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On Nonlinear Urysohn Integral Equation Via Measures of Noncompactness and Numerical Method to Solve It

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Abstract In this study, we present the existence of solutions for Urysohn integral equations. By using the techniques of noncompactness measures, we employ the basic fixed point theorems such as Petryshyn's fixed point theorem to obtain the mentioned aim in Banach algebra. Then this paper presents a numerical approach based on Haar wavelets to solve the equation. This numerical method does not lead to a nonlinear algebraic equations system. Conducting numerical experiments confirm the theoretical results of the applied method and endorse the accuracy of the method.

Keywords Urysohn integral equations · Haar wavelet · Iterative method · Noncompactness measures · Fixed point theorems

Mathematics Subject Classification (2010) 45Gxx · 45G10 · 34A12

1 Introduction

The mathematical modeling of physical phenomena, many problems in applied mathematics, engineering, mechanics, mathematical physics and many other fields can be turned into integral equations of the second type [1–5]. In this research, we will consider nonlinear Urysohn Fredholm integral equations of the second kind (NUFIEs) of the form

$$u(s) = f(s) + \lambda \int_a^b h(s, x, u(x))dx, \qquad s \in [a, b], \tag{1}$$

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where the functions $f(s)$ and $h(s, x, u(x))$ are known and $u(s)$ is a solution to be determined. Investigation on existence theorems for some nonlinear functional–integral equations has been presented in other references such as [6–13].

First of all, we try to prove the existence of the solution for Eq. 1. To get these results, we use the technique of measure of noncompactness and Petryshyn's fixed point theorem [14] (instead of Darbo'stheorem) that has been analyzed as a generalization of Darbos fixed theorem [15]. The idea of using the Petryshyn fixed point theorem in order to investigate the existence of solution of nonlinear functional integral equations for the first time was introduced in [13] by Kazemi et al. (2016). Regarding the fact that we cannot solve the nonlinear Urysohn Fredholm integral equations defined by Eq. (1) to find an exact solution, numerical techniques are employed to estimate an approximated solution. Recently, some the numerical methods including, block-pulse functions (BPFs) [16], operational matrices [17], triangular functions (TFs) [18], Chebyshev polynomials [19], Least squares approximation method [20], wavelet method [21] and Bernoulli polynomials [22] have been proposed to obtain approximate solutions of these equations. In the methods mentioned above, the integral equation is transformed into a system of nonlinear algebraic equations which has to be solved with iterative methods. It is cumbersome to solve these systems, or the solution may be unreliable. So, in the present paper, we apply the successive approximations method based on the Haar wavelets to estimate a numerical solution for Eq. (1). Our method does not consist of reducing the solution of Eq. (1) to a set of algebraic equations. Also, numerical methods have been presented for solving integral equations including successive approximations method based on quadrature rules [23–28].

The structure of this article is divided into five sections. In Section 2, we present some definitions and preliminary results about the concept of measure of noncompactness. In Section 3, using the technique of a suitable measure of noncompactness in the Banach algebra, we prove an existence theorem for Eq. (1). Also, the convergence of the method of successive approximations used to approximate the solution of the Eq. (1), is described in this section. In order to confirm the theoretical results and show the accuracy of the method, some numerical examples in Section 4 are considered. Section 5 includes the conclusion of the proposed method.

2 Preliminaries

In this section, we recall some notations, definitions and theorems to obtain all results of this work. Let $(E, \|.\|)$ be a Banach space. We write

$$\bar{B_r} = \{x \in E : \| x \| \leq r\}$$

for the closed ball and $\partial\bar{B_r} = \{x \in E : \| x \| = r\}$ for the sphere in E around 0 with radius $r > 0$.

The symbol $\bar{M}, ConvM$ will denote the closure and closed convex hull of a subset M of E, respectively. We denote the standard algebraic operations on sets by the symbols λM and $M+N$. Moreover, let $\mathbf{m}_E$ indicate the family of all nonempty and bounded subsets of E and $\mathbf{n}_E$ indicate the family of all nonempty and relatively compact subsets.

