonstrate that $S = B$. Let us demonstrate that $D(B) = \mathscr{E}\Big(= D(S)\Big)$. Clearly, for any $t \in \Omega_r^*$ and $i \in \mathbb{N}$,

$$\frac{J(t)f_i - f_i}{t} = \Big(\frac{e^{t\mu_i} - 1}{t}\Big)f_i.$$

Thus, for any $t \in \Omega_r^*$ and for all $i \in \mathbb{N}$,

$$\Big(\frac{J(t)f_i - f_i}{t}\Big) = \Big(\frac{e^{t\mu_i} - 1}{t}\Big)f_i.$$

Hence for any $u = \sum_{i\in\mathbb{N}} u_i f_i \in \mathscr{E}$, $t \in \Omega_r^*$,

$$|u_i|^q \left\| \frac{J(t)f_i - f_i}{t} \right\|_q \leq \frac{|u_i|^q \|f_i\|_q}{|t|^q} \to 0 \text{ as } i \to \infty. \tag{3}$$

Then $D(B) = \Big\{ u = (u_i)_{i\in\mathbb{N}} : \lim_{i\to\infty} |u_i|^q \Big\| \Big(\frac{J(t)f_i - f_i}{t}\Big) \Big\|_q = 0 \Big\}$. To complete the proof, it suffices to demonstrate that

$$\Big(\forall i \in \mathbb{N}\Big) \lim_{t\to 0} \Big\| Sf_i - \Big(\frac{J(t)f_i - f_i}{t}\Big) \Big\|_q = 0.$$

Using $\lim_{t\to 0} \Big(\frac{e^{t\mu_i} - 1}{t}\Big) = \mu_i$ and then $S = B$ is the infinitesimal generator of the C_0-group $(J(t))_{t\in\Omega_r}$. □

In the next theorem, $\mathscr{E}$ is a non-Archimedean quasi-Banach space over $\mathbb{Q}_p$ with the power q.

Theorem 2. *Let S be a linear continuous operator on $\mathscr{E}$ such that $\|S\|_q < r$ with $r = p^{\frac{-1}{p-1}}$. Hence S is the infinitesimal generator of an uniformly continuous semigroup of continuous linear operators $(J(t))_{t\in\Omega_r}$.*

Proof. Let $S \in \mathscr{B}(\mathscr{E})$ with $\|S\|_q < r$ and $r = p^{\frac{-1}{p-1}}$. Set for each $s \in \Omega_r$,

$$J(s) = e^{sS} = \sum_{n\in\mathbb{N}} \frac{(sS)^n}{n!}, \tag{4}$$

then $(J(s))_{s\in\Omega_r}$ is an uniformly continuous semigroup. In fact, the series (4) converges in norm and defines a family of continuous linear operators on $\mathscr{E}$ by $|s|^q\|S\|_q < r$ and it is easy to chek that $J(0) = I$ and for any $t, s \in \Omega_r$, $J(s+t) = J(s)J(t)$. It remains to demonstrate that $(J(s))_{s\in\Omega_r}$ given above is a C_0 and uniformly continuous group on $\mathscr{E}$. Indeed, $(\forall s \in \Omega_r^*)$ one has $J(s) - I = sS(\sum_{n\in\mathbb{N}} \frac{(sS)^n}{(n+1)!})$, hence for all $x \in \mathscr{E}$, $\|J(s)x - x\|_q \leq |s|^q\|S\|_q\|\zeta_s x\|_q$ where $\zeta_s = \sum_{n\in\mathbb{N}} \frac{(sS)^n}{(n+1)!}$. Then $(J(s))_{s\in\Omega_r}$ is a C_0-group on $\mathscr{E}$. The uniformly continuous property results by $\|J(s) - I\|_q \leq |s|^q\|S\|_q\|\zeta_s\|_q$ where $\zeta_s = \sum_{n\in\mathbb{N}} \frac{(sS)^n}{(n+1)!}$, then

$$\lim_{s\to 0} \|J(s) - I\|_q = 0. \tag{5}$$

Now for all $s \in \Omega_r^*$,

$$\left\|\frac{J(s)-I}{s}-S\right\|_q = \left\|\frac{e^{sS}-I}{s}-S\right\|_q \leq |s|^q \|S\|_q \|\xi_s\|_q,$$

where $\xi_s = \sum_{n=0}^{\infty} \frac{s^n S^{n+1}}{(n+2)!}$ converges. Consequently,

$$\lim_{s\to 0} \|\frac{J(s)-I}{s} - S\|_q = 0. \tag{6}$$

Hence, $(J(s))_{s\in\Omega_r}$ given above is a C_0 and uniformly continuous group of continuous linear operators whose infinitesimal generator is S. □

Remark 2. *(i) Note that the mapping $\Omega_r \longrightarrow \mathscr{B}(\mathscr{E}), t \mapsto J(t) = e^{tS}$ is analytic. So $\frac{dJ(t)}{dt} = SJ(t) = J(t)S$.*

(ii) If char$(\mathbb{K}) = 0$ and char$(k) = p$, then the Theorem 2 remains valid.

(iii) If char$(\mathbb{K}) = 0$ and char$(k) = 0$, then the Theorem 2 remains valid when $r = 1$.

Example 3. *Let S be the multiplication operator on $\mathscr{E} = C(\mathbb{Z}_p, \mathbb{Q}_p)$ defined by for any $(w \in C(\mathbb{Z}_p, \mathbb{Q}_p))$ $Sw = Q(x)w$, $w(0) = w_0$ where $Q = \sum_{n=0}^{\infty} s_n f_n(x) \in C(\mathbb{Z}_p, \mathbb{Q}_p)$, $s_n \in \mathbb{Q}_p$, assume that $\|Q\|_q = \sup_n |s_n|^q < r\left(= p^{\frac{-1}{p-1}}\right)$. Utilizing (i) of Remark 2, the function defined by $(\forall t \in \Omega_r)$ $w(t) = \sum_{n\in\mathbb{N}} \left(\frac{(tS)^n}{n!}\right) w_0$, for certain $w_0 \in \mathscr{E}$ is the solution to the homogenuous p-adic differential equation*

$$\begin{cases} \frac{d}{dt} w(t) = Q(t)w(t), & t \in \Omega_r, \\ w(0) = w_0. \end{cases}$$

Theorem 3. *Let $\mathscr{E}$ be a non-Archimedean quasi-Banach space over $\mathbb{Q}_p$ with the power q. Let $(J(s))_{s\in\Omega_r}$ be a C_0-group on $\mathscr{E}$ with for any $s \in \Omega_r, \|J(s)\| \leq M$, where $M > 0$ and let S be its infinitesimal generator. Then, for all $u \in D(S)$, $J(s)u \in D(S)$ for any $s \in \Omega_r$. Also*

$$\frac{dJ(s)}{ds} u = SJ(s)u = J(s)Su.$$

Proof. Let $u \in D(S), t \in \Omega_r^*$ and $s \in \Omega_r$. Utilizing the Definition 13 and the boundedness of the C_0-group $(J(s))_{s\in\Omega_r}$, hence

$$\frac{J(t)-I}{t} J(s)u = J(s)\frac{J(t)-I}{t} u \to J(s)Su \text{ while } t \to 0. \tag{7}$$

Consequently, $J(s)Su \in D(S)$ and $SJ(s)u = J(s)Su$ from (7). Furthermore, since $J(s)(\frac{J(t)-I}{t})u \to J(s)Su$ while $t \to 0$, it follows that the right derivate of $J(s)u$ is $J(s)Su$. Thus, to complete the proof, we demonstrate that for each $s \in \Omega_r^*$, the left derivate of $J(s)u$ exists and is $J(s)Su$. We have:

$$\lim_{t\to 0} \frac{J(s)u - J(s-t)u}{t} - SJ(s)u = \lim_{t\to 0} (J(s-t))(\frac{J(t)u-u}{t} - Su) + \lim_{t\to 0} (J(s-t)Su - J(s)Su). \tag{8}$$

