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Some Edge Cut Sets and an Upper Bound for Edge Tenacity of Organic Compounds C_nH_{2n+2}

Gholam-Hasan Shirdel* ·
Boshra Vaez-Zadeh

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Abstract The graphs play an important role in our daily life. For example, the urban transport network can be represented by a graph, as the intersections are the vertices and the streets are the edges of the graph. Suppose that some edges of the graph are removed, the question arises, how damaged the graph is. There are some criteria for measuring the vulnerability of graph; the tenacity is the best criteria for measuring it. In this paper, we find some edge cut sets for organic compounds C_nH_{2n+2} and obtain an upper bound for $T_e(C_nH_{2n+2})$ by these edge cut sets.

Keywords Edge Tenacity · Edge cut sets · Organic compounds C_nH_{2n+2} · Tree

Mathematics Subject Classification (2010) 97k30

1 Introduction

Let $G = (V(G), E(G))$ be a graph. Consider some edges of G are deleted, so a problem is mentioned, how much the graph G is damaged. Some items are important to check the vulnerability of G, (1) the number of elements that are omitted, (2) the number of remaining connected subgraphs, (3) and the number of edges in the largest components of the graph after deleting the edges.

*Corresponding author
G.H. Shirdel
Department of Mathematics, Faculty of Sciences University of Qom, Qom, Iran.
Tel.: +989122147930
E-mail: shirdel81math@gmail.com

B. Vaez-Zadeh
Department of Mathematics, Faculty of Sciences University of Qom, Qom, Iran.

With these items some criteria are defined, edge connectivity, toughness, scattering, integrity, tenacity, rupture degree, etc. The connectivity is a parameter defined based on Quantity (1). Both toughness and scattering number take into account Quantities (1) and (2). The integrity is defined based on Quantities (1) and (3). Both the tenacity and rupture degree take into account all the three Quantities [2]. Some researchers compare these criteria and result shows the tenacity and the edge-tenacity are best criteria for measuring the vulnerability of G [3], [5], [6], [8]. We try on Edge Tenacity of C_nH_{2n+2} by using the definition of Edge Tenacity. In section 2 we obtain the Edge Tenacity for Methane, Ethan and Propane. Obtaining all edge cut sets is difficult so we reach some edge cut sets, and we calculate these YAFs, we work it in section 3. In section 4 we compare these amounts by $\frac{1}{2}$ or other case and obtain a bound for Edge Tenacity of C_nH_{2n+2}.

2 Background

Definition 1 Consider a graph $G = (V, E)$, V is the set of vertices of G and E is the set of edges of G. Show the number of vertices by ν and the number of edges of G by ϵ. We call G is empty if $\epsilon = 0$. If ν and ϵ are finite G is called finite.
A tree is a connected acyclic graph. For every tree, $\epsilon = \nu - 1$. The graph of C_nH_{2n+2} is a tree.

Definition 2 A cut edge of G is an edge e such that $\omega(G - e) > \omega(G)$. Let X and Y be subsets of $V(G)$ (not necessary disjoint). We denote by $[X, Y]$, the set of edges of G with one end in X and the other end in Y. When X is a proper subset of $V(G)$ and $Y = V\backslash X$, the set $[X, Y]$ is called an edge cut of G.

Definition 3 Tenacity of a graph is defined as

$$T(G) = min\{\frac{|S| + \tau(G - S)}{\omega(G - S)}\},$$

where the minimization is over S, vertex cut set of G; $|S|$ is the number of vertices in S, $\tau(G - S)$ is the number of vertices in the largest component of $G-S$, and $\omega(G-S)$ is the number of components of $G-S$. A Connected graph G is called T-tenacious, if for any subset S of vertices of G with $\omega(G-S) \geq 1$, we have $|S| + \tau(G - S) \geq T.\omega(G - S)$. If G is not a complete graph, then there is a largest T such that G is T-tenacious; this T is the tenacity of G. On the other hand, a complete graph contains no vertex cutset and so it is T-tenacious for every T. Accordingly, we define $T(K_p) = \infty$ for every p ($p \geq 1$). A set $S \subset V(G)$ is said to be a T-set of G if $T(G) = \frac{|S|+\tau(G-S)}{\omega(G-S)}$ [2].
Edge tenacity is defined similarly,

$$T_e(G) = min\{\frac{|E| + \tau(G - E)}{\omega(G - E)}\}, \tag{1}$$

where the minimization is over edge cut sets E, $\tau(G-S)$ is the number of edges in the largest component of $G-S$ and $\omega(G-S)$ is the number of components of $G-S$.

In this paper, the amount of $\frac{|E|+\tau(G-E)}{\omega(G-E)}$ is called the *" YIELD AMOUNT OF FRACTION (YAF) "*, made by a set E, and will be represented by a symbol *YAF(E)*.

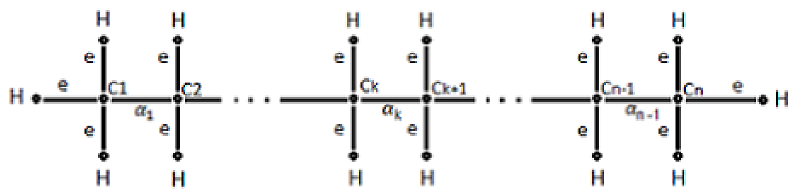

Fig. 1 Graph of C_nH_{2n+2}

As we see the edge tenacity of a graph is depended on edge cut sets of this graph, so first we obtain some edge cut sets of C_nH_{2n+2}. Assign to the Carbon atom by C_1, C_2, $\cdots$, C_n; and assign to the edge joining Carbon atom and Hydrogen atom by e, and the edge joining two Carbon atoms by α. The graph of C_nH_{2+2} is in Figure 1.

3 Some example of Edge Tenacity of C_nH_{2n+2}

Proposition 1 *Every edge in a tree is a cut edge [1].*

The graph of C_nH_{2n+2} is a tree, so all edges are cut edges. We work on all edges of the graph. First, we obtain the edge tenacity for simple cases.