Definition 1 [29] If M is a bounded subset of a Banach space E, let $\alpha(M)$ denote the (Kuratowski) measure of noncompactness of M, that is,

$$\alpha(M) = \inf\{\sigma > 0 : M \text{ may be covered by finitely many sets of diameter} \leq \sigma\}. \tag{2}$$

Definition 2 [30] The Hausdorff (or ball) measure of noncompactness

$$\mu(M) = \inf\{\epsilon > 0 : \text{there exists a finite } \epsilon - \text{net for } M \text{ in } E\}, \tag{3}$$

where by a finite ε-net for M in E we mean, as usual, a set $\{r_1, r_2, \ldots, r_m\} \subset E$ such that the balls $B_\varepsilon(E; r_1), B_\varepsilon(E; r_2), \ldots, B_\varepsilon(E; r_m)$ over M. These measures of noncompactness are mutually equivalent in the sense that

$$\mu(M) \leq \alpha(M) \leq 2\mu(M) \tag{4}$$

for any bounded set $M \subset E$.

Theorem 1 *[14] Let E be a Banach space, $\lambda \in \mathbf{R}$ and $M, N \in \mathbf{m}_E$ bounded. Then*

(i) $\mu(M) = 0$ *if and only if* $M \in \mathbf{n}_E$;

(ii) $\mu(M) = 0$ *if and only if* $M \in \mathbf{n}_E$;

(iii) $M \subseteq N$ *implies* $\mu(M) \leq \mu(N)$;

(iv) $\mu(\bar{M}) = \mu(ConvM) = \mu(M)$;

(v) $\mu(M \cup N) = \max\{\mu(M), \mu(N)\}$;

(vi) $\mu(\lambda M) =\mid \lambda \mid \mu(M)$, *where* $\lambda M = \{\lambda m : m \in M, \lambda \in \mathbf{R}\}$;

(vii) $\mu(M+N) \leq \mu(M) + \mu(N)$, *where* $M + N = \{m + n : m \in M, n \in N\}$.

In what follows, we will work in the space $C[a, b]$ consisting of all real-valued functions and continuous on the interval $[a, b]$. The space $C[a, b]$ is equipped with the standard norm

$$\|x\| = \sup\{|x(t)| : t \in [a, b]\}. \tag{5}$$

Recall that the modulus of continuity of a function $u \in C[a, b]$ is defined by

$$\omega(u, \sigma) = \sup\{|u(x) - u(y)| : |x - y| \leq \sigma\}. \tag{6}$$

We have then $w(u, \sigma) \to 0$, as $\sigma \to 0$, since u is uniformly continuous on $[a, b]$. More generally, if this limit relation holds uniformly for u running over some bounded set $M \subset C$, then M is equicontinuous, and vice versa.

Theorem 2 *[30] On the space $C[a,b]$, the measures of noncompactness* (3) *is equivalent to*

$$\mu(M) = \lim_{\sigma \to 0} \sup_{u \in M} \omega(u, \sigma) \tag{7}$$

for all bounded sets $M \subset C[a,b]$.

For our purpose we use equation (7) in the rest of the paper. Closely associated with the measures of noncompactness is the concept of k-set contraction.

Definition 3 [31] Let $\Gamma : E \to E$ be a continuous mapping of E. Γ is called a k-set contraction if for all $B \subset E$ with B bounded, $\Gamma(B)$ is bounded and $\beta(\Gamma B) \leq k\beta(B), 0 < k < 1$. if

$$\beta(\Gamma B) < \beta(B), \text{ for all } \beta(B) > 0, \tag{8}$$

then Γ is called densifying or condensing map. A k-set contraction with $k \in (0,1)$, is densifying, but converse is not true.

Theorem 3 *[14] Assume that $\Gamma : \bar{B}_r \to E$ be a densifying mapping which satisfying the boundary condition,*

$$If\ \Gamma(x) = kx,\ for\ some\ x\ in\ \partial\ B_r\ then\ k \leq 1, \tag{9}$$

then the set of fixed points of Γ in $\bar{B}_r$ is nonempty. This is known by Petryshyn's fixed point theorem.