Clearly,

$$\lim_{t\to 0} (J(s-t))(\frac{J(t)u-u}{t} - Su) = 0,$$

by for certain M and any $t \in \Omega_r, \|J(t)\| \leq M$. Utilizing the strong continuity of the group $(J(s))_{s\in\Omega_r}$, it deduces that

$$\lim_{t\to 0} (J(s-t)Su - J(s)Su) = 0.$$

Consequently,

$$\lim_{t\to 0} (\frac{J(s)u - J(s-t)u}{t} - SJ(s)u) = 0,$$

and so the left derivate of $J(s)u$ $J(s)Su$. This completes the proof. □

The C_0-groups can be applied to the several p-adic differential equations that may be modeled as a p-adic abstract Cauchy problem on a non-Archimean quasi-Banach space, thanks to Theorem 3, we get:

Remark 3. *One of the consequences of Theorem 3 is that the function $v(s) = J(s)x$, $s \in \Omega_r$ for certain $x \in D(S)$, is the solution to the homogeneous p-adic differential equation given by*

$$\begin{cases} \frac{du(s)}{ds} = Su(s), \quad s \in \Omega_r, \\ u(0) = x, \end{cases}$$

where $S : D(S) \subset \mathscr{E} \to \mathscr{E}$ is the infinitesimal generator of the C_0-group $(J(s))_{s\in\Omega_r}$ and $u : \Omega_r \to D(S)$ is an $\mathscr{E}$-valued function.

We continue with the following example:

Example 4. *Let $\mathscr{E}$ be a non-Archimedean quasi-Banach space over $\mathbb{Q}_p$ with the power q and $S \in \mathscr{B}(\mathscr{E})$ with $\|S\|_q < r$ and $r = p^{\frac{-1}{p-1}}$, hence for any $s \in \Omega_r$, $J(s) = e^{sS} = \sum_{n\in\mathbb{N}} \frac{(sS)^n}{n!}$ satisfied the conditions of Definition 13, we will demonstrate that, for any $s \in \Omega_r$,*

$$J(s) = \sum_{n\in\mathbb{N}} \frac{(sS)^n}{n!}. \tag{9}$$

Clearly, the series (9) converges in norm and defines a family of continuous linear operators on $\mathscr{E}$ by $|s|^q\|S\|_q < r$ and it is easy to chek that $J(0) = I$ and for any $t, s \in \Omega_r$, $J(s+t) = J(s)J(t)$. It remains to demonstrate that $(J(s))_{s\in\Omega_r}$ given above is a C_0 and uniformly continuous group. Indeed, $(\forall s \in \Omega_r^*)$ one has

$$J(t) - I = tS(\sum_{n\in\mathbb{N}} \frac{(tS)^n}{(n+1)!}),$$

hence for all $x \in \mathscr{E}$,

$$\|J(t)x - x\|_q \leq |t|^q\|S\|_q\|\zeta_t x\|_q,$$

where

$$\zeta_t = \sum_{n\in\mathbb{N}} \frac{(tS)^n}{(n+1)!},$$

and thus $(J(t))_{t\in\Omega_r}$ is a C_0-group. The uniformly continuous property results by

$$\|J(t) - I\|_q \leq |t|^q\|S\|_q\|\zeta_t\|_q,$$

where

$$\zeta_t = \sum_{n\in\mathbb{N}} \frac{(tS)^n}{(n+1)!},$$

then

$$\lim_{t\to 0} \|J(t) - I\|_q = 0. \tag{10}$$

Now for all $t \in \Omega_r^*$,

$$\left\|\frac{J(t) - I}{t} - S\right\|_q = \left\|\frac{e^{tS} - I}{t} - S\right\|_q \leq |t|^q\|S\|_q\|\xi_t\|_q$$

where $\xi_t = \sum_{n=0}^{\infty} \frac{t^n S^{n+1}}{(n+2)!}$ converges. Consequently,

$$\lim_{t\to 0} \|\frac{J(t) - I}{t} - S\|_q = 0. \tag{11}$$

Hence, $(J(t))_{t\in\Omega_r}$ given above is a C_0 and uniformly continuous group of continuous linear operators whose infinitesimal generator is S.

Proposition 3. *Let $\mathscr{E}$ be a non-Archimedean quasi-Banach space over $\mathbb{Q}_p$ with the power q and $S, B \in \mathscr{B}(\mathscr{E})$ with $\max\{\|S\|_q, \|B\|_q\} < r$ and $SB = BS$ where $r = p^{\frac{-1}{p-1}}$. We set $J(t) = e^{tS}$ and $S(t) = e^{tB}$ for each $t \in \Omega_r$. Then we have:*

(i) $J(t)S(t) = S(t)J(t)$ for any $t \in \Omega_r$,

(ii) For every $x \in \mathscr{E}$, $\frac{dW(t)}{dt}x = (S+B)x$ where $W(t) = J(t)S(t)$ for any $t \in \Omega_r$.

Proof. (i) Since $SB = BS$, then for all $t \in \Omega_r$, we have

$$\begin{aligned} J(t)S(t) &= \sum_{k=0}^{\infty} \frac{t^k S^k}{k!} \cdot \sum_{k=0}^{\infty} \frac{t^k B^k}{k!} \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{t^{n-k} S^{n-k}}{(n-k)!} \cdot \frac{t^k B^k}{k!} \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{C_n^k t^n S^{n-k} B^k}{n!} \\ &= \sum_{n=0}^{\infty} \frac{t^n (S+B)^n}{n!} \\ &= S(t)J(t). \end{aligned}$$

(ii) Utilizing (i), we get $W(t) = e^{t(S+B)}$ for any $t \in \Omega_r$, so for all $h \in \Omega_r^*$ and $x \in \mathscr{E}$, we obtain

$$\begin{aligned} \frac{W(t+h)x - W(t)x}{h} &= \frac{W(t)W(h)x - W(t)x}{h} \\ &= W(t)\frac{W(h)x - x}{h} \\ &= \frac{W(h) - I}{h} W(t)x. \end{aligned}$$

Utilizing Example 4, $(W(t))_{t\in\Omega_r}$ is a C_0-group of generator $S+B$. Since $S+B$ is bounded, then $D(S+B) = \mathscr{E}$. Consequently, for all $x \in \mathscr{E}$,

$$\lim_{h\to 0} \frac{W(h)x - x}{h} = (S+B)x.$$

Hence, for each $x \in \mathscr{E}$,

$$\frac{dW(t)}{dt} x = \lim_{h\to 0} \frac{W(t+h)x - W(t)x}{h} = (S+B)W(t)x.$$

□

Lemma 1. *Let $\mathscr{E}$ be a non-Archimedean quasi-Banach space over $\mathbb{Q}_p$ with the power q. Let $S \in \mathscr{B}(\mathscr{E})$ be invertible and $S, B \in \mathscr{B}(\mathscr{E})$ such that $\max\{\|S\|_q, \|B\|_q\} < r = p^{\frac{-1}{p-1}}$. We put $S = W^{-1}BW$, then for each $t \in \Omega_r$, $e^{tS} = W^{-1}e^{tB}W$.*

