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Regularity of Bounded Tri-Linear Maps and the Fourth Adjoint of a Tri-Derivation

Abotaleb Sheikhali* · **Ali Ebadian** ·
Kazem Haghnejad Azar

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Abstract In this Article, we give a simple criterion for the regularity of a tri-linear mapping. We provide if $f : X \times Y \times Z \longrightarrow W$ is a bounded tri-linear mapping and $h : W \longrightarrow S$ is a bounded linear mapping, then f is regular if and only if hof is regular. We also shall give some necessary and sufficient conditions such that the fourth adjoint D^{****} of a tri-derivation D is again tri-derivation.

Keywords Fourth adjoint · Regular · Tri-derivation · Tri-linear

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1 Introduction and preliminaries

Richard Arens showed in [3] that a bounded bilinear map $m : X \times Y \longrightarrow Z$ on normed spaces, has two natural different extensions m^{***}, m^{r***r} from $X^{**} \times Y^{**}$ into Z^{**}. When these extensions are equal, m is called Arens regular. A Banach algebra A is said to be Arens regular, if its product $\pi(a, b) = ab$

*Corresponding author
A. Sheikhali
Department of Mathematics, Payame Noor University (PNU), Tehran, Iran.
E-mail: Abotaleb.sheikhali.20@gmail.com

A. Ebadian
Faculty of Science, Department of Mathematics, Urmia University, Urmia, Iran.
E-mail: Ebadian.ali@gmail.com

K. Haghnejad Azar
Department of Mathematics and Applications, University of Mohaghegh Ardabili, Ardabil, Iran.
E-mail: Haghnejad@aut.ac.ir

considered as a bilinear mapping $\pi : A \times A \longrightarrow A$ is Arens regular. The first and second Arens products of A^{**} by symbols $\Box$ and $\Diamond$ respectively defined by

$$a^{**}\Box b^{**} = \pi^{***}(a^{**}, b^{**}) \quad , \quad a^{**}\Diamond b^{**} = \pi^{r****r}(a^{**}, b^{**}).$$

Some characterizations for the Arens regularity of bounded bilinear map m and Banach algebra A are proved in [1], [2], [3], [4], [5], [9], [11], [14] and [15]. Suppose X, Y, Z, W and S are normed spaces and $f : X \times Y \times Z \longrightarrow W$ is a bounded tri-linear mapping. In this paper we first define regularity of f map and showing that f is regular if and only if $f^{***r*}(X^{**}, W^{*}, Z) \subseteq Y^{*}$ and $f^{*****}(W^{*}, X^{**}, Y^{**}) \subseteq Z^{*}$. Also we show that for a bounded tri-linear map $f : X \times Y \times Z \longrightarrow W$ and a bounded linear operator $h : W \longrightarrow S$, f is regular if and only if hof is regular.

The natural extensions of f are as follows:

1. $f^{*} : W^{*} \times X \times Y \longrightarrow Z^{*}$, given by $\langle f^{*}(w^{*}, x, y), z\rangle = \langle w^{*}, f(x, y, z)\rangle$ where $x \in X,\ y \in Y,\ z \in Z,\ w^{*} \in W^{*}$ (f^{*} is said the adjoint of f and is a bounded tri-linear map).
2. $f^{**} = (f^{*})^{*} : Z^{**} \times W^{*} \times X \longrightarrow Y^{*}$, given by $\langle f^{**}(z^{**}, w^{*}, x), y\rangle = \langle z^{**}, f^{*}(w^{*}, x, y)$ where $x \in X,\ y \in Y,\ z^{**} \in Z^{**},\ w^{*} \in W^{*}$.
3. $f^{***} = (f^{**})^{*} : Y^{**} \times Z^{**} \times W^{*} \longrightarrow X^{*}$, given by $\langle f^{***}(y^{**}, z^{**}, w^{*}), x\rangle = \langle y^{**}, f^{**}(z^{**}, w^{*}, x)\rangle$ where $x \in X,\ y^{**} \in Y^{**},\ z^{**} \in Z^{**},\ w^{*} \in W^{*}$.
4. $f^{****} = (f^{***})^{*} : X^{**} \times Y^{**} \times Z^{**} \longrightarrow W^{**}$, given by $\langle f^{****}(x^{**}, y^{**}, z^{**}), w^{*}\rangle = \langle x^{**}, f^{***}(y^{**}, z^{**}, w^{*})\rangle$ where $x^{**} \in X^{**},\ y^{**} \in Y^{**},\ z^{**} \in Z^{**},\ w^{*} \in W^{*}$.

Now let $f^{r} : Z \times Y \times X \longrightarrow W$ be the flip of f defined by $f^{r}(z, y, x) = f(x, y, z)$, for every $x \in X, y \in Y$ and $z \in Z$. Then f^{r} is a bounded tri-linear map and it may extends as above to $f^{r****} : Z^{**} \times Y^{**} \times X^{**} \longrightarrow W^{**}$. When f^{****} and f^{r****r} are equal, then f is said to be regular. For bounded tri-linear maps, we have naturally six different Aron-Berner extensions to the bidual spaces based on six different elements in S3 and compeletly regularity should be defined accordingly to the equality of all these six Aron-Berner extensions. See [12].

Suppose A is a Banach algebra and $\pi_1 : A \times X \longrightarrow X$ is a bounded bilinear map. The pair (π_1, X) is said to be a left Banach $A-$module when $\pi_1(\pi_1(a, b), x) = \pi_1(a, \pi_1(b, x))$, for each $a, b \in A$ and $x \in X$. A right Banach $A-$module may is defined similarly. Let $\pi_2 : X \times A \longrightarrow X$ be a bounded bilinear map. The pair (X, π_2) is said to be a right Banach $A-$module if $\pi_2(x, \pi_2(a, b)) = \pi_2(\pi_2(x, a), b)$. A triple (π_1, X, π_2) is said to be a Banach $A-$module if (X, π_1) and (X, π_2) are left and right Banach $A-$modules, respectively, and $\pi_1(a, \pi_2(x, b)) = \pi_2(\pi_1(a, x), b)$. Let (π_1, X, π_2) be a Banach $A-$module. Then $(\pi_2^{r*r}, X^{*}, \pi_1^{*})$ is the dual Banach $A-$module of (π_1, X, π_2).

A bounded linear mapping $D_1 : A \longrightarrow X^{*}$ is said to be a derivation if for each $a, b \in A$

$$D_1(\pi(a, b)) = \pi_1^{*}(D_1(a), b) + \pi_2^{r*r}(a, D_1(b)).$$

A bounded bilinear map $D_2 : A \times A \longrightarrow X(\text{or } X^*)$ is called a bi-derivation, if for each a, b, c and $d \in A$

$$D_2(\pi(a,b),c) = \pi_1(a, D_2(b,c)) + \pi_2(D_2(a,c),b),$$
$$D_2(a,\pi(b,c)) = \pi_1(b, D_2(a,c)) + \pi_2(D_2(a,b),c).$$

Let $D_1 : A \longrightarrow A^*$ be a derivation. Dales, Rodriguez and Velasco, in [7] showed that $D_1^{**} : (A^{**}, \Box) \longrightarrow A^{***}$ is a derivation if and only if $\pi^{r****}(D_1^{**}(A^{**}), A^{**}) \subseteq A^*$. In [13], S. Mohamadzadeh and H. Vishki extends this and showed that second adjont $D_1^{**} : (A^{**}, \Box) \longrightarrow A^{***}$ is a derivation if and only if $\pi_2^{****}(D_1^{**}(A^{**}), X^{**}) \subseteq A^*$ and which $D_1^{**} : (A^{**}, \Diamond) \longrightarrow A^{***}$ is a derivation if and only if $\pi_1^{r****}(D_1^{**}(A^{**}), X^{**}) \subseteq A^*$.