This property allows us to characterize solution of the integral Eq. (1) and will be used in the next section.

3 Main results

In this section, we will study the existence of the nonlinear functional Eq. (1) for $u \in C[a,b]$ under the following assumptions:

(H1) $f \in C(\mathbf{R}, \mathbf{R}), u \in C([a,b], \mathbf{R}), h \in C([a,b] \times [a,b] \times \mathbf{R}, \mathbf{R})$,

(H2) There exists a constant c such that
$|h(s,x,u(x)) - h(s,x,\bar{u}(x))| \leq c|u(x) - \bar{u}(x)|$;

(H3) (Bounded condition) There exists $r_0 \geq 0$ such that the following bounded condition is satisfied
$\sup\{|f(s) + \lambda \int_a^b h(s,x,u(x))dx| : s,x \in I,\ u \in [-r_0, r_0]\} \leq r_0$,

The following result is obtained by using the above hypotheses.

Theorem 4 *Under the assumption (H1)-(H3) above, Eq.* (1) *has at least one solution in the Banach space $E = C([a,b])$.*

Proof To prove this result using Theorem 3 as our main tool, we need to define operator $T : B_{r_0} \to E$ in the following way

$$(Tx)(t) = f(s) + \lambda \int_a^b h(s, x, u(x))dx, \tag{10}$$

Now, we show that the operator T is continuous on the ball B_{r_0}. To do this, consider $\sigma > 0$ and take arbitrary $u, v \in B_{r_0}$ such that $\| u - v \| \leq \sigma$. Then for $s \in I$, we get

$$\begin{aligned} |(Tu)(s) - (Tv)(s)| &= |\lambda \int_a^b h(s, x, u(x))dx - \lambda \int_a^b h(s, x, v(x))dx| \\ &\leq \lambda \int_a^b |h(s, x, u(x)) - h(s, x, v(x))|dx \\ &\leq \lambda c(b - a)\|u - v\| \end{aligned}$$

Thus, the above estimate shows that the operator T is continuous on B_{r_0}.
Now, we will prove that the operator T satisfies densifying condition with respect to the measure μ as defined in (7). To do this, we choose a fixed arbitrary $\sigma > 0$. Let us take $u \in M$ and M is bounded subset of E, $s_1, s_2 \in [a, b]$ such that without loss of generality we may assume that $s_1 \leq s_2$ with $s_2 - s_1 \leq \sigma$, we obtain

$$\begin{aligned} |(Tu)(s_2) - (Tu)(s_1)| &= |f(s_2) + \lambda \int_a^b h(s_2, x, u(x))dx - f(s_1) - \lambda \int_a^b h(s_1, x, u(x))dx| \\ &\leq |f(s_2) - f(s_1)| + \lambda \int_a^b |h(s_2, x, u(x)) - h(s_1, x, u(x))|dx \end{aligned}$$

For simplicity we use the following notations:

$$\omega_h([a, b], \sigma) = \sup\{|h(s_2, x, u) - h(s_1, x, u)| : |s_2 - s_1| \leq \sigma, s_2, s_1 \in I_a,\ u \in [-r_0, r_0]\}, \tag{11}$$

and

$$\omega_f([a, b], \sigma) = \sup\{f(s_2) - f(s_1)| : |s_2 - s_1| \leq \sigma, s_2, s_1 \in I_a\} \tag{12}$$

The above inequality yields the following estimate

$$\omega(Tu, \sigma) \leq (b - a)\lambda\omega_h([a, b], \sigma) + \omega_f([a, b], \sigma),$$

In view of our assumptions we infer that the functions $f(s)$ and $h(s, x, u(x))$ are continuous on the sets $[a, b]$ and $[a, b] \times [a, b] \times \mathbf{R}$, respectively. Hence we deduce that $\omega_f([a, b], \sigma)$ and $\omega_h([a, b], \sigma) \to 0$ as $\sigma \to 0$. This means T is a densifying map. Finally, investigation of condition 9 is remained. Suppose $x \in \partial\bar{B}_{r_0}$.
If $Tx = kx$ then we have $kr_0 = k\|x\| = \|Tx\|$ and by condition (H3) we concluded that

$$|Tx(t)| = |f(s) + \lambda \int_a^b h(s, x, u(x))dx| \leq r_0, \tag{13}$$

for all $t \in I_a$, hence $\|Tx\| \leq r_0$, so this shows $k \leq 1$. The proof is complete.