Proof. Since $S^k = W^{-1}B^kW$ for all $k \in \mathbb{N}$ and since W and W^{-1} are continuous operators, we obtain

$$\begin{aligned} e^{tS} &= \sum_{k=0}^{\infty} \frac{t^k S^k}{k!} \\ &= \sum_{k=0}^{\infty} \frac{t^k W^{-1} B^k W}{k!} \\ &= W^{-1} \left(\sum_{k=0}^{\infty} \frac{t^k B^k}{k!} \right) W \\ &= W^{-1} e^{tB} W. \end{aligned}$$

□

Remark 4. *Let $\mathscr{E}$ be a non-Archimedean quasi-Banach space over $\mathbb{Q}_p$ with the power q. Let $S \in \mathscr{B}(\mathscr{E})$, then for any $t \in \Omega_r$, $e^{tS} = \sum_{k=0}^{\infty} \frac{t^k S^k}{k!}$ where $r = \frac{p^{\frac{-1}{p-1}}}{r(S)}$ and $r(S) = \lim_{n\longrightarrow\infty} \|S^n\|^{\frac{1}{nq}}$.*

Definition 14. *Let $\mathscr{E}$ be a non-Archimedean quasi-Banach space over $\mathbb{K}$ with the power q. Let $(J(t))_{t\in\Omega_r}$ be a group of operators on $\mathscr{E}$, we set*

$$Y = \{x \in \mathscr{E} : \lim_{t\longrightarrow 0} \|J(t)x - x\|_q = 0\}.$$

Proposition 4. *Let $\mathscr{E}$ be a non-Archimedean quasi-Banach space over $\mathbb{K}$ with the power q. Let $(J(t))_{t\in\Omega_r}$ be a group of operators on $\mathscr{E}$. Then*

(i) Y is a vector subspace of $\mathscr{E}$.

(ii) If $(J(t))_{t\in\Omega_r}$ is a C_0-group on $\mathscr{E}$, then $Y=\mathscr{E}$.

Proof. (i) $0\in Y$, then $Y\neq\emptyset$, it is easy to check that for each $x,y\in Y$ and $\lambda\in\mathbb{K}$, we get $\lambda x+y\in Y$.

(ii) Utilizing (i), $Y\subseteq\mathscr{E}$ for the opposite inclusion, let $x\in\mathscr{E}$, by assumption $(J(t))_{t\in\Omega_r}$ is a C_0-group on $\mathscr{E}$, then for each $x\in\mathscr{E}$, $\lim\limits_{t\longrightarrow 0}\|J(t)x-x\|_q=0$. Consequently, $x\in Y$. Hence $Y=\mathscr{E}$.

□

Theorem 4. *Let $\mathscr{E}$ be a non-Archimedean quasi-Banach space over $\mathbb{K}$ with the power q. Let $(J(s))_{s\in\Omega_r}$ be a C_0-group on $\mathscr{E}$. There exist C and $\delta>0$ such that for any $s\in\Omega_\delta$ with $\delta<r$, $\|J(s)\|_q\leq C$.*

Proof. We demonstrate that there is $\delta<r$ with $\|J(s)\|_q$ is bounded for any $0\leq|s|\leq\delta$. If this is false, then there is a sequence $(s_n)_n$ satisfying $(\forall n\in\mathbb{N})s_n\in\Omega_\delta$, $\lim\limits_{n\longrightarrow\infty}s_n=0$ and $\|J(s_n)\|_q\geq n$. Utilizing the uniform boundedness theorem, it follows that for certain $x\in\mathscr{E}$, $\|J(s_n)x\|_q$ is unbounded contrary to (2). Then $\|J(t)\|_q\leq C$ for $s\in\Omega_\delta$. □

Definition 15. *Let $\mathscr{E}$ be a non-Archimedean quasi-Banach space over $\mathbb{K}$ with the power q. Let $(J(s))_{s\in\Omega_r}$ be a C_0-group on $\mathscr{E}$, $(J(s))_{s\in\Omega_r}$ is called a group of contractions on $\mathscr{E}$ if for any $s\in\Omega_r$, $\|J(s)\|_q\leq 1$.*

Lemma 2. *Let $\mathscr{E}$ be a non-Archimedean quasi-Banach space over $\mathbb{K}$ with the power q. Let $(J(s))_{s\in\Omega_r}$ be a group of contractions. Then for all $u\in\mathscr{E}$, the function $s\to J(s)u$ is continuous from Ω_r into $\mathscr{E}$.*

Proof. Let $s,h\in\Omega_r$ and $x\in\mathscr{E}$. The continuity of $s\mapsto J(s)u$ follows from

$$\|J(s+h)u-J(s)u\|_q\leq\|J(s)\|_q\|J(h)u-u\|_q$$

and

$$\|J(s-h)u-J(s)u\|_q\leq\|J(s-h)\|_q\|u-J(h)u\|_q$$

while $h\to 0$. □

Proposition 5. *Let $\mathscr{E}$ be a non-Archimedean quasi-Banach space over $\mathbb{K}$ with the power q. If $(J(s))_{s\in\Omega_r}$ is a C_0-group of contractions on $\mathscr{E}$. Hence for any $s\in\Omega_r$, $\|J(s)\|_q=1$.*

Let S be an infinitesimal generator of a C_0-group $(J(s))_{s\in\Omega_r}$ on $\mathscr{E}$ satisfying $(\forall s\in\Omega_r)\|J(s)\|_q\leq M$, we define, for any $u\in\mathscr{E}$, $|u|_1=\sup\limits_{s\in\Omega_r}\|J(s)u\|_q$. We conclude the next proposition:

Theorem 5. *Let $\mathscr{E}$ be a non-Archimedean quasi-Banach space over $\mathbb{K}$ with the power q. Let S be the infinitesimal generator of a C_0-group $(J(s))_{s\in\Omega_r}$ on $\mathscr{E}$ with for any $s\in\Omega_r$, $\|J(s)\|_q\leq M$, then $|\cdot|_1$ is a non-Archimedean quasi-norm on $\mathscr{E}$ which is equivalent to the original quasi-norm $\|\cdot\|_q$ on $\mathscr{E}$ and $(J(s))_{s\in\Omega_r}$ is a C_0-group of contractions on $\mathscr{E}$ equipped with the quasi-norm $|\cdot|_1$.*

Proof. We have $\|J(0)\|_q=1$ and $(\forall t\in\Omega_r)$ $\|J(t)\|_q\leq M$, then $(\forall x\in\mathscr{E})\,\|x\|_q\leq|x|_1\leq M\|x\|_q$. Hence $|\cdot|_1$ is a quasi-norm on $\mathscr{E}$ which is equivalent to the original quasi-norm $\|\cdot\|_q$ on $\mathscr{E}$. Furthermore, for all $x\in\mathscr{E}$ and for each $t\in\Omega_r$, $|J(t)x|_1=\sup\limits_{s\in\Omega_r}\|J(s)J(t)x\|_q\leq\sup\limits_{s\in\Omega_r}\|J(s)x\|_q=|x|_1$. □

We have the following lemma:

Lemma 3. *Let $\mathscr{E}$ be a non-Archimedean quasi-Banach space over $\mathbb{K}$ with the power q. Let T be a continuous linear operator on $\mathscr{E}$ over $\mathbb{Q}_p$ such that $\|T\|_q<r$ where $r=p^{\frac{-1}{p-1}}$. Then for every $x\in\mathscr{E}$ and all $t\in\Omega_r$, $\|e^{t(T-I)}x-T^n\|_q\leq\|x-Tx\|_q$*

Proof. Let $k,n \geq 0$ be two integers. If $k \geq n$, then for all $x \in \mathscr{E}$, we have

$$\begin{aligned} \|T^k x - T^n x\|_q &= \| \sum_{j=n}^{k-1} \left(T^{j+1}x - T^j x \right) \|_q \\ &\leq \max_{n \leq j \leq k-1} \|T^j\|_q \|x - Tx\|_q \\ &\leq \|x - Tx\|_q. \end{aligned}$$

Hence

$$\|T^k x - T^n x\|_q \leq \|x - Tx\|_q. \quad (12)$$

From the symmetry of the estimate with respect to k and n, it is clear that (12) holds also for $n > k$. For $k = n$, we have equality, and therefore (12) is valid for all integers $k,n \geq 0$. Now, let $t \in \Omega_r$ and $k,n \geq 0$, we have for all $x \in \mathscr{E}$,

$$\begin{aligned} \|e^{t(T-I)}x - T^n x\|_q &= \|e^{-t} \sum_{k=0}^{\infty} \frac{t^k}{k!} (T^k x - T^n x)\|_q \\ &\leq \|e^{-t} \sum_{k=0}^{\infty} \frac{t^k}{k!} \|_q \|_q T^k x - T^n x\|_q \\ &\leq \|x - Tx\|_q. \end{aligned}$$

□

We can see easily the following lemma.