Edge Tenacity of CH_4

The graph of Methane doesn't have any edge of the type α and all of its edges are of the type e. Let E be a subset of $E(G)$ that has i edge of the type e, so $|E| = i$, $1 \leq i \leq 4$, $\omega(G-E) = 1+i$ and $\tau(G-E) = 4-i$, thus

$$T_e(CH_4) = min_E\{\frac{|E| + \tau(G-E)}{\omega(G-E)}\} = min_{1\leq i\leq 4}\{\frac{4}{1+i}\} = \frac{4}{5}.$$

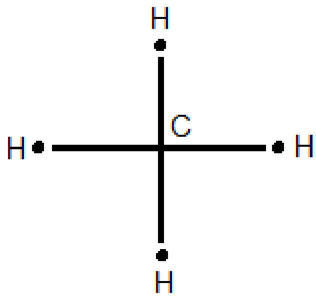

Fig. 2 Graph of methane

Edge Tenacity of C_2H_6

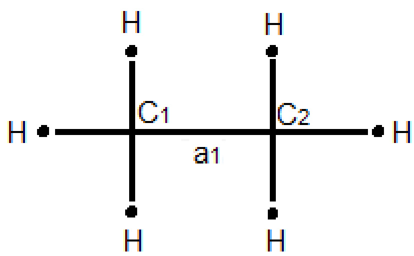

Fig. 3 Graph of ethane

The graph of Ethane has an edge of the type α and six edges of the type e, see Figure 3. If E just has i edges of the type e, then

$$YAF(E) = \frac{\epsilon}{1+i} = \frac{7}{1+i}, 1 \leq i \leq 6.$$

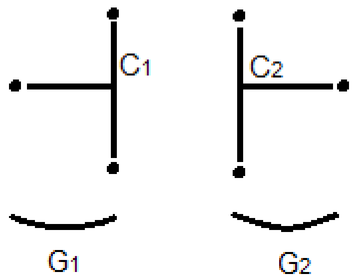

Fig. 4 Graph of $C_2H_6 - \alpha$

Consider $E_0 = \{\alpha_1\}$, so

$$|E_0| = 1, \omega_0 = \omega(G - E_0) = 2, \tau_0 = \tau(G - E_0) = 3,$$

and $YAF(E_0) = 2$, see Figure 4.
Assume $E = E_0 \cup A$, that A contains 1 edge of the type e, so

$$|E| = |E_0| + 1, \tau(G - E) = \tau_0, \omega(G - E) = \omega_0 + 1,$$

then $YAF(E) = \frac{5}{3}$.
Assume $E = E_0 \cup A$, that A contains 2 edges of the type e. If both edges e belong to G_1 or belong to G_2, then

$$|E| = |E_0| + 2, \omega = \omega_0 + 2, \tau = \tau_0,$$

and thus $YAF(E) = \frac{6}{4}$. If one edge e belongs to G_1 and the other edge belongs to G_2, then

$$|E| = |E_0| + 2, \omega = \omega_0 + 2, \tau = \tau_0 - 1,$$

and thus $YAF(E) = \frac{5}{4}$.
Assume $E = E_0 \cup A$, that A contains three edges of the type e, so

$$|E| = |E_0| + 3, \omega = \omega_0 + 3.$$

If these edges belong to one component of $G - E_0$, so $\tau = \tau_0$ and then $YAF(E) = \frac{7}{5}$. If one edge belongs to a component and other edges belong to the other component, so $\tau = \tau_0 - 1$ and then $YAF(E) = \frac{6}{5}$.
Assume $E = E_0 \cup A$, that A contains four edges of the type e, so

$$|E| = |E_0| + 4, \omega = \omega_0 + 4.$$

If three edges e belong to one component of $G - E_0$, so $\tau = \tau_0 - 1$ and then $YAF(E) = \frac{7}{6}$. If two edges e belong to one component of $G - E_0$ and other edges belong to other component, so $\tau = \tau_0 - 2$ and then $YAF(E) = 1$.
Assume $E = E_0 \cup A$, that A contains five edges of the type e, so

$$|E| = |E_0| + 5, \omega = \omega_0 + 5.$$

In this case, three edges of the type e are in a component and other edges belong to other component, then $\tau = \tau_0 - 2 = 1$, and thus $YAF(E) = 1$.
Assume $E = E_0 \cup A$, that A contains all of edges of the type e, so

$$|E| = |E_0| + 6, \omega = \omega_0 + 6, \tau = \tau_0 - 3 = 0,$$

thus $YAF(E) = \frac{7}{8}$.
We checked all edge cut sets and their YAFs. By Definition 3, we have

$$T_e(C_2H_6) = min_{\{E|E \text{ is an edge cut set}\}}\{\frac{|E| + \tau(G - E)}{\omega(G - E)}\} = \frac{7}{8}.$$

Edge Tenacity of C_3H_8

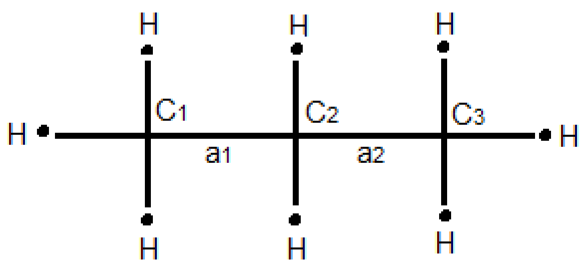

Fig. 5 Graph of C_3H_8

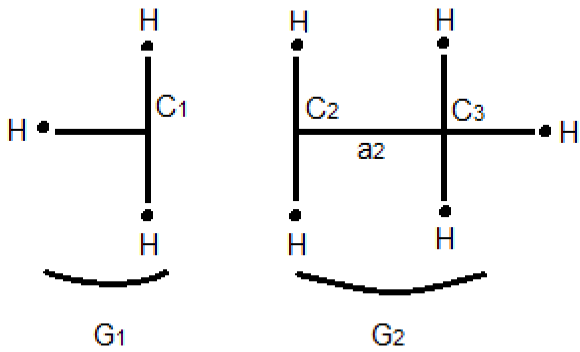

Fig. 6 Graph of $C_3H_8 - \{\alpha_1\}$

Graph of molecule propane is symmetric, so all cases for α_1 is true for α_2.