A. Erfanian Attar et al, provide condition such that the third adjoint D_2^{***} of a bi-derivation $D_2 : A\times A \longrightarrow X$ (or X^*) is again a bi-derivation, see [8]. For a Banach $A-$module (π_1, X, π_2), the fourth adjoint D^{****} of a tri-derivation $D : A \times A \times A \longrightarrow X^*$ is trivially a tri-linear extension of D. A problem which is of interest is under what conditions we need that D^{****} is again a tri-derivation. In section 4 we will extend above mentioned result. A bounded trilinear mapping $f : X \times Y \times Z \longrightarrow W$ is said to factor if it is surjective, that is $f(X \times Y \times Z) = W$.

Throughout the article, we usually identify a normed space with its canonical image in its second dual.

2 Regularity of bounded tri-linear maps

Theorem 1 *Let $f : X \times Y \times Z \longrightarrow W$ be a bounded tri-linear map. Then f is regular if and only if*

$$w^*-\lim_\alpha w^*-\lim_\beta w^*-\lim_\gamma f(x_\alpha, y_\beta, z_\gamma) = w^*-\lim_\gamma w^*-\lim_\beta w^*-\lim_\alpha f(x_\alpha, y_\beta, z_\gamma),$$

where $\{x_\alpha\}, \{y_\beta\}$ and $\{z_\gamma\}$ are nets in X, Y and Z which converge to $x^{**} \in X^{**}, y^{**} \in Y^{**}$ and $z^{**} \in Z^{**}$ in the w^*-topologies, respectively.

Proof. For every $w^* \in W^*$ we have

$$\begin{aligned}
&\langle f^{****}(x^{**}, y^{**}, z^{**}), w^*\rangle = \langle x^{**}, f^{***}(y^{**}, z^{**}, w^*)\rangle \\
&= \lim_\alpha \langle f^{***}(y^{**}, z^{**}, w^*), x_\alpha\rangle = \lim_\alpha \langle y^{**}, f^{**}(z^{**}, w^*, x_\alpha)\rangle \\
&= \lim_\alpha \lim_\beta \langle f^{**}(z^{**}, w^*, x_\alpha), y_\beta\rangle = \lim_\alpha \lim_\beta \langle z^{**}, f^*(w^*, x_\alpha, y_\beta)\rangle \\
&= \lim_\alpha \lim_\beta \lim_\gamma \langle f^*(w^*, x_\alpha, y_\beta), z_\gamma\rangle = \lim_\alpha \lim_\beta \lim_\gamma \langle f(x_\alpha, y_\beta, z_\gamma), w^*\rangle.
\end{aligned}$$

Therefore $f^{****}(x^{**}, y^{**}, z^{**}) = w^* - \lim_\alpha w^* - \lim_\beta w^* - \lim_\gamma f(x_\alpha, y_\beta, z_\gamma)$. In the other hands $f^{r****r}(x^{**}, y^{**}, z^{**}) = w^* - \lim_\gamma w^* - \lim_\beta w^* - \lim_\alpha f(x_\alpha, y_\beta, z_\gamma)$, and proof follows. □

In the following theorem, we provide a criterion concerning to the regularity of a bounded tri-linear map.

Theorem 2 *For a bounded tri-linear map $f : X \times Y \times Z \longrightarrow W$ the following statements are equivalent:*

1. *f is regular.*
2. *$f^{*****}(W^{***}, X^{**}, Y^{**}) = f^{r*****r}(W^{***}, X^{**}, Y^{**})$.*
3. *$f^{***r*}(X^{**}, W^{*}, Z) \subseteq Y^{*}$ and $f^{*****}(W^{*}, X^{**}, Y^{**}) \subseteq Z^{*}$.*

Proof. (1) $\Rightarrow$ (2), if f is regular, then $f^{****} = f^{r****r}$. For every $x^{**} \in X^{**}, y^{**} \in Y^{**}, z^{**} \in Z^{**}$ and $w^{***} \in W^{***}$ we have

$$\langle f^{*****}(w^{***}, x^{**}, y^{**}), z^{**}\rangle = \langle w^{***}, f^{****}(x^{**}, y^{**}, z^{**})\rangle$$
$$= \langle w^{***}, f^{r****r}(x^{**}, y^{**}, z^{**})\rangle = \langle f^{r*****r}(w^{***}, x^{**}, y^{**}), z^{**}\rangle.$$

as claimed.

(2) $\Rightarrow$ (1), let $f^{*****} = f^{r*****r}$, then for every $w^{*} \in W^{*}$,

$$\langle f^{r****r}(x^{**}, y^{**}, z^{**}), w^{*}\rangle = \langle f^{r*****r}(w^{*}, x^{**}, y^{**}), z^{**}\rangle$$
$$= \langle f^{*****}(w^{*}, x^{**}, y^{**}), z^{**}\rangle = \langle f^{****}(x^{**}, y^{**}, z^{**}), w^{*}\rangle.$$

It follows that f is regular.

(1) $\Rightarrow$ (3), assume that f is regular and $x^{**} \in X^{**}, y^{**} \in Y^{**}, z \in Z, w^{*} \in W^{*}$. Then we have

$$\langle f^{***r*}(x^{**}, w^{*}, z), y^{**}\rangle = \langle f^{****}(x^{**}, y^{**}, z), w^{*}\rangle$$
$$= \langle f^{r****r}(x^{**}, y^{**}, z), w^{*}\rangle = \langle f^{r**}(x^{**}, w^{*}, z), y^{**}\rangle.$$

Therefore $f^{***r*}(x^{**}, w^{*}, z) = f^{r**}(x^{**}, w^{*}, z) \in Y^{*}$. So $f^{***r*}(X^{**}, W^{*}, Z) \subseteq Y^{*}$. A similar argument shows that $f^{*****}(w^{*}, x^{**}, y^{**}) = f^{r***r}(w^{*}, x^{**}, y^{**}) \in Z^{*}$. Thus $f^{*****}(W^{*}, X^{**}, Y^{**}) \subseteq Z^{*}$, as claimed.