Lemma 4. *Let $\mathscr{E}$ be a non-Archimedean quasi-Banach space over $\mathbb{K}$ with the power q. Let $(J(s))_{s\in\Omega_r}$ be a C_0-group of infinitesimal generator S on $\mathscr{E}$. For all $s \in \Omega_r$, the range space and the null space for $J(s)$ are respectively: $R(J(s)) = \{J(s)u : u \in \mathscr{E}\} = \mathscr{E}$ and $N(J(s)) = \{u \in \mathscr{E} : J(s)u = 0\} = \{0\}$.*

Definition 16. *Let $\mathscr{E}$ be a non-Archimedean quasi-Banach space over $\mathbb{K}$ with the power q. A C_0-group $(J(s))_{s\in\Omega_r}$ of infinitesimal generator S on $\mathscr{E}$ is called differentiable at $s \in \Omega_r^*$ if for all $u \in \mathscr{E}$, the mapping $s \mapsto S(s)u$ is differentiable at s. The C_0-group $(J(s))_{s\in\Omega_r}$ is called differentiable if it is differentiable at any $s \in \Omega_r^*$.*

Theorem 6. *Let $\mathscr{E}$ be a non-Archimedean quasi-Banach space over $\mathbb{K}$ with the power q. Let $(J(s))_{s\in\Omega_r}$ be a differentiable C_0-group of infinitesimal generator S on $\mathscr{E}$. Then $R(J(s)) \subset D(S)$ for all $s \in \Omega_r^*$.*

Proof. Utilizing $(J(s))_{s\in\Omega_r}$ is differentiable, then for all $u \in \mathscr{E}$, the mapping $s \mapsto S(s)u$ is differentiable on Ω_r^*. Hence $\lim_{h\to 0} \frac{J(s+h)u - J(s)u}{h}$ exists in $\mathscr{E}$. Consequently, $R(J(s)) \subset D(S)$ for all $s \in \Omega_r^*$. □

Theorem 7. *Let $\mathscr{E}$ be a non-Archimedean quasi-Banach space over $\mathbb{K}$ with the power q. Let $(J(s))_{s\in\Omega_r}$ be a C_0-group of contractions of infinitesimal generator S on $\mathscr{E}$ such that for any $s \in \Omega_r^*, R(J(s)) \subset D(S)$, then for all $u \in \mathscr{E}$, $\frac{dJ(s)}{ds}u = SJ(s)u = J(s)Su$.*

Proof. Let $u \in \mathscr{E}$ and $s \in \Omega_r^*$. Utilizing $J(s)u \in D(S)$, then

$$\begin{aligned} \frac{dJ(s)u}{ds} &= \lim_{h\to 0} \frac{J(s+h)u - J(s)u}{h} \\ &= \lim_{h\to 0} J(s) \frac{J(h)u - u}{h} \\ &= \lim_{h\to 0} \frac{J(h) - I}{h} J(s)u \\ &= SJ(s)u \\ &= J(s)Su. \end{aligned}$$

□

Remark 5. *Since $(J(s))_{s\in\Omega_r}$ is a C_0-group on $\mathscr{E}$, we have $R(J(s)) = \mathscr{E}$ for all $s \in \Omega_r$. From Theorem 6, $D(S) = \mathscr{E}$. This shows that the infinitesimal generator of a differentiable C_0-group is bounded.*

We define the next definition:

Definition 17. *Let $\mathscr{E}$ and $\mathscr{F}$ two non-Archimedean quasi-Banach spaces over $\mathbb{K}$ with the power q. For any $A \in \mathscr{B}(\mathscr{E})$ and $S \in \mathscr{B}(\mathscr{E})$, the operator $A \oplus S$ is defined on $\mathscr{E} \oplus \mathscr{F} = \{(u,v) : u \in \mathscr{E}, v \in \mathscr{F}\} = \{u \oplus v : u \in \mathscr{E},\ v \in \mathscr{F}\}$ endowing with the non-Archimedean quasi-norm $\|u \oplus v\|_q = \max(\|u\|_q, \|v\|_q)$ by*

$$(\forall u \oplus v \in \mathscr{E} \oplus \mathscr{F}), (A \oplus S)(u \oplus v) = Au \oplus Sv = (Au, Sv).$$

We get:

Theorem 8. *Let $\mathscr{E}$ be a non-Archimedean quasi-Banach space over $\mathbb{K}$ with the power q. Let $(J(s))_{s\in\Omega_r}$ be a C_0-group of generator J on $\mathscr{E}$. Let $S(t) = J(t) \oplus I$ for all $t \in \Omega_r$. Then we have*

(i) $(S(s))_{s\in\Omega_r}$ *is a C_0-group on $\mathscr{E} \oplus \mathscr{E}$,*

(ii) The generator of $(S(s))_{s\in\Omega_r}$ is the operator S defined on $D(S) = D(J) \oplus \mathscr{E}$ such that for any $u \in D(J)$, $v \in \mathscr{E}$, $S(u \oplus v) = Ju \oplus 0$.

Proof. (i) Utilizing $(J(s))_{s\in\Omega_r}$ is a C_0-group of generator J on $\mathscr{E}$, hence

$$S(0) = J(0) \oplus I = I \oplus I = I_{\mathscr{E}\oplus\mathscr{E}}.$$

Let $u \oplus v \in \mathscr{E} \oplus \mathscr{E}$ and $t, s \in \Omega_r$, we obtain

$$\begin{aligned} S(t+s)(u \oplus v) &= J(t+s)(u) \oplus v \\ &= J(t)J(s)(u) \oplus v \\ &= (J(t) \oplus I)(J(s)(u) \oplus v) \\ &= S(t)((J(s) \oplus I)(u \oplus v)) \\ &= S(t)S(s)(u \oplus v). \end{aligned}$$

Also

$$\begin{aligned} \lim_{s\to 0} \|S(s)(u \oplus v) - u \oplus v\|_q &= \lim_{s\to 0} \|(J(s)u - u) \oplus 0\|_q \\ &= \lim_{s\to 0} \max\left(\|J(s)u - u\|_q, 0\right) \\ &= \lim_{s\to 0} \|J(s)u - u\|_q \\ &= 0. \end{aligned}$$

So $(S(s))_{s\in\Omega_r}$ is a C_0-group on $\mathscr{E} \oplus \mathscr{E}$.