3.1 Haar wavelet

In the previous section, we proved the existence of solution for Eq. (1). Now, we are going to approximate the solution of Eq. (1) by successive approximations method based on the Haar wavelets.

Definition 4 [32] The Haar scaling function also, called the father wavelet, is defined on the interval $[a, b)$ as

$$\varphi(x) = \begin{cases} 1, & a \leq x < b, \\ 0, & \text{otherwise.} \end{cases} \tag{14}$$

Definition 5 [32] The mother wavelet for Haar wavelets family is also defined on the interval $[a, b)$ as follows

$$\psi(x) = \begin{cases} 1, & a \leq x < \frac{a+b}{2}, \\ -1, & \frac{a+b}{2} \leq x < b, \\ 0, & \text{otherwise.} \end{cases} \tag{15}$$

All the other functions in the Haar wavelets family are defined on subintervals of $[a, b)$ and are generated from $\psi(x)$ by the operations of dilation and translation. Each function in the Haar wavelets family defined for $x \in [a, b)$ except the scaling function can be expressed as

$$H_i(x) = \Psi(2^j - k) = \begin{cases} 1, & \zeta \leq x < \eta, \\ -1, & \eta \leq x < \xi, \\ 0, & \text{otherwise,} \end{cases} \tag{16}$$

where $i = 2, 3, \ldots, 2N$ and

$$\zeta = a + (b-a)\frac{k}{n}, \quad \eta = a + (b-a)\frac{k+0.5}{n}, \quad \xi = a + (b-a)\frac{k+1}{n}. \tag{17}$$

In the above definition the integer $n = 2^j$, $j = 0, 1, \ldots, J$ shows the level of the wavelet and $k = 0, 1, \ldots, n-1$ is the translation parameter. The maximal level of resolution is the integer J.

The wavelet numbers i is calculated according the formula $i = n + k + 1$. In the case of minimal values $n = 1, k = 0$, we have $i = 2$. The maximum of i is $i = 2N = 2^{J+1}$. For $i = 1, 2$, the function $H_1(x)$ is called scaling function whereas $H_2(x)$ is the mother wavelet for the Haar wavelet family.

Proposition 1 *[21] Let $f : [a, b] \to \mathbf{R}$ be continuous integrable function. Consider the integral*

$$I = \int_a^b f(x)dx \tag{18}$$

over the $[a, b]$. Using the quadrature formula with respect to Haar wavelets the above integral can be approximated as follows:

$$I \simeq \frac{(b-a)}{2N} \sum_{i=1}^{2N} f(a + (b-a)\frac{2i-1}{4N}), \qquad i = 1, 2, \ldots, 2N. \tag{19}$$

Definition 6 For $L \geq 0$, the function $f : \mathbf{R} \to \mathbf{R}$ is L-Lipschitz if

$$|f(x_1) - f(x_2)| \leq L|x_1 - x_2|, \qquad \forall x_1, x_2 \in [a, b] \tag{20}$$

Theorem 5 *Let $f : [a, b] \to \mathbf{R}$ be integrable function on $[a, b]$ of L-Lipschitz type. Then the following inequality holds:*

$$\left| \int_a^b f(x)dx - \frac{(b-a)}{2N} \sum_{i=1}^{2N} f\big(a + (b-a)\frac{2i-1}{4N}\big) \right| \leq L\frac{(b-a)^2}{4N}, \tag{21}$$

where $N = 2^J$ is the maximal level of resolution of Haar wavelets.