(3) $\Rightarrow$ (1), let $\{x_\alpha\}, \{y_\beta\}$ and $\{z_\gamma\}$ are nets in X, Y and Z which converge to x^{**}, y^{**} and z^{**} in the $w^{*}-$topologies, respectively. For every $w^{*} \in W^{*}$ we have

$$\langle f^{r****r}(x^{**}, y^{**}, z^{**}), w^{*}\rangle = \lim_\gamma \lim_\beta \lim_\alpha \langle f(x_\alpha, y_\beta, z_\gamma), w^{*}\rangle$$
$$= \lim_\gamma \lim_\beta \lim_\alpha \langle f^{***}(y_\beta, z_\gamma, w^{*}), x_\alpha\rangle = \lim_\gamma \lim_\beta \langle x^{**}, f^{***}(y_\beta, z_\gamma, w^{*})$$
$$= \lim_\gamma \lim_\beta \langle x^{**}, f^{***r}(w^{*}, z_\gamma, y_\beta)\rangle = \lim_\gamma \lim_\beta \langle f^{***r*}(x^{**}, w^{*}, z_\gamma), y_\beta\rangle$$
$$= \lim_\gamma \langle f^{***r*}(x^{**}, w^{*}, z_\gamma), y^{**}\rangle = \lim_\gamma \langle x^{**}, f^{***r}(w^{*}, z_\gamma, y^{**})\rangle$$
$$= \lim_\gamma \langle x^{**}, f^{***}(y^{**}, z_\gamma, w^{*})\rangle = \lim_\gamma \langle f^{****}(x^{**}, y^{**}, z_\gamma), w^{*}\rangle$$
$$= \lim_\gamma \langle f^{*****}(w^{*}, x^{**}, y^{**}), z_\gamma\rangle = f^{*****}(w^{*}, x^{**}, y^{**}), z^{**}\rangle$$
$$= \langle f^{****}(x^{**}, y^{**}, z^{**}), w^{*}\rangle.$$

It follows that f is regular and this completes the proof. □

Corollary 1 *For a bounded tri-linear map $f : X \times Y \times Z \longrightarrow W$ the following statements are equivalent:*

1. *f is regular.*
2. *$f^{r******r} = f^{*******}$.*
3. *$f^{r****r*}(Z^{**}, W^*, X) \subseteq Y^*$ and $f^{*****}(W^*, Z^{**}, Y^{**}) \subseteq X^*$.*

Proof. The mapping f is regular if and only if f^r is regular. Therefore by Theorem 2, the desired result is obtained. □

Corollary 2 *For a bounded tri-linear map $f : X \times Y \times Z \longrightarrow W$, if from X, Y or Z at least two reflexive then f is regular.*

Proof. Without having to enter the whole argument, let Y and Z are reflexive. Since Y is reflexive, $Y^* = Y^{***}$. Therefore

$$f^{***r*}(X^{**}, W^*, Z^{**}) \subseteq Y^{***} = Y^* \qquad (2-1)$$

In the other hands, since Z is the reflexive space, thus

$$f^{*****}(W^{***}, X^{**}, Y^{**}) \subseteq Z^{***} = Z^* \qquad (2-2)$$

Now Using (2-1), (2-2) and Theorem 2, the result holds. □

Corollary 3 *Let bounded tri-linear map $f : X \times Y \times Z \longrightarrow W$ be regular. Then*

1. *If $f^{***r*}(X^{**}, W^*, Z)$ factors, then Y is reflexive space.*
2. *If $f^{*****}(W^*, X^{**}, Y^{**})$ factors, then Z is reflexive space.*
3. *If $f^{*****r*}(W^*, Z, Y)$ factors, then X is reflexive space.*

Proof. (1) Let f be regular. It follows that $f^{***r*}(X^{**}, W^*, Z) \subseteq Y^*$. In the other hands, $f^{***r*}(X^{**}, W^*, Z)$ is factor. So for each $y^{***} \in Y^{***}$ there exist $x^{**} \in X^{**}, w^* \in W^*$ and $z \in Z$ such that $f^{***r*}(x^{**}, w^*, z) = y^{***}$. Therefore $Y^{***} \subseteq Y^*$.

(2) The proof similar to (1).

(3) Enough show that $f^{*****r*}(W^*, Z, Y) \subseteq X^*$ whenever f is regular. For every $x^{**} \in X^{**}, y \in Y, z \in Z$ and $w^* \in W^*$ we have

$$\begin{aligned} &\langle f^{*****r*}(w^*, z, y), x^{**}\rangle = \langle w^*, f^{****}(x^{**}, y, z)\rangle \\ &= \langle f^{r*****r}(x^{**}, y, z), w^*\rangle = \langle f^{r*}(w^*, z, y), x^{**}\rangle. \end{aligned}$$

Therefore $f^{*****r*}(w^*, z, y) = f^{r*}(w^*, z, y) \in X^*$. The rest of proof has similar argument such as (1). □

Corollary 4 *If I_X, I_Y and I_Z are weakly compact identity mapping, then all of them and all of their adjoints are regular.*

Example 1 1. Let G be a compact group. Let $1 < p, q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$. Then by [10, Sections 2.4 and 2.5], we conclude that $L^1(G) \star L^p(G) \subset L^p(G)$ and $L^p(G) \star L^q(G) \subset L^r(G)$ where $(g \star h)(x) = \int_G g(y)h(y^{-1}x)dy$ for $x \in G$. Since the Banach spaces $L^p(G)$ and $L^q(G)$ are reflexive, thus by corollary 2 we conclude that the bounded tri-linear mapping

$$f : L^1(G) \times L^p(G) \times L^q(G) \longrightarrow L^r(G)$$

defined by $f(k, g, h) = (k \star g) \star h$, is regular for every $k \in L^1(G), g \in L^p(G)$ and $h \in L^q(G)$.

2. Let G be a locally compact group. We know from [16] that $L^1(G)$ is regular if and only if it is reflexive or G is finite. It follows that for every finite locally compact group G, by corollary 2, the bounded tri-linear mapping $f : L^1(G) \times L^1(G) \times L^1(G) \longrightarrow L^1(G)$ defined by $f(k, g, h) = k \star g \star h$, is regular for every k, g and $h \in L^1(G)$.
3. C^*-algebras are standard examples of Banach algebras that are Arens regular, see[6]. We know that a C^*-algebra is reflexive if and only if it is of finite dimension. Since if A is a finite dimension C^*-algebra, then by corollary 2, we conclude that the bounded tri-linear mapping $f : A \times A \times A \longrightarrow A$ is regular.
4. Let G be a locally compact group and let $M(G)$ be measure algebra of G, see [10, Section 2.5]. Let the convolution for $\mu_1, \mu_2 \in M(G)$ defined by

$$\int \psi d(\mu_1 * \mu_2) = \int \int \psi(xy) d\mu_1(x) d\mu_2(y), \quad (\psi \in C_0(G)).$$

We have

$$\begin{aligned} \int \psi d(\mu_1 * (\mu_2 * \mu_3)) &= \int \int \int \psi(xyz) d\mu_1(x) d\mu_2(y) d\mu_3(z) \\ &= \int \psi d((\mu_1 * \mu_2) * \mu_3) \end{aligned}$$

for μ_1, μ_2 and $\mu_3 \in M(G)$. Therefore convolution is associative. Now we define the bounded tri-linear mapping

$$f : M(G) \times M(G) \times M(G) \longrightarrow M(G)$$

by $f(\mu_1, \mu_2, \mu_3) = \int \psi d(\mu_1 * \mu_2 * \mu_3)$. If G is finite, then f is regular.

3 Some results for regularity

Dales, Rodriguez-Palacios and Velasco in [7, Theorem 4.1], for a bonded bilinear map $m : X \times Y \longrightarrow Z$ have shown that, $m^{r*r***} = m^{***r*r}$ if and only if both m and m^{r*} are Arens regular. Now in the following we study it in general case.

Remark 1 In the next theorem, f^n is $n-$th adjoint of f for each $n \in N$.