Consider $E_0 = \{\alpha_1\}$. Name the components of $G - E_0$ by G_1 and G_2, so

$$|E_0| = 1, \omega_0 = \omega(G - E_0) = 2, \tau_0 = \tau(G - E_0) = 6,$$

thus $YAF(E_0) = \frac{7}{2}$.
Assume $E = E_0 \cup A$, that A contains one edge of the type e,

$$|E| = |E_0| + 1, \omega = \omega_0 + 1.$$

The amount of τ is changed, depending on the edge e belongs to G_1 or G_2 . If $e \in E(G_1)$, so $\tau = \tau_0$ and then $YAF(E) = \frac{8}{3}$. If $e \in E(G_2)$, so $\tau = \tau_0 - 1$ and then $YAF(E) = \frac{7}{3}$.
Assume $E = E_0 \cup A$, that A contains two edges of the type e ,then

$$|E| = |E_0| + 2, \omega = \omega_0 + 2.$$

If both edges belong to G_1, so $\tau = \tau_0$ and $YAF(E) = \frac{9}{4}$. If both edges belong to G_2, so $\tau = \tau_0 - 2$ and $YAF(E) = \frac{7}{4}$. If one edge belongs to G_1 and other one belongs to G_2, then $\tau = \tau_0 - 1$ and $YAF(E) = \frac{8}{4}$.
Assume $E = E_0 \cup A$, that A contains three edges of the type e, thus

$$|E| = |E_0| + 3, \omega = \omega_0 + 3.$$

If three edges belong to G_1, so $\tau = \tau_0$ and thus $YAF(E) = \frac{10}{5}$. If exactly two edges belong to G_1, so $\tau = \tau_0 - 1$ and then $YAF(E) = \frac{9}{5}$. If just one edge belongs to G_1, so $\tau = \tau_0 - 2$ and then $YAF(E) = \frac{8}{5}$. If no one edges belong to G_1, we have $\tau = \tau_0 - 3$ and therefore $YAF(E) = \frac{7}{5}$.
Assume $E = E_0 \cup A$, that A contains four edges of the type e, then

$$|E| = |E_0| + 4, \omega = \omega_0 + 4.$$

If four edges are in $E(G_2)$, so $\tau = |E(G_1)| = 3$ and thus $YAF(E) = \frac{8}{6}$. If three are in $E(G_2)$, so $\tau = \tau_0 - 3 = 3$ and then $YAF(E) = \frac{8}{6}$. If two edges are in $E(G_2)$, so $\tau = \tau_0 - 2 = 4$ and thus $YAF(E) = \frac{9}{6}$. If just one edge is in $E(G_2)$, so $\tau = \tau_0 - 1 = 5$ and then $YAF(E) = \frac{10}{6}$.
Assume $E = E_0 \cup A$, that A contains five edges of the type e, therefore

$$|E| = |E_0| + 5, \omega = \omega_0 + 5.$$

If two edges belong to $E(G_2)$, so $\tau = \tau_0 - 2 = 4$ and then $YAF(E) = \frac{10}{7}$. If three edges belong to $E(G_2)$, so $\tau = \tau_0 - 3 = 3$ and thus $YAF(E) = \frac{9}{7}$. If four edges belong to $E(G_2)$, so $\tau = 2$ and then $YAF(E) = \frac{8}{7}$. If five edges belong to $E(G_2)$, so $\tau = |E(G_1)| = 3$ and thus $YAF(E) = \frac{9}{7}$.
Assume $E = E_0 \cup A$, that A contains six edges of the type e, therefore

$$|E| = |E_0| + 6, \omega = \omega_0 + 6.$$

If three edges are in $E(G_2)$, so $\tau = \tau_0 - 3 = 3$ and $YAF(E) = \frac{10}{8}$. If four edges are in $E(G_2)$, so $\tau = \tau_0 - 4 = 2$ and $YAF(E) = \frac{9}{8}$. If five edges are in $E(G_2)$, so $\tau = 2$ and $YAF(E) = \frac{9}{8}$.
Assume $E = E_0 \cup A$, that A contains seven edges of the type e, so

$$|E| = |E_0| + 7, \omega = \omega_0 + 7.$$

If four edges belong to $E(G_2)$, so $\tau = 2$ and thus $YAF(E) = \frac{10}{9}$. If five edges belong to $E(G_2)$, so $\tau = 1$ and $YAF(E) = 1$.
Assume $E = E_0 \cup A$, that A contains eight edges of the type e, therefore

$$|E| = |E_0| + 8, \omega = \omega_0 + 8, \tau = 1,$$

and then $YAF(E) = 1$.
Consider $E_0 = \{\alpha_1, \alpha_2\}$, so

$$|E_0| = 2, \omega_0 = 3, \tau_0 = 3$$

and $YAF(E_0) = \frac{5}{3}$.
If $E = E_0 \cup A$, that A contains one edge of the type e, so

$$|E| = |E_0| + 1, \omega = \omega_0 + 1, \tau = \tau_0,$$

therefore $YAF(E) = \frac{6}{4}$.
Assume $E = E_0 \cup A$, that A contains two edges of the type e, so