Proof

$$\begin{aligned}
&\left| \int_a^b f(x)dx - \frac{(b-a)}{2N} \sum_{i=1}^{2N} f\big(a + (b-a)\frac{2i-1}{4N}\big) \right| \\
&\qquad = \left| \int_a^b f(x)dx - \int_a^b \frac{1}{2N} \sum_{i=1}^{2N} f\big(a + (b-a)\frac{2i-1}{4N}\big)dx \right| \\
&\qquad \leq \frac{1}{2N} \int_a^b \sum_{i=1}^{2N} \big|f(x) - f\big(a + (b-a)\frac{2i-1}{4N}\big)\big|dx \\
&\qquad \leq \frac{1}{2N} \int_a^b \sum_{i=1}^{2N} \big(L|x - (a + (b-a)\frac{2i-1}{4N})|\big)dx
\end{aligned}$$

According to the $x \in [a + (b-a)\frac{2i-1}{4N}, a + \frac{b-a}{4N}(2i))$ we get

$$\left| \int_a^b f(x)dx - \frac{(b-a)}{2N} \sum_{i=1}^{2N} f\big(a + (b-a)\frac{2i-1}{4N}\big) \right| \leq L\frac{(b-a)^2}{4N}. \tag{22}$$

Thus, the proof is complete.

3.2 Numerical method

Here, for solving the equation

$$u(s) = f(s) + \lambda \int_a^b h(s, x, u(x))dx, \qquad s \in [a, b], \tag{23}$$

we try to discretize the integral equation by the quadrature formula for the above integral as

$$\int_a^b h(s, x, u(x))dx = \delta \sum_{i=1}^{2N} h\big(s, t_i, u(t_i)\big) + L\frac{(b-a)^2}{4N}, \tag{24}$$

where

$$t_i = a + \frac{b-a}{4N}(2i-1), i = 1, \dots, 2N, \tag{25}$$

with

$$\delta = \frac{b-a}{2N}. \tag{26}$$

Then by substituting in the Hammerstein integral equation, we have

$$S_N(u) \simeq f(s) + \lambda\delta \sum_{i=1}^{2N} h\big(s, t_i, u(t_i)\big), \qquad s \in [a,b]. \tag{27}$$

Now, let

$$\begin{aligned} u_0(s) &= f(s), \\ u_m(s) &= f(s) + \delta \sum_{i=1}^{2N} h\big(s, a + \frac{b-a}{4N}(2i-1), u_{m-1}(a + \frac{b-a}{4N}(2i-1))\big). \end{aligned}$$

It is easy to derive from numerical examples that

$$\lim_{m\to\infty} u_m(s) = u(s), \qquad s \in [a,b].$$

4 Numerical experiments

We have applied our method on some numerical examples, to observe the accuracy and efficiency of the present method for solving NUFIEs. In order to analyze the error of the method we introduce notations

$$\|e_N\|_\infty := \|u^* - u_m^{(N)}\|_\infty = \max\{| \ u^*(s_j) - u_m(s_j) \mid j = \overline{0, 2N}\} \tag{28}$$

The experimental rate of convergence for the following examples is also calculated which is difined as (Chapter 2, [33]):

$$\text{Ratio} = \frac{\|u^* - u_m^{(N)}\|_\infty}{\|u^* - u_m^{(2N)}\|_\infty}, \tag{29}$$

and

$$\rho_N = \log_2\left(\frac{\|e_N\|_\infty}{\|e_{2N}\|_\infty}\right) \tag{30}$$

where ρ_N estimates the convergence rate. Also u^* and $\bar{u}_m$ are the exact solution and approximate solution of the Eq. (1), respectively. Moreover, the number of iterations, m, and the error $\|e_n\|_\infty$ are inserted in Tables 2 and 4. The absolute value of the errors for different values of N, is reported. These tables show that increasing N the error significantly is reduced. In all examples, we choose the tolerance $\varepsilon = 10^{-15}$ to stop the iterations, i.e. fixed point iterations stop when $\|u_m - u_{m-1}\| < \varepsilon$.