Theorem 3 *If f and f^{rn} are reular, then $f^{4rnr} = f^{rnr4}$.*

Proof. Since f is regular, so $f^{4r} = f^{r4}$. Therefore $f^{4rn} = f^{r(n+4)}$. In the other hands, regularity of f^{rn} follows that $f^{r(n+4)} = f^{rnr4r}$. Thus $f^{rnr4r} = f^{4rn}$ and this completes the proof. □

Theorem 4 *Let $f : X \times Y \times Z \longrightarrow W$ be a bounded tri-linear mapping. Then*

1. *$f^{****r**r} = f^{r**r****}$ if and only if both f and f^{r**} are regular.*
2. *$f^{****r***r} = f^{r***r****}$ if and only if both f and f^{r***} are regular.*

Proof. We prove only (1), the other part has the same argument. If both f and f^{r**} are regular, then by applying Theorem 3, for $n = 2$, $f^{****r**r} = f^{r**r****}$.

Conversely, suppose that $f^{****r**r} = f^{r**r****}$. First we show that f is regular. Let $\{z_\gamma\}$ is net in Z which converge to $z^{**} \in Z^{**}$ in the w^*-topologies. Then for every $x^{**} \in X^{**}, y^{**} \in Y^{**}$ and $w^* \in W^*$ we have

$$\begin{aligned}
&\langle f^{****}(x^{**}, y^{**}, z^{**}), w^* \rangle = \langle f^{*****r}(z^{**}, y^{**}, x^{**}), w^* \rangle \\
&= \langle f^{****r**r}(z^{**}, w^*, x^{**}), y^{**} \rangle = \langle f^{r**r****}(z^{**}, w^*, x^{**}), y^{**} \rangle \\
&= \lim_\gamma \langle y^{**}, f^{r**r}(z_\gamma, w^*, x^{**}) \rangle = \langle f^{r****r}(x^{**}, y^{**}, z^{**}), w^* \rangle.
\end{aligned}$$

Therefore f is regular. Now we show that f^{r**} is regular. Let $\{x^{**}_\alpha\}$ be net in X^{**} which converge to $x^{****} \in X^{****}$ in the w^*-topologies. Then for every $y^{**} \in Y^{**}, z^{**} \in Z^{**}$ and $w^{***} \in W^{***}$ we have

$$\begin{aligned}
&\langle f^{r**r*****}(x^{****}, w^{***}, z^{**}), y^{**} \rangle = \langle f^{r**r****}(z^{**}, w^{***}, x^{****}), y^{**} \rangle \\
&= \langle f^{****r**r}(z^{**}, w^{***}, x^{****}), y^{**} \rangle = \lim_\alpha \langle w^{***}, f^{****}(x^{**}_\alpha, y^{**}, z^{**}) \rangle \\
&= \lim_\alpha \langle w^{***}, f^{r****r}(x^{**}_\alpha, y^{**}, z^{**}) \rangle = \lim_\alpha \langle w^{***}, f^{r****}(z^{**}, y^{**}, x^{**}_\alpha) \rangle \\
&= \langle f^{r******}(x^{****}, w^{***}, z^{**}), y^{**} \rangle.
\end{aligned}$$

It follows that f^{r**} is regular and this completes the proof. □

Arens has shown [3] that a bounded bilinear map m is regular if and only if for each $z^* \in Z^*$, the bilinear form $z^* om$ is regular. In the next theorem we give an important characterization of regularity bounded tri-linear mappings.

Lemma 1 *Suppose X, Y, Z, W and S are normed spaces and $f : X \times Y \times Z \longrightarrow W$ and $h : W \longrightarrow S$ are bounded tri-linear mapping and bounded linear mapping, respectively. Then we have*

1. *$h^{**} of^{****} = (hof)^{****}$.*
2. *$h^{**} of^{r****r} = (hof)^{r****r}$.*

Proof. Let $\{x_\alpha\}, \{y_\beta\}$ and $\{z_\gamma\}$ be nets in X, Y and Z which converge to $x^{**} \in X^{**}, y^{**} \in Y^{**}$ and $z^{**} \in Z^{**}$ in the w^*-topologies, respectively. For each $s^* \in S^*$ we have

$$\begin{aligned}
&\langle h^{**}of^{****}(x^{**}, y^{**}, z^{**}), s^*\rangle = \langle h^{**}(f^{****}(x^{**}, y^{**}, z^{**})), s^*\rangle \\
&= \langle f^{****}(x^{**}, y^{**}, z^{**}), h^*(s^*)\rangle = \lim_\alpha \lim_\beta \lim_\gamma \langle h^*(s^*), f(x_\alpha, y_\beta, z_\gamma)\rangle \\
&= \lim_\alpha \lim_\beta \lim_\gamma \langle s^*, h(f(x_\alpha, y_\beta, z_\gamma))\rangle = \lim_\alpha \lim_\beta \lim_\gamma \langle s^*, hof(x_\alpha, y_\beta, z_\gamma)\rangle \\
&= \langle (hof)^{****}(x^{**}, y^{**}, z^{**}), s^*\rangle.
\end{aligned}$$

Hence $h^{**}of^{****}(x^{**}, y^{**}, z^{**}) = (hof)^{****}(x^{**}, y^{**}, z^{**})$. A similar argument applies for (2). □

Theorem 5 *Let $f : X \times Y \times Z \longrightarrow W$ and $h : W \longrightarrow S$ be bounded tri-linear mapping and bounded linear mapping, respectively. Then f is regular if and only if hof is regular.*

Proof. Assume that f is regular. Then for every $x^{**} \in X^{**}, y^{**} \in Y^{**}, z^{**} \in Z^{**}$ and $s^* \in S^*$ we have

$$\begin{aligned}
&\langle h^{**}(f^{r*****r}(x^{**}, y^{**}, z^{**})), s^*\rangle = \langle f^{r*****r}(x^{**}, y^{**}, z^{**}), h^*(s^*)\rangle \\
&= \langle f^{****}(x^{**}, y^{**}, z^{**}), h^*(s^*)\rangle = \langle h^{**}(f^{****}(x^{**}, y^{**}, z^{**})), s^*\rangle.
\end{aligned}$$

Therefore $h^{**}of^{r*****r}(x^{**}, y^{**}, z^{**}) = h^{**}of^{****}(x^{**}, y^{**}, z^{**})$ and by applying Lemma 1, we implies that

$$(hof)^{r*****r}(x^{**}, y^{**}, z^{**}) = (hof)^{****}(x^{**}, y^{**}, z^{**}).$$

It follows that hof is regular.

For the converse, suppose that hof is regular. By contradiction, let f be not regular. Thus there exist $x^{**} \in X^{**}, y^{**} \in Y^{**}$ and $z^{**} \in Z^{**}$ such that $f^{****}(x^{**}, y^{**}, z^{**}) \neq f^{r*****r}(x^{**}, y^{**}, z^{**})$. Therefore we have

$$\begin{aligned}
&(hof)^{****}(x^{**}, y^{**}, z^{**}) = w^* - \lim_\alpha w^* - \lim_\beta w^* - \lim_\gamma (hof)(x_\alpha, y_\beta, z_\gamma) \\
&= \lim_\alpha \lim_\beta \lim_\gamma \langle f(x_\alpha, y_\beta, z_\gamma), h\rangle = \langle f^{****}(x^{**}, y^{**}, z^{**}), h\rangle \\
&\neq \langle f^{r*****r}(x^{**}, y^{**}, z^{**}), h\rangle = \lim_\gamma \lim_\beta \lim_\alpha \langle f(x_\alpha, y_\beta, z_\gamma), h\rangle \\
&= w^* - \lim_\gamma w^* - \lim_\beta w^* - \lim_\alpha (hof)(x_\alpha, y_\beta, z_\gamma) \\
&= (hof)^{r*****r}(x^{**}, y^{**}, z^{**}).
\end{aligned}$$

It follows that $(hof)^{****}(x^{**}, y^{**}, z^{**}) \neq (hof)^{r*****r}(x^{**}, y^{**}, z^{**})$. □

Another interesting case of regularity is in the following.