$$|E| = |E_0| + 2, \omega = \omega_0 + 2.$$

If both edges belong to $E(G_1)$ or $E(G_2)$, then $\tau = \tau_0$ and $YAF(E) = \frac{7}{5}$. If one edge belongs to $E(G_1)$ or $E(G_3)$ and other one belongs to $E(G_2)$, then $\tau = \tau_0$ and $YAF(E) = \frac{7}{5}$. If one edge belongs to $E(G_1)$ and other edge belongs to $E(G_3)$, then $\tau = \tau_0 - 1$ and $YAF(E) = \frac{6}{5}$.
Assume $E = E_0 \cup A$, that A contains three edges of the type e, therefore

$$|E| = |E_0| + 3, \omega = \omega_0 + 3.$$

If three edges are in $E(G_1)$ or $E(G_3)$, so $\tau = \tau_0$ and then $YAF(E) = \frac{8}{6}$. If three edges are in $E(G_1 \cup G_2)$ or $E(G_3) \cup E(G_2)$, so $\tau = \tau_0$ and then $YAF(E) = \frac{8}{6}$. If three edges belong to $E(G_1) \cup E(G_3)$ (not only in one), so $\tau = \tau_0 - 1 = 2$ and $YAF(E) = \frac{7}{6}$.
Assume $E = E_0 \cup A$, that A contains four edges of the type e, therefore

$$|E| = |E_0| + 4, \omega = \omega_0 + 4.$$

If these edges are in $E(G_1) \cup E(G_2)$ or $E(G_2) \cup E(G_3)$, then $\tau = \tau_0$ and $YAF(E) = \frac{9}{7}$. If these edges are in $E(G_1) \cup E(G_3)$, so $\tau = 2$ and then $YAF(E) = \frac{8}{7}$.
Assume $E = E_0 \cup A$, that A contains five edges of the type e, therefore

$$|E| = |E_0| + 5, \omega = \omega_0 + 5.$$

If these edges belong to $E(G_1) \cup E(G_3)$, so $\tau = 2$, and then $YAF(E) = \frac{9}{8}$. If these edges belong to $E(G_1) \cup E(G_2)$ or $E(G_2) \cup E(G_3)$, so $\tau = \tau_0 = 3$ and then $YAF(E) = \frac{10}{8}$.
Assume $E = E_0 \cup A$, that A contains six edges of the type e, so

$$|E| = |E_0| + 6, \omega = \omega_0 + 6.$$

If no one edges belong to $E(G_2)$, so $\tau = 2$ and thus $YAF(E) = \frac{10}{9}$. If just one edge belongs to $E(G_2)$, so $\tau = 1$ and then $YAF(E) = 1$. If two edges are in $E(G_2)$ and three edges belong to $E(G_1)$ or $E(G_3)$, so $\tau = 2$ and then $YAF(E) = \frac{10}{9}$. If two edges belong to $E(G_2)$, two edges belong to $E(G_1)$ and two edges belong to $E(G_3)$, so $\tau = 1$ and thus $YAF(E) = 1$.
Assume $E = E_0 \cup A$, that A has seven edges of the type e, so

$$|E| = |E_0| + 7, \omega = \omega_0 + 7.$$

In this case one edge is or two edges are in $E(G_2)$, by any way, $\tau = 1$ and therefore $YAF(E) = 1$.
Assume $E = E_0 \cup A$, that A has eight edges of the type e, so

$$|E| = |E_0| + 8, \omega = \omega_0 + 8, \tau = 0,$$

therefore $YAF(E) = \frac{10}{11} < 1$.

Therefore by Definition 3,

$$T_e(C_3H_8) = \frac{10}{11}.$$

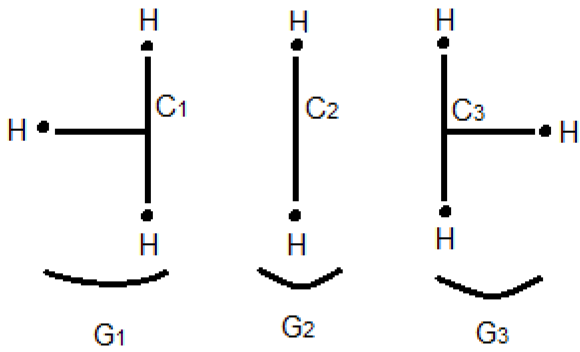

Fig. 7 Graph of $C_3H_8 - \{\alpha_1, \alpha_2\}$

4 Some Edge-Cut Sets and their YAFs for General Cases

By Definition 3 we need all edge-cut sets, But it is hard to find all the cut edge sets. Therefore we use some edge-cut set and reach these YAFs. We work on $n \geq 4$.

Theorem 1 *Consider* $E_0 = \{\alpha_1\}$ *and* $E = E_0 \cup A$ *that* A *has* s *edges of the type* e*,* $0 \leq s \leq 2n + 2$ *, then*

$$YAF(E) \geq \frac{|E_0| + \tau_0}{\omega_0 + s}.$$

Proof. The graph of $G - E_0$ has two components, also

$$|E_0| = 1, \omega_0 = 2, \tau_0 = 3(n-1).$$

Consider $E = E_0 \cup A$, that A has s edges of the type e, and i edges of A incident with C_1, that $0 \leq i \leq 3$, so

$$|E| = |E_0| + s, \omega(G - E) = \omega_0 + s, \tau = \tau_0 - (s - i),$$

therefore

$$YAF(E) = \frac{|E| + \tau(G-E)}{\omega(G-E)} = \frac{|E_0| + s + \tau_0 - (s-i)}{\omega_0 + s} = \frac{|E_0| + \tau_0 + i}{\omega_0 + s} \geq \frac{|E_0| + \tau_0}{\omega_0 + s}.$$

□

Note. The above relations are commised firmly for $E_0 = \{\alpha_{n-1}\}$.