(ii) Let $u \in D(J)$ and $v \in \mathscr{E}$, we get

$$\begin{aligned} \lim_{s\to 0} \frac{S(s)(u \oplus v) - u \oplus v}{s} &= \lim_{s\to 0} \frac{J(s)(u) \oplus v - u \oplus v}{s} \\ &= \lim_{s\to 0} \frac{(J(s)(u) - u) \oplus 0}{s} \\ &= Ju \oplus 0. \end{aligned}$$

Then $D(S) = D(J) \oplus \mathscr{E}$ and $S(u \oplus v) = J(u) \oplus 0$ for any $u \in D(J)$.

□

Theorem 9. *Let $\mathscr{E}$ be a non-Archimedean quasi-Banach space over $\mathbb{K}$ with the power q. Let $(A(s))_{s\in\Omega_r}$ and $(B(s))_{s\in\Omega_r}$ be two C_0-groups on $\mathscr{E}$ of generators respectively A and B. We put $T(s) = A(s) \oplus B(s)$ for all $s \in \Omega_r$. Then we get* *(i)* $(T(s))_{s\in\Omega_r}$ *is a* C_0*-group on* $\mathscr{E}\oplus\mathscr{E}$*.*

(ii) The generator of $(T(s))_{s\in\Omega_r}$ *is the operator* T *defined on* $D(T)=D(A)\oplus D(B)$ *by* $T(u\oplus v)=Au\oplus Bv$ *for any* $(u,v)\in\mathscr{E}^2$*.*

Proof. (i) Let $u\oplus v\in\mathscr{E}\oplus\mathscr{E}$ and $s,t\in\Omega_r$. Utilizing $(A(s))_{s\in\Omega_r}$ and $(B(s))_{s\in\Omega_r}$ are C_0-groups on $\mathscr{E}$, hence

$$T(0)(u\oplus v)=A(0)u\oplus B(0)v=Iu\oplus Iv=u\oplus v,$$

then $T(0)=I\oplus I=I_{\mathscr{E}\oplus\mathscr{E}}$.

We get also:

$$\begin{aligned}T(t+s)(u\oplus v) &= A(t+s)u\oplus B(t+s)v\\ &= A(t)A(s)u\oplus B(t)B(s)v\\ &= (A(t)\oplus B(t))(A(s)x\oplus B(s)v)\\ &= T(t)(A(s)\oplus B(s)(u\oplus v))\\ &= T(t)T(s)(u\oplus v).\end{aligned}$$

So $T(s+t)=T(s)T(t)$. However

$$\begin{aligned}\lim_{s\to 0}\|T(s)(u\oplus v)-u\oplus v\|_q &= \lim_{s\to 0}\|A(s)u\oplus B(s)v-u\oplus v)\|_q\\ &= \lim_{s\to 0}\|(A(s)u-u)\oplus(B(s)v-v)\|_q\\ &= \lim_{s\to 0}\max(\|A(s)u-u\|_q,\|B(s)u-u\|_q)\\ &= 0.\end{aligned}$$

So $(T(s))_{s\in\Omega_r}$ is a C_0-group on $\mathscr{E}\oplus\mathscr{E}$.

(ii) If $u\in D(A)$ and $v\in D(B)$, then

$$\begin{aligned}\lim_{s\to 0}\frac{T(s)(u\oplus v)-u\oplus v}{s} &= \lim_{s\to 0}\frac{(A(s)u-u)\oplus(B(s)v-v)}{s}\\ &= Au\oplus Bv.\end{aligned}$$

So $D(T)=D(A)\oplus D(B)$ and $T(u\oplus v)=Au\oplus Bv$.

□

Utilizing Theorem 8 and Theorem 9, we get:

Theorem 10. *Let* $\mathscr{E}$ *be a non-Archimedean quasi-Banach space over* $\mathbb{K}$ *with the power* q*. Let* $(J(s))_{s\in\Omega_r}$ *be a contraction* C_0*-group of generator* J *on* $\mathscr{E}$*. Let* $S(s)=J(s)\oplus I$ *for any* $s\in\Omega_r$*. Hence*

(i) $(S(s))_{s\in\Omega_r}$ *is a* C_0*-group of contractions on* $\mathscr{E}\oplus\mathscr{E}$*,*

(ii) The generator of $(S(s))_{s\in\Omega_r}$ *is the operator* S *defined on* $D(S)=D(J)\oplus\mathscr{E}$ *such that* $S(u\oplus v)=Ju\oplus 0$ *for all* $u\in D(J)$*,* $v\in\mathscr{E}$*.*

Theorem 11. *Let* $\mathscr{E}$ *be a non-Archimedean quasi-Banach space over* $\mathbb{K}$ *with the power* q*. Let* $(A(s))_{s\in\Omega_r}$ *and* $(B(s))_{s\in\Omega_r}$ *be two* C_0*-groups on* $\mathscr{E}$ *of a generators respectively* A *and* B*. We set* $T(s)=A(s)\oplus B(s)$ *for any* $s\in\Omega_r$*. Then*

(i) $(T(s))_{s\in\Omega_r}$ *is a* C_0*-group on* $\mathscr{E}\oplus\mathscr{E}$*.*

(ii) The generator of $(T(s))_{s\in\Omega_r}$ *is the operator* S *defined on* $D(S)=D(A)\oplus D(B)$ *by* $S(u\oplus v)=Au\oplus Bv$ *for any* $(u,v)\in D(A)\times D(B)$*.*

Now, we define the concept of C-groups of operators as follows.

Definition 18. *Let $\mathscr{E}$ be a non-Archimedean quasi-Banach space over $\mathbb{K}$ with the power q. Let $C \in \mathscr{B}(\mathscr{E})$ be invertible. A family $(J(s))_{s\in\Omega_r} \subset \mathscr{B}(\mathscr{E})$ is called a C-group if*

(i) $CJ(t+s) = J(t)J(s)$ for any $t,s \in \Omega_r$ and $J(0) = C$,

(ii) For any $u \in \mathscr{E}$, $J(\cdot)u : \Omega_r \longrightarrow \mathscr{E}$ is continuous.

The generator S of a C-group $(J(s))_{s\in\Omega_r}$ is defined by

$$D(S) = \{u \in \mathscr{E} : \lim_{s\to 0} \frac{J(s)u - Cu}{s} \text{ exists }\},$$

and

$$Sx = C^{-1} \lim_{s\to 0} \frac{J(s)u - Cu}{s}, \text{ for any } u \in D(S),$$

is called the infinitesimal generator of the C-group $(J(s))_{s\in\Omega_r}$.

Remark 6. *Let $\mathscr{E}$ be a non-Archimedean quasi-Banach space over $\mathbb{K}$ with the power q. Let $(J(s))_{s\in\Omega_r}$ be a C_0-group of infinitesimal generator J on $\mathscr{E}$ and let $C \in \mathscr{B}(\mathscr{E})$ be invertible such that for any $s \in \Omega_r$, $CJ(s) = J(s)C$ then for all $s \in \Omega_r$, $S(s) = T(s)C$ is a C-group of infinitesimal generator J on $\mathscr{E}$. In this sense, the Definition 18 generalizes the definition of the C_0-group.*

Remark 7. *Let $\mathscr{E}$ be a free non-Archimedean quasi-Banach space with the power q, let $(J(s))_{s\in\Omega_r}$ be a C-group of linear operators of infinitesimal generator J on $\mathscr{E}$, from Remark 1, J may or may not be a continuous linear operator on $\mathscr{E}$.*

We get the next theorem.