Theorem 6 *Let X, Y, Z, W and S be Banach spaces, $f : X \times Y \times Z \longrightarrow W$ be a bounded tri-linear mapping and $x \in X, y \in Y, z \in Z$. Then*

1. *Let $g_1 : S \times Y \times Z \longrightarrow W$ be a bounded tri-linear mapping and let $h_1 : X \longrightarrow S$ be a bounded linear mapping such that $f(x, y, z) = g_1(h_1(x), y, z)$. If h_1 is weakly compact, then $f^{*****r*}(W^{***}, Z^{**}, Y^{**}) \subseteq X^*$.*

2. *Let $g_2 : X \times S \times Z \longrightarrow W$ be a bounded tri-linear mapping and let $h_2 : Y \longrightarrow S$ be a bounded linear mapping such that $f(x, y, z) = g_2(x, h_2(y), z)$. If h_2 is weakly compact, then $f^{****r*}(X^{**}, W^*, Z^{**}) \subseteq Y^*$.*
3. *Let $g_3 : X \times Y \times S \longrightarrow W$ be a bounded tri-linear mapping and let $h_3 : Z \longrightarrow S$ be a bounded linear mapping such that $f(x, y, z) = g_3(x, y, h_3(z))$. If h_3 is weakly compact, then $f^{*****}(W^{***}, X^{**}, Y^{**}) \subseteq Z^*$.*

Proof. We prove only (1), the other parts have the same argument. For every $x \in X, y \in Y, z \in Z$ and $w^* \in W^*$ we have

$$\langle f^*(w^*, x, y), z\rangle = \langle w^*, f(x, y, z)\rangle = \langle w^*, g_1(h_1(x), y, z)\rangle = \langle g_1^*(w^*, h_1(x), y), z\rangle.$$

Therefore $f^*(w^*, x, y) = g_1^*(w^*, h_1(x), y)$, and implies that for every $z^{**} \in Z^{**}$,

$$\begin{aligned}&\langle f^{**}(z^{**}, w^*, x), y\rangle = \langle z^{**}, f^*(w^*, x, y)\rangle \\ &= \langle z^{**}, g_1^*(w^*, h_1(x), y)\rangle = \langle g_1^{**}(z^{**}, w^*, h_1(x)), y\rangle.\end{aligned}$$

So $f^{**}(z^{**}, w^*, x) = g_1^{**}(z^{**}, w^*, h_1(x))$ and implies that for every $y^{**} \in Y^{**}$,

$$\begin{aligned}&\langle f^{***}(y^{**}, z^{**}, w^*), x\rangle = \langle y^{**}, f^{**}(z^{**}, w^*, x)\rangle = \langle y^{**}, g_1^{**}(z^{**}, w^*, h_1(x))\rangle \\ &= \langle g_1^{***}(y^{**}, z^{**}, w^*), h_1(x)\rangle = \langle h_1^*(g_1^{***}(y^{**}, z^{**}, w^*)), x\rangle.\end{aligned}$$

Thus $f^{***}(y^{**}, z^{**}, w^*) = h_1^*(g_1^{***}(y^{**}, z^{**}, w^*))$ and implies that for every $x^{**} \in X^{**}$,

$$\begin{aligned}&\langle f^{****}(x^{**}, y^{**}, z^{**}), w^*\rangle = \langle x^{**}, f^{***}(y^{**}, z^{**}, w^*)\rangle \\ &= \langle x^{**}, h_1^*(g_1^{***}(y^{**}, z^{**}, w^*))\rangle = \langle h_1^{**}(x^{**}), (g_1^{***}(y^{**}, z^{**}, w^*)\rangle \\ &= \langle g_1^{****}(h_1^{**}(x^{**}), y^{**}, z^{**}), w^*\rangle.\end{aligned}$$

Therefore for every $w^{***} \in W^{***}$ we have

$$\begin{aligned}&\langle f^{****r*}(w^{***}, z^{**}, y^{**}), x^{**}\rangle = \langle w^{***}, f^{****r}(z^{**}, y^{**}, x^{**})\rangle \\ &= \langle w^{***}, f^{****}(x^{**}, y^{**}, z^{**})\rangle = \langle w^{***}, g_1^{****}(h_1^{**}(x^{**}), y^{**}, z^{**})\rangle \\ &= \langle w^{***}, g_1^{****r}(z^{**}, y^{**}, h_1^{**}(x^{**}))\rangle = \langle g_1^{****r*}(w^{***}, z^{**}, y^{**}), h_1^{**}(x^{**})\rangle \\ &= \langle h_1^{***}(g_1^{****r*}(w^{***}, z^{**}, y^{**})), x^{**}\rangle.\end{aligned}$$

Therefore $f^{****r*}(w^{***}, z^{**}, y^{**}) = h_1^{***}(g_1^{****r*}(w^{***}, z^{**}, y^{**}))$. The weak compactness of h_1 implies that of h_1^*, from which we have $h_1^{***}(S^{***}) \subseteq X^*$. Thus $h_1^{***}(g_1^{****r*}(w^{***}, z^{**}, y^{**})) \in X^*$ and this completes the proof. □

This theorem, combined with Theorem 2, yields the next result.

Corollary 5 *With the assumptions Theorem 6, if h_2 and h_3 are weakly compact, then f is regular.*

Proof. Both h_2 and h_3 are weakly compact, so by Theorem 6 we have

$$f^{***r*}(X^{**}, W^*, Z^{**}) \subseteq Y^* \quad , \quad f^{*****}(W^{***}, X^{**}, Y^{**}) \subseteq Z^*.$$

In particular

$$f^{***r*}(X^{**}, W^*, Z) \subseteq Y^* \quad , \quad f^{*****}(W^*, X^{**}, Y^{**}) \subseteq Z^*.$$

Now by Theorem 2, f is regular. □

The converse of previous result is not true in general sense as following corollary.

Corollary 6 *With the assumptions Theorem 6, if f is regular and both g_2^{***r*} and g_3^{*****} are factors, then h_2 and h_3 are weakly compact.*

Proof. Since $f^{***r*}(X^{**}, W^*, Z^{**}) = h_2^{***}(g_2^{***r*}(X^{**}, W^*, Z^{**}))$, so $h_2^{***}(g_2^{***r*}(X^{**}, W^*, Z^{**})) \subseteq Y^*$. In the other hands g_2^{***r*} is factors, so implies that $h_2^{***}(S^{***}) \subseteq Y^*$. Therefore h_2^* is weakly compact and implies that h_2 is weakly compact. The other part has the same argument for h_3. □

4 The fourth adjoint of a tri-derivation

Definition 1 Let (π_1, X, π_2) be a Banach $A-$module. A bounded tri-linear mapping $D : A \times A \times A \longrightarrow X$ is said to be a tri-derivation when

1. $D(\pi(a, d), b, c) = \pi_2(D(a, b, c), d) + \pi_1(a, D(d, b, c))$,
2. $D(a, \pi(b, d), c) = \pi_2(D(a, b, c), d) + \pi_1(b, D(a, d, c))$,
3. $D(a, b, \pi(c, d)) = \pi_2(D(a, b, c), d) + \pi_1(c, D(a, b, d))$,

for each $a, b, c, d \in A$. If (π_1, X, π_2) is a Banach $A-$module, then $(\pi_2^{r*r}, X^*, \pi_1^*)$ is the dual Banach $A-$module of (π_1, X, π_2). Therefore a bounded tri-linear mapping $D : A \times A \times A \longrightarrow X^*$ is a tri-derivation when

1. $D(\pi(a, d), b, c) = \pi_1^*(D(a, b, c), d) + \pi_2^{r*r}(a, D(d, b, c))$,
2. $D(a, \pi(b, d), c) = \pi_1^*(D(a, b, c), d) + \pi_2^{r*r}(b, D(a, d, c))$,
3. $D(a, b, \pi(c, d)) = \pi_1^*(D(a, b, c), d) + \pi_2^{r*r}(c, D(a, b, d))$.