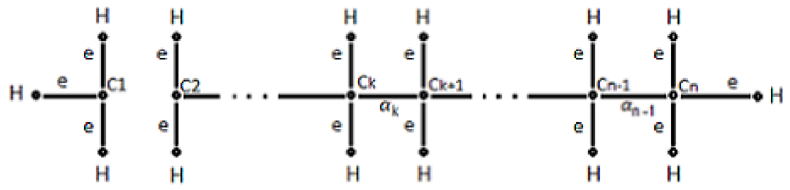

Fig. 8 Graph of $G - \{\alpha_1\}$

Theorem 2 *Consider* $E_0 = \{\alpha_1, \alpha_2\}$ *and* $E = E_0 \cup A$ *that* A *has* s *edges of the type* e, $0 \leq s \leq 2n+2$ *, then*

$$YAF(E) \geq \frac{|E_0| + \tau_0}{\omega_0 + s}.$$

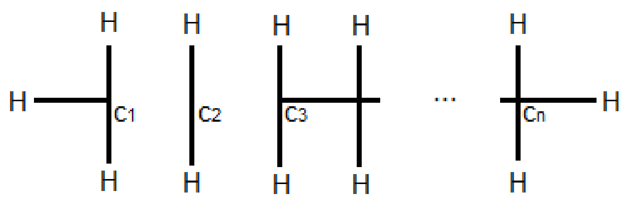

Fig. 9 Graph of $G - \{\alpha_1, \alpha_2\}$

Proof. The graph of $G - E_0$ is in Figure 9.

$$|E_0| = 2, \omega(G - E_0) = 3, \tau(G - E_0) = 3(n-2), YAF(E_0) = \frac{3n-4}{3}.$$

Let E^1 be a set of edges of the type e don't incident with C_1 and/or C_2. Consider $E = E_0 \cup A$, that A has s edges of the type e and let $|A \cap E^1| = i$, $(0 \leq i \leq s)$, so

$$|E| = |E_0| + s, \omega(G - E) = \omega_0 + s, \tau(G - E) = \tau_0 - i.$$

Therefore

$$YAF(E) = \frac{|E| + \tau(G-E)}{\omega(G-E)} = \frac{|E_0| + s + \tau_0 - i}{\omega_0 + s} \geq \frac{|E_0| + \tau_0}{\omega_0 + s}$$

□

Note. If $E_0 = \{\alpha_{n-1}, \alpha_{n-2}\}$ then the above relations are established.

Theorem 3 *Consider* $E_0 = \{\alpha_k\}$ *and* $E = E_0 \cup A$ *that* A *has* s *edges of the type* e, $0 \leq s \leq 2n+2$, *then*

$$YAF(E) \geq \frac{|E_0| + \tau_0}{\omega_0 + s}.$$

Proof. $E_0 = \{\alpha_k\}$, so

$$|E_0| = 1, \omega(G - E_0) = 2.$$

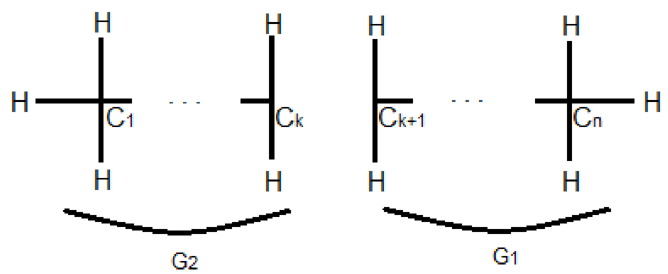

Fig. 10 Graph of $G - \{\alpha_k\}$

Name the components of $G - E_0$ by G_1 and G_2.

$$|E(G_1)| = 3(n-k), |E(G_2)| = 3k.$$

Also $|E(G_1)| \geq |E(G_2)|$ iff $n \geq 2k$, therefore if $n \geq 2k$ so $\tau_0 = \tau(G - E_0) = |E(G_1)| = 3(n-k)$ and if $n < 2k$ then $\tau_0 = \tau(G - E_0) = |E(G_2)| = 3k$ (Note the function τ_0 is continues). So

$$|E_0| = 1, \omega_0 = 2, \tau_0 = \begin{cases} 3(n-k) & , n \geq 2k, \\ 3k & , n < 2k. \end{cases}$$

Consider $E = E_0 \cup A$ that A has s edges of the type e, so

$$|E| = |E_0| + s, \omega(G - E) = \omega_0 + s.$$

Assume i edges of A are in $E(G_1)$ and j edges of A are in $E(G_2)$ $(i+j=s)$. Without loss of generality assume $|E(G_1)| \geq |E(G_2)|$, so $\tau_0 = |E(G_1)|$. G_1 translates to G_1' and G_2 translates to G_2' after removing E from G, so

$$|E(G_1')| = |E(G_1)| - i = \tau_0 - i, |E(G_2')| = |E(G_2)| - j.$$

Note $|E(G_1)| + |E(G_2)| + 1 = \epsilon$ thus $|E(G_2')| = 3n - \tau_0 - j$. For reaching $\tau(G-E)$, we see $|E(G_1')| \geq |E(G_2')|$ iff $2\tau_0 \geq 3n - j + i$, therefore if $2\tau_0 \geq 3n - j + i$ then $\tau(G-E) = |E(G_1')| = \tau_0 - i$ and

$$YAF(E) = \frac{|E_0| + \tau_0 + j}{\omega_0 + s} \geq \frac{|E_0| + \tau_0}{\omega_0 + s}.$$

and if $2\tau_0 < 3n - j + i$, then $\tau(G-E) = |E(G_2')| = 3n - \tau_0 - j$, note $i \leq s$ so $\tau_0 < 3n - \tau_0 + i$ and thus

$$YAF(E) = \frac{|E_0| + s + 3n - \tau_0 - j}{\omega_0 + s} \geq \frac{|E_0| + \tau_0}{\omega_0 + s}.$$