Table 1 Absolute errors of the itrative method for Example 1.

s	Exact solution	$e_j, 2N = 8$	$e_j, 2N = 16$	$e_j, 2N = 32$
0.1	1.99	1.1274×10^{-7}	2.7122×10^{-7}	2.8641×10^{-8}
0.3	1.91	2.8843×10^{-7}	5.0062×10^{-7}	7.3253×10^{-8}
0.5	1.75	6.0475×10^{-7}	8.4570×10^{-7}	1.5354×10^{-7}
0.7	1.51	1.1165×10^{-6}	1.3381×10^{-6}	2.8342×10^{-7}
0.9	1.19	1.8865×10^{-6}	2.0131×10^{-6}	4.7881×10^{-7}

Table 2 Rate of convergence and order of convergence for Example 1.

J	N	$2N$	m	$\|e_N\|_\infty$	Ratio	ρ_N
1	2	4	6	3.5629×10^{-5}	–	–
2	4	8	7	1.1253×10^{-5}	3.1661	1.6627
3	8	16	7	2.9856×10^{-6}	3.7690	1.9141
4	16	32	7	7.5763×10^{-7}	3.9407	1.9784
5	32	64	7	1.9012×10^{-7}	3.9850	1.9945

Example 1 Consider nonlinear Fredholm integral equation ([34], Example 3)

$$u(s,t) = f(s) + \int_0^1 sx\sqrt{u(x)}dx, \qquad (s) \in [0,1], \tag{31}$$

where

$$f(s) = 2 - \frac{1}{3}(2\sqrt{2} - 1)s - s^2,$$

and exact solution

$$u(s) = 2 - s^2.$$

In order to find an approximation solution of u^* using above numerical scheme, we choose $2N = 8, 2N = 16, 2N = 32$ and $\varepsilon = 10^{-15}$. The exact solution and the absolute errors $e_{2N}(s_j) = e_j$ are displayed in Table 1.
Also, for $N \in \{2, 4, 8, 16, 32\}$, we test the rate of convergence, the order of convergence and computational results are given in Table 2.

Example 2 Consider the nonlinear integral equation [35]

$$u(s,t) = f(s) + \int_0^1 e^{s-2x}(u(x))^3 dx, \qquad (s) \in [0,1], \tag{32}$$

where

$$f(s) = e^{s+1},$$

and exact solution

$$u(s) = e^s.$$

Table 3 Absolute errors of the itrative method for Example 2.

s	Exact solution	$e_j, 2N = 8$	$e_j, 2N = 16$	$e_j, 2N = 32$
0.1	1.1051	1.3836 $\times 10^{-8}$	3.4843 $\times 10^{-9}$	8.7273 $\times 10^{-10}$
0.3	1.3498	5.0706 $\times 10^{-8}$	1.2767 $\times 10^{-8}$	3.1977 $\times 10^{-9}$
0.5	1.6487	1.2969 $\times 10^{-7}$	3.2652 $\times 10^{-8}$	8.1780 $\times 10^{-9}$
0.7	2.0137	2.7187 $\times 10^{-7}$	6.8440 $\times 10^{-8}$	1.7140 $\times 10^{-8}$
0.9	2.4596	5.0188 $\times 10^{-7}$	1.2632 $\times 10^{-7}$	3.1638 $\times 10^{-8}$

Table 4 Rate of convergence and order of convergence for Example 2.

J	N	$2N$	m	$\|e_N\|_\infty$	Ratio	ρ_N
1	2	4	6	3.3020×10^{-6}	–	–
2	4	8	7	8.4790×10^{-7}	3.8943	1.9613
3	8	16	7	2.1340×10^{-7}	3.9732	1.9903
4	16	32	7	5.3445×10^{-8}	3.9928	1.9974
5	32	64	7	1.3301×10^{-8}	4.0181	2.0065

For $2N = 8$, $2N = 16$, $2N = 32$, and $\varepsilon = 10^{-15}$, the following results are obtained (see Table 3). Also, for $N \in \{2, 4, 8, 16, 32\}$, we test the rate of convergence, the order of convergence and computational results are given in Table 4.

5 Conclusions

In this research, numerical method for numerical solution of nonlinear Urysohn integral equations based on Haar wavelet has been suggested. This method is very simple and involves lower computation. By using the techniques of non-compactness measures and Petryshyn's fixed point, in the Theorem 4 sufficint conditions for the existence solution of Eq. (1) are presented. To illustrate the efficiency of the presented method, two examples are given.

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