It can also be written, a bounded tri-linear mapping $D : A \times A \times A \longrightarrow A$ is said to be a tri-derivation when

1. $D(\pi(a, d), b, c) = \pi(D(a, b, c), d) + \pi(a, D(d, b, c))$,
2. $D(a, \pi(b, d), c) = \pi(D(a, b, c), d) + \pi(b, D(a, d, c))$,
3. $D(a, b, \pi(c, d)) = \pi(D(a, b, c), d) + \pi(c, D(a, b, d))$.

Example 2 Let A be a Banach algebra, for any $a, b \in A$ the symbol $[a, b] = ab - ba$ stands for multiplicative commutator of a and b. Let $M_{n\times n}(C)$ be the Banach algebra of all $n \times n$ matrix and $A = \{\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} \in M_{n\times n}(C) |\ x, y \in C\}$. Then A is Banach algebra with the norm

$$\| a \| = (\Sigma_{i,j} |\alpha_{ij}|^2)^{\frac{1}{2}} \quad , \ (a = (\alpha_{ij}) \in A).$$

We define $D : A \times A \times A \longrightarrow A$ to be the bounded tri-linear map given by

$$D(a,b,c) = [\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, abc] \quad , \quad (a,b,c \in A).$$

Then for $a = \begin{pmatrix} x_1 & y_1 \\ 0 & 0 \end{pmatrix}, b = \begin{pmatrix} x_2 & y_2 \\ 0 & 0 \end{pmatrix}, c = \begin{pmatrix} x_3 & y_3 \\ 0 & 0 \end{pmatrix}$ and $d = \begin{pmatrix} x_4 & y_4 \\ 0 & 0 \end{pmatrix} \in A$ we have

$$\begin{aligned}
&D(\pi(a,d),b,c) = D(\begin{pmatrix} x_1x_4 & x_1y_4 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} x_2 & y_2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} x_3 & y_3 \\ 0 & 0 \end{pmatrix}) \\
&= [\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} x_1x_2x_3x_4 & x_1x_2x_4y_3 \\ 0 & 0 \end{pmatrix}] = \begin{pmatrix} 0 & -x_1x_2x_3x_4 \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & -x_1x_2x_3 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_4 & y_4 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} x_1 & y_1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -x_2x_3x_4 \\ 0 & 0 \end{pmatrix} \\
&= (\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & x_1x_2x_3 \\ 0 & 0 \end{pmatrix}) \begin{pmatrix} x_4 & y_4 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} x_1 & y_1 \\ 0 & 0 \end{pmatrix} (\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & x_2x_3x_4 \\ 0 & 0 \end{pmatrix}) \\
&= (\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1x_2x_3 & x_1x_2y_3 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} x_1x_2x_3 & x_1x_2y_3 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}) \begin{pmatrix} x_4 & y_4 \\ 0 & 0 \end{pmatrix} \\
&+ \begin{pmatrix} x_1 & y_1 \\ 0 & 0 \end{pmatrix} (\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_2x_3x_4 & x_2x_4y_3 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} x_2x_3x_4 & x_2x_4y_3 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}) \\
&= [\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} x_1x_2x_3 & x_1x_2y_3 \\ 0 & 0 \end{pmatrix}] \begin{pmatrix} x_4 & y_4 \\ 0 & 0 \end{pmatrix} \\
&+ \begin{pmatrix} x_1 & y_1 \\ 0 & 0 \end{pmatrix} [\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} x_2x_3x_4 & x_2x_4y_3 \\ 0 & 0 \end{pmatrix}] \\
&= [\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} x_1 & y_1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_2 & y_2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_3 & y_3 \\ 0 & 0 \end{pmatrix}] \begin{pmatrix} x_4 & y_4 \\ 0 & 0 \end{pmatrix} \\
&+ \begin{pmatrix} x_1 & y_1 \\ 0 & 0 \end{pmatrix} [\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} x_4 & y_4 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_2 & y_2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_3 & y_3 \\ 0 & 0 \end{pmatrix}] \\
&= D(\begin{pmatrix} x_1 & y_1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} x_2 & y_2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} x_3 & y_3 \\ 0 & 0 \end{pmatrix}) \begin{pmatrix} x_4 & y_4 \\ 0 & 0 \end{pmatrix} \\
&+ \begin{pmatrix} x_1 & y_1 \\ 0 & 0 \end{pmatrix} D(\begin{pmatrix} x_4 & y_4 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} x_2 & y_2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} x_3 & y_3 \\ 0 & 0 \end{pmatrix}) \\
&= \pi(D(a,b,c),d) + \pi(a, D(d,b,c)).
\end{aligned}$$

Similarly, we have $D(a, \pi(b,d), c) = \pi(D(a,b,c),d) + \pi(b, D(a,d,c))$ and $D(a,b, \pi(c,d)) = \pi(D(a,b,c),d) + \pi(c, D(a,b,d))$. Thus D is tri-derivation.

Now, we provide a necessary and sufficient condition such that the fourth adjoint D^{****} of a tri-derivation $D : A \times A \times A \longrightarrow X$ is again a tri-derivation. For the fourth adjoint D^{****} of a tri-derivation $D : A \times A \times A \longrightarrow X$, we are faced with the case eight:

$$
\begin{array}{ll}
(case1) & D^{****}:(A^{**},\Box)\times(A^{**},\Box)\times(A^{**},\Box)\longrightarrow X^{**},\\
(case2) & D^{****}:(A^{**},\Diamond)\times(A^{**},\Box)\times(A^{**},\Box)\longrightarrow X^{**},\\
(case3) & D^{****}:(A^{**},\Box)\times(A^{**},\Diamond)\times(A^{**},\Box)\longrightarrow X^{**},\\
(case4) & D^{****}:(A^{**},\Box)\times(A^{**},\Box)\times(A^{**},\Diamond)\longrightarrow X^{**},\\
(case5) & D^{****}:(A^{**},\Diamond)\times(A^{**},\Diamond)\times(A^{**},\Box)\longrightarrow X^{**},\\
(case6) & D^{****}:(A^{**},\Diamond)\times(A^{**},\Box)\times(A^{**},\Diamond)\longrightarrow X^{**},\\
(case7) & D^{****}:(A^{**},\Box)\times(A^{**},\Diamond)\times(A^{**},\Diamond)\longrightarrow X^{**},\\
(case8) & D^{****}:(A^{**},\Diamond)\times(A^{**},\Diamond)\times(A^{**},\Diamond)\longrightarrow X^{**}.
\end{array}
$$

In the following, we prove the state of case 1. The remaining state are proved in the same way.