□

Theorem 4 *Consider $E_0 = \{\alpha_1, \alpha_k\}$ $(3 \leq k \leq n-2)$, and $E = E_0 \cup A$ that A has s edges of the type e, $0 \leq s \leq 2n+2$, then*

$$YAF(E) \geq \frac{|E_0| + \tau_0}{\omega_0 + s}.$$

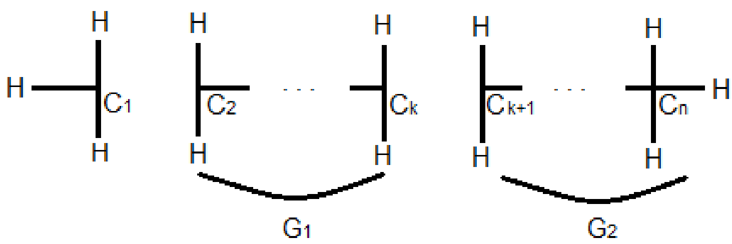

Fig. 11 Graph of $G - \{\alpha_1, \alpha_k\}$

Proof. Graph of $G - \{\alpha_1, \alpha_k\}$ is in Figure 11.

$$|E_0| = 2, \omega_0 = 3.$$

Also $|E(G_1)| = 3(k-1)-1$ and $|E(G_2)| = 3(n-k)$ and $|E(G_1)| \geq |E(G_2)|$ iff $3n \leq 6k-4$. Thus if $3n \leq 6k-4$ then $\tau_0 = |E(G_1)| = 3(k-1)-1$ and if $3n > 6k-4$ then $\tau_0 = |E(G_2)| = 3(n-k)$. So

$$|E_0| = 2, \omega_0 = 3, \tau_0 = \begin{cases} 3(k-1)-1 \text{ , } 3n \leq 6k-4 \\ 3(n-k) \quad\quad \text{ , } 3n > 6k-4 \end{cases}.$$

Consider $E = E_0 \cup A$, that A has s edges of the type e. Let E^1 be a set of edges of the form e incident with C_1. Without reducing the generality, assume G_1 is the largest component of $G - E_0$, therefore $|E(G_1)| = \tau_0$ and we have $|E(G_1)| + |E(G_2)| + 5 = \epsilon$, so $|E(G_2)| = 3n - \tau_0 - 4$.
Let $|A \cap E^1| = l$, $|A \cap E(G_1)| = i$ and $|A \cap E(G_2)| = j$, so $i + j + l = s$. G_1 is changed to G'_1 by deleting i edges from G_1 and G_2 is changed by deleting j edges of the from G_2, so

$$|E(G'_1)| = |E(G_1)| - i, |E(G'_2)| = |E(G_2)| - j,$$

$$|E| = |E_0| + s, \omega(G - E) = \omega_0 + s.$$

Note $|E(G'_1)| \geq |E(G'_2)$ iff $2\tau_0 \geq 3n+i-j-4$. Therefore if $2\tau_0 \geq 3n+i-j-4$, then $\tau(G-E) = |E(G'_1)| = \tau_0 - i$, so

$$YAF(E) = \frac{|E| + \tau(G-E)}{\omega(G-E)} = \frac{|E_0| + s + \tau_0 - i}{\omega_0 + s} \geq \frac{|E_0| + \tau_0}{\omega_0 + s},$$

and if $2\tau_0 < 3n + i - j - 4$, then $\tau(G - E) = |E(G'_2)| = 3n - \tau_0 - 4 - j$, so

$$YAF(E) = \frac{|E| + \tau(G-E)}{\omega(G-E)} = \frac{|E_0| + s + 3n - \tau_0 - 4 - j}{\omega_0 + s}$$

because $2\tau_0 < 3n + i - j - 3$ and $i \leq s$, we have $2\tau_0 < 3n + s - j - 4$ so $s + 3n - \tau_0 - 4 - j \geq \tau_0$, therefore

$$YAF(E) \geq \frac{|E_0| + \tau_0}{\omega_0 + s}.$$

□

Note Butane doesn't have this edge-cut set, so we have this set for $n \geq 5$.

Theorem 5 *Consider* $E_0 = \{\alpha_k, \alpha_{k+1}\}$ *and* $E = E_0 \cup A$ *that* A *has* s *edges of the type* e, $0 \leq s \leq 2n+2$ *, then*

$$YAF(E) \geq \frac{|E_0| + \tau_0}{\omega_0 + s}.$$

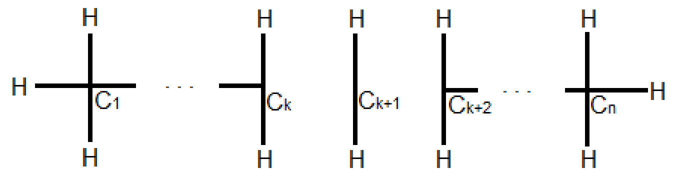

Fig. 12 Graph of $G-\{\alpha_k,\alpha_{k+1}\}$

Proof. The graph of $G-E_0$ is in Figure 12,

$$|E_0|=2,\omega_0=3.$$

Also $|E(G_1)|=3k$ and $|E(G_2)|=3(n-(k+1))$, so $|E(G_1)|\geq|E(G_2)|$ iff $n\leq 2k+1$. If $2k\geq n-1$, then $\tau_0=3k$ and if $2k<n-1$, then $\tau_0=3(n-(k+1))$, so

$$|E_0|=2,\omega_0=3,\tau_0=\begin{cases}3k & ,2k\geq n-1,\\ 3(n-(k+1)) & ,2k<n-1.\end{cases}$$