Theorem 12. *Let $\mathscr{E}$ be a non-Archimedean quasi-Banach space over $\mathbb{K}$ with the power q. Let $(J(s))_{s\in\Omega_r}$ be a C-group on $\mathscr{E}$ with there is $M > 0$ such that for any $s \in \Omega_r$, $\|J(s)\| \leq M$, and let J be its infinitesimal generator. Hence, for all $u \in D(J), s \in \Omega_r$, $J(s)u \in D(J)$. Furthermore*

$$\frac{dJ(s)}{ds} u = JJ(s)u = J(s)Ju.$$

Proof. Let $u \in D(J)$, $s \in \Omega_r^*$ and $s \in \Omega_r$. Utilizing Definition 18 and the boundedness of the C-group $(J(s))_{s\in\Omega_r}$, it follows that

$$\frac{J(t)J(s)u - CJ(s)u}{t} = J(s)\frac{J(t)u - Cu}{t} \to J(s)CJx = CJ(s)Jx \text{ as } t \to 0. \tag{13}$$

So $J(s)Jx \in D(J)$ and $JJ(s)u = J(s)Ju$, from (13). Note that

$$\frac{CJ(t+s)u - CJ(s)u}{t} = \frac{J(t)J(s)u - CJ(s)u}{t},$$

so that

$$\frac{dCJ(s)}{ds} u = \lim_{t\to 0} \frac{CJ(t+s)u - CJ(s)u}{t},$$

exists and equals $CJ(s)Jx$. Furthermore, from invertibility of C, we have

$$\frac{dJ(s)}{ds} u = J(s)Ju = JJ(s)u.$$

This completes the proof. □

The C-groups can be applied to the several p-adic differential equations that may be modeled as a p-adic abstract Cauchy problem on a non-Archimean quasi-Banach space, thanks to Theorem 12, we get:

Remark 8. *One of the consequences of Theorem 12 is that the function $v(s) = J(s)u$, $s \in \Omega_r$ for certain $u \in D(J)$, is the solution to the homogeneous p-adic differential equation given by*

$$\begin{cases} \frac{du(s)}{ds} = Ju(s), & s \in \Omega_r, \\ u(0) = Cu, \end{cases}$$

where $J : D(J) \subset \mathscr{E} \to \mathscr{E}$ is the infinitesimal generator of the C-group $(J(s))_{s\in\Omega_r}$ and $u : \Omega_r \to D(J)$ is $\mathscr{E}$-valued function.

We get the next example.

Example 5. *Let $\mathscr{E}$ be a non-Archimedean quasi-Banach space over $\mathbb{Q}_p$ with the power q. Let $S, C \in \mathscr{B}(\mathscr{E})$ such that C is invertible, $SC = CS$ and $\|S\|_q < r$ with $r = p^{\frac{-1}{p-1}}$, then for any $s \in \Omega_r$, $J(s) = Ce^{sS}$, in particular if $C = (I-S)^{-1}$, is a C-group of continuous linear operators on $\mathscr{E}$. In fact*

(i) $J(0) = C$.

(ii) *For any* $t, s \in \Omega_r, J(t)J(s) = Ce^{tS}Ce^{sS} = C^2e^{(t+s)S} = CJ(s+t)$.

(iii) *For all* $u \in \mathscr{E}$, $J(\cdot)u : \Omega_r \to \mathscr{E}$ *is continuous.*

Proposition 6. *Let $\mathscr{E}$ be a non-Archimedean quasi-Banach space over $\mathbb{K}$ with the power q. Let $(J(s))_{s\in\Omega_r}$ be a C_1-group of infinitesimal generator J on $\mathscr{E}$ and $C_2 \in \mathscr{B}(\mathscr{E})$ be invertible such that for any $s \in \Omega_r, C_2J(s) = J(s)C_2$, then $(C_2J(s))_{s\in\Omega_r}$ is a C_1C_2-group on $\mathscr{E}$.*

Proof. Set for any $s \in \Omega_r$, $S(s) = C_2J(s)$, then $(S(s))_{s\in\Omega_r}$ is a C_1C_2-group on $\mathscr{E}$. In fact

(i) $S(0) = C_2J(0) = C_1C_2$,

(ii) For any $s, t \in \Omega_r$,

$$\begin{aligned} S(s)S(t) &= C_2J(s)C_2J(t) \\ &= J(s)J(t)C_2^2 \\ &= C_1J(s+t)C_2^2 \\ &= C_1C_2^2J(s+t) \\ &= C_1C_2S(s+t). \end{aligned}$$

(iii) For each $u \in \mathscr{E}$, $S(\cdot)u : \Omega_r \to \mathscr{E}$ is continuous.

So, $(S(s))_{s\in\Omega_r}$ is a C_1C_2-group on $\mathscr{E}$. □

We get the next example.

Example 6. *Suppose that $\mathbb{K} = \mathbb{Q}_p$ and $r = p^{\frac{-1}{p-1}}$, let $\mathscr{E}$ be a free non-Archimedean quasi-Banach space over $\mathbb{Q}_p$ and $(f_i)_{i\in\mathbb{N}}$ a base of $\mathscr{E}$. Define for any $s \in \Omega_r$, $u \in \mathscr{E}$ such that $u = \sum_{i\in\mathbb{N}} u_i f_i$,*

$$J(s)u = \sum_{i\in\mathbb{N}} (1-\mu_i)e^{s\mu_i}u_i f_i,$$

where $(\mu_i)_{i\in\mathbb{N}} \subset \Omega_r$. It is easy to check that the family $(J(s))_{s\in\Omega_r}$ is well defined on $\mathscr{E}$.

We have the following proposition.

Proposition 7. *The family $(J(t))_{t\in\Omega_r}$ of continuous linear operators given above is a C-group of continuous linear operators, whose infinitesimal generator is the continuous diagonal operator J defined by $Ju = \sum_{i\in\mathbb{N}} \mu_i u_i f_i$ for any $u = \sum_{i\in\mathbb{N}} u_i f_i \in \mathscr{E}$.*

Proof. Define for each $s \in \Omega_r$, $i \in \mathbb{N}$,

$$J(s)f_i = (1-\mu_i)e^{s\mu_i}f_i = \left(\sum_{n\in\mathbb{N}} \frac{(1-\mu_i)\mu_i^n s^n}{n!}\right)f_i,$$

where $(\mu_j)_{j\in\mathbb{N}} \subset \Omega_r$. From for any $i \in \mathbb{N}$, $s\mu_i \in \Omega_r$, we get for all $s \in \Omega_r$, $u \in \mathscr{E}$, $\|J(s)u\|_q \le \sup_{i\in\mathbb{N}} \left|(1-\mu_i)e^{s\mu_i}\right|_p^q \|u\|_q < \infty$, then $\left(\forall s \in \Omega_r\right)$ $\|J(s)\|_q$ is finite. Hence the family $(J(s))_{s\in\Omega_r}$ is well defined on $\mathscr{E}$. Furthermore,

(i) $J(0) = I - J$, (since J is a diagonal operator on $\mathscr{E}$, we have $\|J\|_q = \sup_{i\in\mathbb{N}} |\mu_i|^q$, thus $\|J\|_q < r < 1$, we have $I - J$ is invertible).

(ii) For all $t,s \in \Omega_r$,

$$\begin{aligned} J(t)J(s) &= (I-J)e^{sJ}(I-J)e^{sJ} \\ &= (I-J)(I-J)e^{(t+s)J} \\ &= (I-J)J(t+s). \end{aligned}$$

(iii) For each $u \in \mathscr{E}$, $S(\cdot)u : \Omega_r \to \mathscr{E}$ is continuous on Ω_r.