Theorem 7 *Let (π_1, X, π_2) be a Banach $A-$module and $D : A\times A\times A \longrightarrow X$ be a tri-derivation. Then $D^{****} : (A^{**},\Box)\times(A^{**},\Box)\times(A^{**},\Box)\longrightarrow X^{**}$ is a tri-derivation if and only if*

1. $\pi_2^{**r*}(D^{****}(A, A, A^{**}), X^*) \subseteq A^*$,
2. $\pi_2^{****}(X^*, D^{****}(A, A^{**}, A^{**})) \subseteq A^*$,
3. $D^{****r*}(\pi_1^{****}(X^*, A^{**}), A^{**}, A^{**}) \subseteq A^*$,
4. $D^{******}(A^{**}, \pi_1^{****}(X^*, A^{**}), A) \subseteq A^*$,
5. $D^{*******}(A^{**}, A^{**}, \pi_1^{****}(X^*, A^{**})) \subseteq A^*$.

Proof. Let $D : A \times A \times A \longrightarrow X$ be a tri-derivation and (1),(2),(3),(4),(5) holds. If $\{a_\alpha\}, \{b_\beta\}, \{c_\gamma\}$ and $\{d_\tau\}$ are bounded nets in A , converging in w^*-topology to a^{**}, b^{**}, c^{**} and $d^{**} \in A^{**}$ respectively, in this case using (2), we conclude that $w^* - \lim_\alpha w^* - \lim_\tau w^* - \lim_\beta w^* - \lim_\gamma \pi_2(D(a_\alpha, b_\beta, c_\gamma), d_\tau) = \pi_2^{***}(D^{****}(a^{**}, b^{**}, c^{**}), d^{**})$. Thus for every $x^* \in X^*$ we have

$$
\begin{aligned}
&\langle D^{****}(\pi^{***}(a^{**}, d^{**}), b^{**}, c^{**}), x^*\rangle\\
&= \lim_\alpha \lim_\tau \lim_\beta \lim_\gamma \langle x^*, D(\pi(a_\alpha, d_\tau), b_\beta, c_\gamma)\rangle\\
&= \lim_\alpha \lim_\tau \lim_\beta \lim_\gamma \langle x^*, \pi_2(D(a_\alpha, b_\beta, c_\gamma), d_\tau) + \pi_1(a_\alpha, D(d_\tau, b_\beta, c_\gamma))\rangle\\
&= \lim_\alpha \lim_\tau \lim_\beta \lim_\gamma \langle x^*, \pi_2(D(a_\alpha, b_\beta, c_\gamma), d_\tau)\rangle\\
&+ \lim_\alpha \lim_\tau \lim_\beta \lim_\gamma \langle x^*, \pi_1(a_\alpha, D(d_\tau, b_\beta, c_\gamma))\rangle\\
&= \langle x^*, \pi_2^{***}(D^{****}(a^{**}, b^{**}, c^{**}), d^{**})\rangle + \langle x^*, \pi_1^{***}(a^{**}, D^{****}(d^{**}, b^{**}, c^{**}))\rangle\\
&= \langle \pi_2^{***}(D^{****}(a^{**}, b^{**}, c^{**}), d^{**}) + \pi_1^{***}(a^{**}, D^{****}(d^{**}, b^{**}, c^{**})), x^*\rangle.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&D^{****}(\pi^{***}(a^{**}, d^{**}), b^{**}, c^{**})\\
&= \pi_2^{***}(D^{****}(a^{**}, b^{**}, c^{**}), d^{**}) + \pi_1^{***}(a^{**}, D^{****}(d^{**}, b^{**}, c^{**})).
\end{aligned}
$$

Applying (1) and (3) respectively, we can deduce that $w^* - \lim_\alpha w^* - \lim_\beta w^* - \lim_\tau w^* - \lim_\gamma \pi_2(D(a_\alpha, b_\beta, c_\gamma), d_\tau) = \pi_2^{***}(D^{****}(a^{**}, b^{**}, c^{**}), d^{**})$ and $w^* - \lim_\alpha w^* - \lim_\beta w^* - \lim_\tau w^* - \lim_\gamma \pi_1(b_\beta, D(a_\alpha, d_\tau, c_\gamma)) = \pi_1^{***}(b^{**}, D^{****}(a^{**}, d^{**}, c^{**}))$. So in similar way, we can deduce that

$$\begin{aligned} &D^{****}(a^{**}, \pi^{***}(b^{**}, d^{**}), c^{**}) \\ &= \pi_2^{***}(D^{****}(a^{**}, b^{**}, c^{**}), d^{**}) + \pi_1^{***}(b^{**}, D^{****}(a^{**}, d^{**}, c^{**})). \end{aligned}$$

Applying (4) and (5), we can write $w^* - \lim_\alpha w^* - \lim_\beta w^* - \lim_\gamma w^* - \lim_\tau \pi_1(c_\gamma, D(a_\alpha, b_\beta, d_\tau)) = \pi_1^{***}(c^{**}, D^{****}(a^{**}, b^{**}, d^{**}))$. Thus

$$\begin{aligned} &D^{****}(a^{**}, b^{**}, \pi^{***}(c^{**}, d^{**})) \\ &= \pi_2^{***}(D^{****}(a^{**}, b^{**}, c^{**}), d^{**}) + \pi_1^{***}(c^{**}, D^{****}(a^{**}, b^{**}, d^{**})). \end{aligned}$$

By comparing equations (4.1), (4.2) and (4.3) follows that $D^{****} : (A^{**}, \Box) \times (A^{**}, \Box) \times (A^{**}, \Box) \longrightarrow X^{**}$ is a tri-derivation.

For the converse, let D and $D^{****} : (A^{**}, \Box) \times (A^{**}, \Box) \times (A^{**}, \Box) \longrightarrow X^{**}$ be tri-derivation. We have to show that (1), (2), (3), (4) and (5) hold. We shall only prove (2) the others parts have similar argument. Fourth adjoint D^{****} is tri-derivation, thus we have

$$\begin{aligned} D^{****}(\pi^{***}(a, d^{**}), b^{**}, c^{**}) = &\ \pi_2^{***}(D^{****}(a, b^{**}, c^{**}), d^{**}) \\ &+ \pi_1^{***}(a, D^{****}(d^{**}, b^{**}, c^{**})). \end{aligned}$$

In the other hands, the mapping D is tri-derivation, which follows that

$$\begin{aligned} D^{****}(\pi^{***}(a, d^{**}), b^{**}, c^{**}) = &\ w^* - \lim_\tau w^* - \lim_\beta w^* - \lim_\gamma \pi_2(D(a, b_\beta, c_\gamma), d_\tau) \\ &+ \pi_1^{***}(a, D^{****}(d^{**}, b^{**}, c^{**})). \end{aligned}$$

Therefore follows that

$$\begin{aligned} &\pi_2^{***}(D^{****}(a, b^{**}, c^{**}), d^{**}) \\ &= w^* - \lim_\tau w^* - \lim_\beta w^* - \lim_\gamma \pi_2(D(a, b_\beta, c_\gamma), d_\tau). \end{aligned}$$