Assume $E=E_0\cup A$, where A contains s edges of the form e. Let E^1 be a set of edges of the type e incident with C_{k+1}. Also consider $|A\cap E^1|=l$ $(0\leq l\leq 2)$, $|A\cap E(G_1)|=i$ and $|A\cap E(G_2)|=j$ (i.e. $i+j+l=s$). Consider G_1 is changed to G'_1 by omitting i edges of the form e, and G_2 is changed to G'_2 by omitting j edges of the form e, $|E(G'_1)|=|E(G_1)|-i$ and $|E(G'_2)|=|E(G_2)|-j$. Without reducing the generality, consider G_1 is the largest component of $G-E_0$, thus $\tau_0=|E(G_1)|$. We have $|E(G_1)|+|E(G_2)|+4=\epsilon$, so $|E(G_2)|=3n-\tau_0-3$. Note $|E(G'_1)|\geq|E(G'_2)|$ iff $2\tau_0\geq 3n+i-j-3$. Therefore if $2\tau_0\geq 3n+i-j-2$ then $\tau=|E(G'_1)|=\tau_0-i$ and

$$YAF(E)=\frac{|E|+\tau(G-E)}{\omega(G-E)}=\frac{|E_0|+s+\tau_0-i}{\omega_0+s}\geq\frac{|E_0|+\tau_0}{\omega_0+s}.$$

If $2\tau_0<3n+i-j-3$ then $\tau=|E(G'_2)|=3n-\tau_0-j-2$ and because $i\leq s$ we have $s+3n-\tau_0-3-j\geq\tau_0$, and

$$YAF(E)=\frac{|E|+\tau(G-E)}{\omega(G-E)}=\frac{|E_0|+s+3n-\tau_0-3-j}{\omega_0+s}\geq\frac{|E_0|+\tau_0}{\omega_0+s}.$$

□

Theorem 6 *Consider $E_0=\{\alpha_1,\cdots,\alpha_k\}$ $(k<n-1)$ and $E=E_0\cup A$ that A has s edges of the type e, $0\leq s\leq 2n+2$, then*

$$YAF(E)\geq\frac{|E_0|+\tau_0}{\omega_0+s}.$$

Proof. We have

$$|E_0| = k, \omega_0 = k + 1, \tau_0 = 3(n - k).$$

Assume $E = E_0 \cup A$ where A has s edges of the type e. Let E^1 be a set of edges of form e incident with C_i, $1 \leq i \leq k$, and let E^2 the set of edges of form e don't incident with C_i, $1 \leq i \leq k$. If $|A \cap E^2| = l$ $(l \leq 2(n - k) + 1 \leq s)$, so $|E| = |E_0| + s$, $\omega = \omega_0 + s$ and $\tau = \tau_0 - l$. Therefore

$$YAF(E) = \frac{|E| + \tau(G - E)}{\omega(G - E)} = \frac{|E_0| + s + \tau_0 - l}{\omega_0 + s} \geq \frac{|E_0| + \tau_0}{\omega_0 + s}.$$

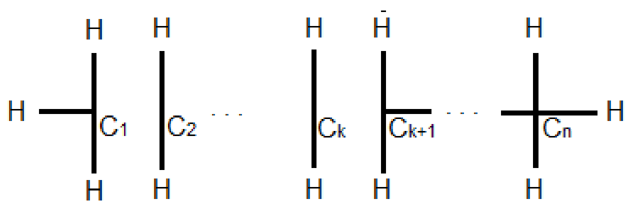

Fig. 13 Graph of $G - \{\alpha_1, \cdots, \alpha_k\}$

□

Theorem 7 *Consider* $E_0 = \{\alpha_1, \alpha_2, \cdots, \alpha_{n-1}\}$ *and* $E = E_0 \cup A$ *that* A *has* s *edges of the type* e, $0 \leq s \leq 2n + 2$ *, then*

$$YAF(E) \geq \frac{|E_0| + \tau_0}{\omega_0 + s}.$$

Proof. We have

$$|E_0| = n - 1, \omega_0 = n, \tau_0 = 3.$$

Assume $E = E_0 \cup A$ that A contains s edges of the type e. Let l edges of A incident with C_1 or C_n $(0 \leq l \leq 6)$, $|E| = |E_0| + s$ and $\omega = \omega_0 + s$. There are some relations between τ, τ_0 and l. Thus $\tau = \tau_0$ or $\tau = \tau_0 - 1$; therefore $\tau \leq \tau_0$. So $\tau + s \geq \tau_0$, and then

$$YAF(E) = \frac{|E| + \tau(G - E)}{\omega(G - E)} = \frac{|E_0| + s + \tau}{\omega_0 + s} \geq \frac{|E_0| + \tau_0}{\omega_0 + s}.$$

□

l	τ
1	$3=\tau_0$
2	$3=\tau_0$ or $2=\tau_0-1$
3	$3=\tau_0$ or $2=\tau_0-1$
4	$2=\tau_0-1$
5	$2=\tau_0-1$
6	$2=\tau_0-1$

Fig. 14 The table of l, τ and τ_0

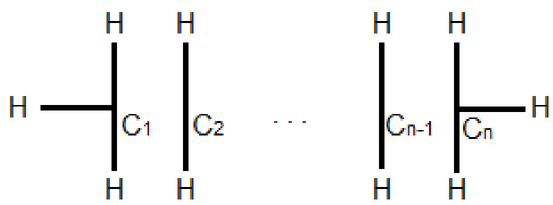

Fig. 15 Graph of $G-\{\alpha_1,\cdots,\alpha_n\}$

5 Finding a best upper bound for Edge Tenacity

As we seen, in every cases of E in the previous section, the yield amount of fraction of E is grater than or equal $\frac{|E_0|+\tau_0}{\omega_0+s}$, $(0 \leq s \leq 2n+2)$. We compare $YAF(E)$ by $\frac{1}{2}$ for every E and we show $T_e(C_nH_{2n+2}) \leq \frac{n+3}{3n+2}$.