Thus $(J(s))_{s\in\Omega_r}$ is a C-group of continuous linear operators on $\mathscr{E}$ where $C = I - J$. Let B be the infinitesimal generator of $(J(s))_{s\in\Omega_r}$. It remains to demonstrate that $J = B$. Let us demonstrate that $D(B) = \mathscr{E}\Big(= D(J)\Big)$. Clearly, for each $s \in \Omega_r^*$ and $i \in \mathbb{N}$,

$$\frac{J(s)f_i - Cf_i}{s} = C\Big(\frac{e^{s\mu_i}-1}{s}\Big)f_i.$$

Thus, for all $s \in \Omega_r^*$ and for all $i \in \mathbb{N}$,

$$C^{-1}\Big(\frac{J(s)e_i - Cf_i}{s}\Big) = \Big(\frac{e^{s\mu_i}-1}{s}\Big)f_i.$$

Hence, for all $u = \sum_{i\in\mathbb{N}} u_i f_i \in \mathscr{E}$, $s \in \Omega_r^*$

$$|u_i|_p^q \left\| C^{-1}\frac{J(s)f_i - Cf_i}{s} \right\|_q \leq \frac{|u_i|_p^q \|f_i\|_q}{|s|_p^q} \to 0 \; as \; i \to \infty. \tag{14}$$

Thus,

$$D(B) = \Big\{u = (u_i)_{i\in\mathbb{N}} : \lim_{i\to\infty} |u_i|_p^q \Big\| C^{-1}\Big(\frac{J(s)f_i - Cf_i}{s}\Big)\Big\|_q = 0\Big\}.$$

To complete the proof, it suffices to demonstrate that

$$\big(\forall i \in \mathbb{N}\big) \;\; \lim_{s\to 0} \Big\| Jf_i - C^{-1}\Big(\frac{J(s)f_i - Cf_i}{s}\Big)\Big\|_q = 0.$$

The latter is actually obvious since $\lim_{s\to 0}\Big(\frac{e^{s\mu_i}-1}{s}\Big) = \mu_i$, and hence $J = B$ is the infinitesimal generator of the C-group $(J(s))_{s\in\Omega_r}$.

□

We introduce the following definition.

Definition 19. *Let $\mathscr{E}$ be a non-Archimedean quasi-Banach space over $\mathbb{K}$ with the power q. Let $(J(t))_{t\in\Omega_r}$ be a C-group of continuous linear operators on $\mathscr{E}$. $(J(t))_{t\in\Omega_r}$ is said to be an uniformly continuous C-group on $\mathscr{E}$ if*

$$\lim_{t\to 0} \|J(t) - C\|_q = 0.$$

Theorem 13. *Let $\mathscr{E}$ be a non-Archimedean quasi-Banach space over $\mathbb{Q}_p$ with the power q. Let $A \in \mathscr{B}(\mathscr{E})$ such that $\|A\|_q < r\Big(= p^{\frac{-1}{p-1}}\Big)$. Then A is the infinitesiml generator of an uniformly continuous C-group of bounded linear operators $(J(s))_{s\in\Omega_r}$.*

Proof. Suppose $\|A\|_q < r\Big(= p^{\frac{-1}{p-1}}\Big)$ and set, for all $s \in \Omega_r$,

$$J(s) = (I-A)e^{sA} = \sum_{n\in\mathbb{N}} \frac{(I-A)(sA)^n}{n!}. \tag{15}$$

Clearly, the series given by (15) converges in norm and defines a family of continuous linear operators on $\mathscr{E}$ by $|t|^q\|A\|_q < r$. Furthermore,

(1) $J(0) = I - A$, (from $\|A\|_q < r < 1$, we have $I - A$ is invertible).

(ii) The same as in Proposition 7.

(iii) For all $x \in \mathscr{E}$, $J(\cdot)x : \Omega_r \to \mathscr{E}$ is continuous.

Thus $(J(t))_{t\in\Omega_r}$ is a C-group of bounded linear operatqors on X where $C = I - A$. For all $t \in \Omega_r$,

$$\begin{aligned} \|J(t)-C\|_q &= \|(I-A)(e^{tA}-I)\|_q \\ &\leq \|I-A\|_q\|e^{tA}-I\|_q \\ &\leq \|e^{tA}-I\|_q. \end{aligned}$$

Hence

$$\lim_{t\to 0}\|J(t)-C\|_q = 0.$$

For all $t \in \Omega_r^*$,

$$\frac{J(t)-C}{t} = C\Big(\frac{e^{tA}-I}{t}\Big).$$

Thus, for all $t \in \Omega_r^*$,

$$C^{-1}\Big(\frac{J(t)-C}{t}\Big) = \Big(\frac{e^{tA}-I}{t}\Big) = \sum_{n=0}^{\infty}\frac{t^nA^{n+1}}{(n+1)!}.$$

Hence, for all $t \in \Omega_r^*$,

$$\Big\|C^{-1}\Big(\frac{J(t)-C}{t}\Big)-A\Big\|_q = \Big\|\frac{e^{tA}-I}{t}-A\Big\|_q \leq |t|^q\|A\|_q\|\xi_t\|_q,$$

where $\xi_t = \displaystyle\sum_{n=0}^{\infty}\frac{t^nA^{n+1}}{(n+2)!}$ converges. Consequently,

$$\lim_{t\to 0}\Big\|C^{-1}\Big(\frac{J(t)-C}{t}\Big)-A\Big\|_q = 0.$$

Then, $(J(t))_{t\in\Omega_r}$ given above is an uniformly continuous C-group of continuous linear operators of infinitesimal generator A. □

Proposition 8. *Let $\mathscr{E}$ be a non-Archimedean quasi-Banach space over $\mathbb{K}$ with the power q. For all invertible operators A and B in $\mathscr{B}(\mathscr{E})$, $A \oplus B$ is invertible on $\mathscr{E} \oplus \mathscr{E}$ and its inverse is denoted by $(A\oplus B)^{-1}$. Furthermore,*

$$(A\oplus B)^{-1} = A^{-1}\oplus B^{-1}.$$

Proof. Let $A, B \in \mathscr{B}(\mathscr{E})$ two invertible operators, then $A^{-1}A = AA^{-1} = I$ and $B^{-1}B = BB^{-1} = I$, therefore for all $(x\oplus y) \in \mathscr{E}\oplus\mathscr{E}$, we have

$$\begin{aligned} \Big(A^{-1}\oplus B^{-1}\Big)\Big(A\oplus B\Big)(x\oplus y) &= \Big(A^{-1}\oplus B^{-1}\Big)(Ax\oplus By) \\ &= (A^{-1}Ax)\oplus(B^{-1}By) \\ &= x\oplus y \end{aligned}$$

and

$$\begin{aligned} \Big(A\oplus B\Big)\Big(A^{-1}\oplus B^{-1}\Big)(x\oplus y) &= \Big(A\oplus B\Big)(A^{-1}x\oplus B^{-1}y) \\ &= \Big(AA^{-1}x\oplus BB^{-1}x\Big) \\ &= x\oplus y. \end{aligned}$$

Thus, for all $x\oplus y \in \mathscr{E}\oplus\mathscr{E}$,

$$\Big(A\oplus B\Big)\Big(A^{-1}\oplus B^{-1}\Big)(x\oplus y) = \Big(A^{-1}\oplus B^{-1}\Big)\Big(A\oplus B\Big)(x\oplus y) = x\oplus y.$$

Then, $A\oplus B$ is invertible on $\mathscr{E}\oplus\mathscr{E}$ and its inverse is $(A\oplus B)^{-1} = A^{-1}\oplus B^{-1}$.