So, for every $d^{**} \in A^{**}$ we have

$$\begin{aligned} &\langle \pi_2^{****}(x^*, D^{****}(a, b^{**}, c^{**})), d^{**} \rangle = \langle x^*, \pi_2^{***}(D^{****}(a, b^{**}, c^{**}), d^{**}) \rangle \\ &= \lim_\tau \lim_\beta \lim_\gamma \langle x^*, \pi_2(D(a, b_\beta, c_\gamma), d_\tau) \rangle = \lim_\tau \lim_\beta \lim_\gamma \langle x^*, \pi_2^r(d_\tau, D(a, b_\beta, c_\gamma)) \rangle \\ &= \lim_\tau \lim_\beta \lim_\gamma \langle \pi_2^{r*}(x^*, d_\tau), D(a, b_\beta, c_\gamma) \rangle = \lim_\tau \lim_\beta \lim_\gamma \langle D^*(\pi_2^{r*}(x^*, d_\tau), a, b_\beta), c_\gamma \rangle \\ &= \lim_\tau \lim_\beta \langle c^{**}, D^*(\pi_2^{r*}(x^*, d_\tau), a, b_\beta) \rangle = \lim_\tau \lim_\beta \langle D^{**}(c^{**}, \pi_2^{r*}(x^*, d_\tau), a), b_\beta \rangle \\ &= \lim_\tau \langle b^{**}, D^{**}(c^{**}, \pi_2^{r*}(x^*, d_\tau), a) \rangle = \lim_\tau \langle D^{***}(b^{**}, c^{**}, \pi_2^{r*}(x^*, d_\tau)), a \rangle \\ &= \lim_\tau \langle D^{****}(a, b^{**}, c^{**}), \pi_2^{r*}(x^*, d_\tau) \rangle = \lim_\tau \langle D^{****}(a, b^{**}, c^{**}), \pi_2^{r*r}(d_\tau, x^*) \rangle \\ &= \lim_\tau \langle \pi_2^{r*r*}(D^{****}(a, b^{**}, c^{**}), d_\tau), x^* \rangle = \lim_\tau \langle \pi_2^{r*r**}(x^*, D^{****}(a, b^{**}, c^{**})), d_\tau \rangle \\ &= \langle \pi_2^{r*r**}(x^*, D^{****}(a, b^{**}, c^{**})), d^{**} \rangle. \end{aligned}$$

As $\pi_2^{r*r**}(x^*, D^{****}(a, b^{**}, c^{**}))$ always lies in A^*, we have reached (2). □

For case 2, fourth adjoint D^{****} of tri-derivation $D : A \times A \times A \longrightarrow X$ is a tri-derivation if and only if

1. $\pi_2^{**r*}(D^{****}(A^{**}, A^{**}, A^{**}), X^*) \subseteq A^*$,
2. $D^{****r*}(\pi_1^{****}(X^*, A^{**}), A^{**}, A^{**}) \subseteq A^*$,
3. $D^{******}(A^{**}, \pi_1^{****}(X^*, A^{**}), A) \subseteq A^*$,
4. $D^{*******}(A^{**}, A^{**}, \pi_1^{****}(X^*, A^{**})) \subseteq A^*$.

For case 3, fourth adjoint D^{****} of tri-derivation $D : A \times A \times A \longrightarrow X$ is a tri-derivation if and only if

1. $\pi_2^{****}(X^*, D^{****}(A, A^{**}, A^{**})) \subseteq A^*$,
2. $D^{******}(A^{**}, \pi_1^{****}(X^*, A^{**}), A) \subseteq A^*$,
3. $D^{*******}(A^{**}, A^{**}, \pi_1^{****}(X^*, A^{**})) \subseteq A^*$.

For case 4, fourth adjoint D^{****} of tri-derivation $D : A \times A \times A \longrightarrow X$ is a tri-derivation if and only if

1. $\pi_2^{**r*}(D^{****}(A, A, A^{**}), X^*) \subseteq A^*$,
2. $\pi_2^{****}(X^*, D^{****}(A, A^{**}, A^{**})) \subseteq A^*$,
3. $D^{****r*}(\pi_1^{****}(X^*, A^{**}), A^{**}, A^{**}) \subseteq A^*$,
4. $D^{*****}(\pi_1^{****}(X^*, A^{**}), A, A) \subseteq A^*$,
5. $D^{******}(A^{**}, \pi_1^{****}(X^*, A^{**}), A) \subseteq A^*$,
6. $D^{*******}(A^{**}, A^{**}, \pi_1^{****}(X^*, A^{**})) \subseteq A^*$.

For case 5, fourth adjoint D^{****} of tri-derivation $D : A \times A \times A \longrightarrow X$ is a tri-derivation if and only if

1. $\pi_2^{**r*}(D^{****}(A^{**}, A^{**}, A^{**}), X^*) \subseteq A^*$,
2. $\pi_2^{****}(X^*, D^{****}(A, A^{**}, A^{**})) \subseteq A^*$,
3. $D^{****r*}(\pi_1^{****}(X^*, A^{**}), A^{**}, A^{**}) \subseteq A^*$,
4. $D^{******}(A^{**}, \pi_1^{****}(X^*, A^{**}), A) \subseteq A^*$,
5. $D^{*******}(A^{**}, A^{**}, \pi_1^{****}(X^*, A^{**})) \subseteq A^*$.

For case 6, fourth adjoint D^{****} of tri-derivation $D : A \times A \times A \longrightarrow X$ is a tri-derivation if and only if

1. $\pi_2^{**r*}(D^{****}(A^{**}, A^{**}, A^{**}), X^*) \subseteq A^*$,
2. $D^{****r*}(\pi_1^{****}(X^*, A^{**}), A^{**}, A^{**}) \subseteq A^*$,
3. $D^{*****}(\pi_1^{****}(X^*, A^{**}), A, A) \subseteq A^*$,
4. $D^{******}(A^{**}, \pi_1^{****}(X^*, A^{**}), A) \subseteq A^*$,
5. $D^{*******}(A^{**}, A^{**}, \pi_1^{****}(X^*, A^{**})) \subseteq A^*$.

For case 7, fourth adjoint D^{****} of tri-derivation $D : A \times A \times A \longrightarrow X$ is a tri-derivation if and only if

1. $\pi_2^{****}(X^*, D^{****}(A, A^{**}, A^{**})) \subseteq A^*$,
2. $\pi_2^{**r*}(D^{****}(A, A, A^{**}), X^*) \subseteq A^*$,
3. $D^{*****}(\pi_1^{****}(X^*, A^{**}), A, A) \subseteq A^*$,
4. $D^{******}(A^{**}, \pi_1^{****}(X^*, A^{**}), A) \subseteq A^*$,

5. $D^{*******}(A^{**}, A^{**}, \pi_1^{****}(X^*, A^{**})) \subseteq A^*$.

For case 8, fourth adjoint D^{****} of tri-derivation $D : A \times A \times A \longrightarrow X$ is a tri-derivation if and only if

1. $\pi_2^{****}(X^*, D^{****}(A, A^{**}, A^{**})) \subseteq A^*$,
2. $\pi_2^{**r*}(D^{****}(A^{**}, A^{**}, A^{**}), X^*) \subseteq A^*$,
3. $D^{****r*}(\pi_1^{****}(X^*, A^{**}), A^{**}, A^{**}) \subseteq A^*$,
4. $D^{*****}(\pi_1^{****}(X^*, A^{**}), A, A) \subseteq A^*$,
5. $D^{******}(A^{**}, \pi_1^{****}(X^*, A^{**}), A) \subseteq A^*$,
6. $D^{*******}(A^{**}, A^{**}, \pi_1^{****}(X^*, A^{**})) \subseteq A^*$.

Remark 2 For adjoint $D^{r*****r}$ of tri-derivation $D : A \times A \times A \longrightarrow X$ we have the same argument.

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