Theorem 8 *Consider* $E_0 = \{\alpha_k\}$ *(*$1 \leq k \leq n-1$*) and* $E = E_0 \cup A$ *that* A *has* s *edges of the type* e *(*$0 \leq s \leq 2n+2$*), then*

$$YAF(E) \geq \frac{1}{2}. \tag{2}$$

Proof. $E_0 = \{\alpha_k\}$, $1 \leq k \leq n-1$. Let $E = E_0 \cup A$, that A has s edges of the type e. Consider $n \geq 2k$, so for all s, $0 \leq s \leq 2n+2$,

$$\frac{|E_0| + \tau_0}{\omega_0 + s} \geq \frac{3n - 3k + 1}{2n + 4},$$

and because $n \geq 2k$, we have $2n > 3k$ and therefore

$$\frac{6n - 6k + 1}{2n + 4} \geq 1,$$

thus $YAF(E) > \frac{1}{2}$.
Now Consider $n < 2k$, so for all s, $0 \leq s \leq 2n+2$, we have

$$\frac{|E_0| + \tau_0}{\omega_0 + s} \geq \frac{3k + 1}{2n + 4},$$

and because $n < 2k$, we have $n + 1 < 3k$ and therefore $6k + 2 > 2n + 4$, thus $YAF(E) > \frac{1}{2}$. □

Theorem 9 *Consider* $E_0 = \{\alpha_1, \alpha_k\}$ *(*$1 \leq k \leq n-1$*) and* $E = E_0 \cup A$ *that* A *has* s *edges of the type* e *(*$0 \leq s \leq 2n+2$*), then*

$$YAF(E) \geq \frac{1}{2}.$$

Proof. Consider $E = E_0 \cup A$ that A has s edges of the type e. Assume $3n \leq 6k - 4$. For all s, $0 \leq s \leq 2n+2$,

$$\frac{|E_0| + \tau_0}{\omega_0 + s} \geq \frac{2 + 3(k-1) - 1}{2n + 5}.$$

On the other hand $3n \leq 6k-4$, so $2n \leq 6k-n-4$ and $n > 5$, thus $6k-n-4 \leq 6k-9$, therefore $2n \leq 6k-9$ and

$$6k - 4 \geq 2n + 5,$$

so $YAF(E) > \frac{1}{2}$.
Assume $3n > 6k - 4$. For all s, $0 \leq s \leq 2n+2$,

$$\frac{|E_0| + \tau_0}{\omega_0 + s} \geq \frac{3n - 3k + 2}{2n + 5},$$

$n \geq 4$, so $4n > 3n + 3$ and $3n + 3 > 6k - 1$, therefore $4n > 6k - 1$ and thus

$$6n - 6k + 4 > 2n + 5$$

and then $YAF(E) > \frac{1}{2}$. □

Theorem 10 *Consider $E_0 = \{\alpha_k, \alpha_{k+1}\}$ $(1 \leq k \leq n-1)$ and $E = E_0 \cup A$ that A has s edges of the type e $(0 \leq s \leq 2n+2)$, then*

$$YAF(E) \geq \frac{1}{2}.$$

Proof. Let $E = E_0 \cup A$ that A has s edges of the type e. Assume $2k \geq n-1$, for all s $(0 \leq s \leq 2n+2)$,

$$\frac{|E_0| + \tau_0}{\omega_0 + s} \geq \frac{3k+2}{2n+5}.$$

$n > 4$, so $3n-3 > 2n+1$ and also $2k \geq n-1$, so $6k \geq 3n-3$, therefore $6k > 2n+1$, thus

$$6k + 4 > 2n + 5,$$

and then $YAF(E) > \frac{1}{2}$.
Assume $2k < n-1$ and $E = E_0 \cup A$ that A has s edges of the type e. For all s,

$$\frac{|E_0| + \tau_0}{\omega_0 + s} \geq \frac{3n-3k-1}{2n+5},$$

$2k < n-1$ and $n > 4$, so $6k + 7 < 3n + 4 < 4n$ and

$$6n - 6k - 2 > 2n + 5,$$

therefore $YAF(E) > \frac{1}{2}$. □

Theorem 11 *Consider $E_0 = \{\alpha_1, \cdots, \alpha_k\}$ and $E = E_0 \cup A$ that A has s edges of the type e $(0 \leq s \leq 2n+2)$, then*

$$YAF(E) \geq \frac{n+2}{3n+2}.$$

Proof. Consider $E_0 = \{\alpha_1, \cdots, \alpha_k\}$. Let $E = E_0 \cup A$, that A has s edges of the type e,

$$\frac{|E_0| + \tau_0}{\omega_0 + s} \geq \frac{3n-2k}{2n+k+3},$$

for all s, $0 \leq s \leq 2n+2$. Let

$$f(k) := \frac{3n-2k}{2n+k+3}$$

the function f is a decreasing function, so

$$k < n-1 \rightarrow f(k) > f(n-1) \rightarrow \frac{3n-2k}{2n+k+3} > \frac{n+2}{3n+2}. \tag{3}$$

□

Theorem 12 *Consider* $E_0 = \{\alpha_1, \cdots, \alpha_{n-1}\}$ *and* $E = E_0 \cup A$ *that* A *has* s *edges of the type* e $(0 \leq s \leq 2n+2)$*, then*

$$YAF(E) \leq \frac{1}{2}.$$

Proof. Consider $E_0 = \{\alpha_1, \cdots, \alpha_{n-1}\}$. Let $E = E_0 \cup A$ that A has s edges of the type e. We have

$$\frac{|E_0| + \tau_0}{\omega_0 + s} \geq \frac{n+2}{3n+2}.$$

Beacause $n > 2$, thus

$$\frac{n+2}{3n+2} \leq \frac{1}{2}. \tag{4}$$

□

The below conclusion is the main result of this paper.

Conclusion. As we seen, for any cut-edge set E, except 3 and 4, therefore

$$YAF(E) \geq \frac{1}{2}.$$

Because of relationship between Theorems 11 and 12 we have

$$\frac{n+2}{3n+2} \leq min\{\frac{1}{2}, \frac{3n-2k}{2n+k+3}\}, k < n-1.$$

Therefore by Definition 3,

$$T_e(C_nH_{2n+2}) \leq \frac{n+3}{3n+2}.$$

By the above we had a best upper bound for Edge tenacity of the organic compounds C_nH_{2n+2}. May be there is an organic compound by the edge tenacity equal to the upper bound we had.

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