□

Example 7. *Let $\mathscr{E}$ be a non-Archimedean quasi-Banach space over $\mathbb{K}$ with the power q. Let $A, B \in \mathscr{B}(\mathscr{E})$ such that* $\max\{\|A\|_q, \|B\|_q\} < r\left(= p^{\frac{-1}{p-1}}\right)$. *Set for all $t \in \Omega_r$,*

$$T(t) = e^{tA} \oplus e^{tB}.$$

It is easy to see that for all $t \in \Omega_r$, $T(t)$ is invertible and $(\forall t \in \Omega_r)$ $T(t)^{-1} = T(-t)$.

We continue with the next theorem.

Theorem 14. *Let $\mathscr{E}$ be a non-Archimedean quasi-Banach space over $\mathbb{K}$ with the power q. Let $(T(t))_{t\in\Omega_r}$ be a C-group of infinitesimal generator A on $\mathscr{E}$. Set, for all $t \in \Omega_r$, $S(t) = T(t) \oplus I$. Then we have*

(i) $(S(t))_{t\in\Omega_r}$ is a $C \oplus I$-group on $\mathscr{E} \oplus \mathscr{E}$,

(ii) The infinitesimal generator of $(S(t))_{t\in\Omega_r}$ is the operator T defined on $D(T) = D(A) \oplus \mathscr{E}$ by for all $x \in D(A)$, $y \in \mathscr{E}$, $T(x \oplus y) = Ax \oplus 0$.

Proof. (i) Since $(T(t))_{t\in\Omega_r}$ is a C-group of infinitesimal generator A on $\mathscr{E}$, then

$$S(0) = T(0) \oplus I = C \oplus I.$$

Let $x \oplus y \in \mathscr{E} \oplus \mathscr{E}$ and $t, s \in \Omega_r$, we have

$$\begin{aligned}
(C\oplus I)S(t+s)(x\oplus y) &= (C\oplus I)T(t+s)(x)\oplus y \\
&= CT(t+s)(x)\oplus y \\
&= T(t)T(s)(x)\oplus y \\
&= (T(t)\oplus I)(T(s)(x)\oplus y) \\
&= S(t)((T(s)\oplus I)(x\oplus y)) \\
&= S(t)S(s)(x\oplus y).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\lim_{t\to 0}\|S(t)(x\oplus y)-(C\oplus I)(x\oplus y)\|_q &= \lim_{t\to 0}\|(T(t)x-Cx)\oplus 0\|_q \\
&= \lim_{t\to 0}\max\left(\|T(t)x-Cx\|_q, 0\right) \\
&= \lim_{t\to 0}\|T(t)x-Cx\|_q \\
&= 0.
\end{aligned}$$

Therefore $(S(t))_{t\in\Omega_r}$ is a $C \oplus I$-group on $\mathscr{E} \oplus \mathscr{E}$.

(ii) Let $x \in D(A)$ and $y \in \mathscr{E}$, we have

$$\begin{aligned}
\lim_{t\to 0}\frac{S(t)(x\oplus y)-(C\oplus I)(x\oplus y)}{t} &= \lim_{t\to 0}\frac{(T(t)(x)-Cx)\oplus 0}{t} \\
&= CAx\oplus 0 = (C\oplus I)(Ax\oplus 0).
\end{aligned}$$

Thus, for all $x \in D(A)$, $y \in \mathscr{E}$ we have

$$(C\oplus I)^{-1}\left(\lim_{t\to 0}\frac{S(t)(x\oplus y)-(C\oplus I)(x\oplus y)}{t}\right) = Ax\oplus 0.$$

Then $D(T) = D(A) \oplus \mathscr{E}$ and $T(x \oplus y) = A(x) \oplus 0$ for all $x \in D(A)$ and for each $y \in \mathscr{E}$.

□

Theorem 15. *Let $\mathscr{E}$ be a non-Archimedean quasi-Banach space over $\mathbb{K}$ with the power q. Let $(A(t))_{t\in\Omega_r}$ and $(B(t))_{t\in\Omega_r}$ be two C_1-group and C_2-group on $\mathscr{E}$ of infinitesimal generators A and B respectively. We set for all $t \in \Omega_r, T(t) = A(t) \oplus B(t)$. Then* *(i)* $(T(t))_{t\in\Omega_r}$ *is a* $C_1\oplus C_2$*-group on* $\mathscr{E}\oplus\mathscr{E}$.

(ii) The infinitesimal generator of $(T(t))_{t\in\Omega_r}$ *is the operator* T *defined on* $D(T)=D(A)\oplus D(B)$ *by* $T(x\oplus y)=Ax\oplus By$ *for all* $(x,y)\in\mathscr{E}^2$.

Proof. (i) Let $x\oplus y\in\mathscr{E}\oplus\mathscr{E}$. Since $(A(t))_{t\in\Omega_r}$ and $(B(t))_{t\in\Omega_r}$ are two C_1-group and C_2-group on $\mathscr{E}$ respectively, then

$$T(0)(x\oplus y)=A(0)x\oplus B(0)y=C_1x\oplus C_2y=(C_1\oplus C_2)(x\oplus y).$$

Hence $T(0)=C_1\oplus C_2$. We have also, for all $(t,s)\in\Omega_r^2$,

$$\begin{aligned}(C_1\oplus C_2)T(t+s)(x\oplus y) &= (C_1\oplus C_2)(A(t+s)x\oplus B(t+s)y)\\ &= C_1A(t+s)x\oplus C_2B(t+s)y\\ &= A(t)A(s)x\oplus B(t)B(s)y\\ &= (A(t)\oplus B(t))\,(A(s)x\oplus B(s)y)\\ &= T(t)\,(A(s)\oplus B(s)\,(x\oplus y))\\ &= T(t)T(s)\,(x\oplus y)\,.\end{aligned}$$

Then, $(C_1\oplus C_2)T(t+s)=T(t)T(s)$. On the other hand,

$$\begin{aligned}\lim_{t\to 0}\|T(t)(x\oplus y)-(C_1\oplus C_2)\,(x\oplus y)\,\|_q &= \lim_{t\to 0}\|A(t)x\oplus B(t)y-C_1x\oplus C_2y\|_q\\ &= \lim_{t\to 0}\|\Big(A(t)x-C_1x\Big)\oplus\Big(B(t)y-C_2y\Big)\|_q\\ &= \lim_{t\to 0}\max(\|A(t)x-C_1x\|_q,\|B(t)y-C_2y\|_q)\\ &= 0.\end{aligned}$$

Therefore, $(T(t))_{t\in\Omega_r}$ is a $C_1\oplus C_2$-group on $\mathscr{E}\oplus\mathscr{E}$.

(ii) Let $x\in D(A)$ and $y\in D(B)$, we have:

$$\begin{aligned}\lim_{t\to 0}\frac{T(t)(x\oplus y)-(C_1\oplus C_2)\,(x\oplus y)}{t} &= \lim_{t\to 0}\frac{(A(t)x-C_1x)\oplus(B(t)y-C_2y)}{t}\\ &= C_1Ax\oplus C_2By\\ &= (C_1\oplus C_2)(Ax\oplus By).\end{aligned}$$

Thus, for all $x\in D(A)$, $y\in D(B)$, we have

$$(C_1\oplus C_2)^{-1}\left(\lim_{t\to 0}\frac{T(t)\,(x\oplus y)-(C_1\oplus C_2)\,(x\oplus y)}{t}\right)=Ax\oplus By.$$

Consequently, $D(T)=D(A)\oplus D(B)$ and $T(x\oplus y)=Ax\oplus By$.

□

Data Availability

The manuscript has no associated data or the data will not be deposited.

Conflicts of Interest

The author declares that there is no conflict of interest.

Ethical Considerations

The author has diligently addressed ethical concerns, such as informed consent, plagiarism, data fabrication, misconduct, falsification, double publication, redundancy, submission, and other related matters.

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